ILLINOIS JOURNAL OF MATHEMATICS
Volume 29, Number 1, Spring 1985

# EQUIVARIANT ISOTOPIES AND SUBMERSIONS

BY

R. LASHOF

## Introduction

Lees' topological immersion theory [1] has been generalized in two ways. In [2] an equivariant immersion theory was developed, and in [3] Gauld developed a submersion theory. These theories have been important in smoothing theory, and for similar reasons it is desirable to have an equivariant submersion theory for the topological category. (A smooth equivariant submersion theory, in fact a smooth equivariant Gromov theory has already been given by Bierstone [4].)

By now the form of such arguements is routine. The key result needed is an equivariant lifting theorem for submersions (Theorem A below). As in [2], we would like to use Siebenmann's deformation of stratified spaces theorem [5] to derive this. In fact, Siebenmann shows that his theorem gives a (non-equivariant) lifting theorem for submersions; and indeed, the same argument would generalize to equivariant submersions with trivial $G$ action on the target space. The problem is that one needs a $G$ isotopy extension theorem in which the parameter space has a non-trivial $G$ action. As is often the case in equivariant theories, this problem is solved by reducing it to the case that the parameter space has a single orbit type. (See the proof of the Fibrewise $G$ deformation theorem in Section 3.) Finally, in trying to follow Gauld's proof of the lifting theorem, it is necessary to understand intersections of equivariant tubes and products (3.1 and Corollary 3, Section 3).

DEFINITION. A *G-manifold* $M^n$ is a second countable Hausdorff $G$-space $M$ such that for each $x \in M$ there is an $n$-dimensional $G_x$ orthogonal representation space $V_x$ and a $G_x$ homeomorphism $h_x$ of $V_x$ onto a neighborhood of $x$ with $h_x(0) = x$, $G_x$ the isotropy subgroup of $x$. We call $h_x$ a $G_x$ *chart*. Because $G$ is finite, this is equivalent to Bredon's notion [6] of a locally smooth $G$-manifold.

DEFINITION. Let $N$ and $Q$ be $G$ manifolds. A $G$ map $f\colon N \to Q$ is a $G$ submersion if for each $x \in N$ we can find a $G_x$ chart $h_x\colon V_x \to N$ and a $G_y$

Received May 28, 1982.

chart $k_y$: $W_y \to Q$, $y = f(x)$, such that the following diagram commutes:

$$\begin{array}{ccc} V_x & \xrightarrow{h_x} & N \\ {\scriptstyle \pi_x}\downarrow & & \downarrow{\scriptstyle f} \\ W_y & \xrightarrow{k_y} & Q \end{array}$$

where $\pi_x$ is a surjective $G_x$-linear map.

*Remark.* If $N$ is a $G$-manifold with boundary, an equivariant map $f$: $N \to Q$ is an equivariant submersion if we can extend $f$ to an equivariant submersion $f$: $\hat{N} \to \hat{Q}$, where $\hat{N}$ is the union of $N$ with an open collar on the boundary.

THEOREM A. *Let $f$: $I^k \times N \to I^k \times Q$, $f(t, x) = (t, f_t(x))$ be a $G$ submersion. Let $B \subset N$ be a compact set. There is an $\varepsilon > 0$, an ambient $G$ isotopy $H_t$ of $N$, $\|t\| < \varepsilon$, and a neighborhood $U$ of $B$, such that $f_t = f_0 H_t$ on $U$.*

Those familiar with the $G$-isotopy extension theorem with non-equivariant parameter space (Section 2) may go directly to Section 3.

## 1. Siebenmann's Theorem

DEFINITION. A *stratified set* is a metrizable space $X$ equipped with a filtration

$$X \supset \cdots \supset X^{(n)} \supset X^{(n-1)} \supset \cdots X^{(-1)} = \emptyset$$

by closed sets called skeleta, such that for each $n \geq 0$, the components of $X^{(n)} - X^{(n-1)}$ are open in $X^{(n)} - X^{(n-1)}$. $X$ is called a TOP stratified set if $X^{(n)} - X^{(n-1)}$ is an $n$-manifold without boundary—called the $n$-stratum of $X$.

If $X$ is a compact stratified set, the open cone $cX$ on $X$ has a natural stratification $(cX)^{(n)} = c(X^{(n-1)})$, $n \geq 1$, $(cX)^{(0)} =$ cone point.

A stratified set $X$ is *locally cone-like* if for each point $x \in X$, say $x \in X^{(n)} - X^{(n-1)}$, there exists an open neighborhood $U$ of $x$ in $X^{(n)} - X^{(n-1)}$, a compact stratified set $L$ of finite dimension, called the link of $x$ in $X$, and an isomorphism of $U \times cL$ onto an open neighborhood of $x$ in $X$. A locally cone-like TOP stratified set is called a *CS set.*

LEMMA 1.1. *Let $M$ be a $G$ manifold. Then $M/G$ is a CS set with the components of the manifolds $M_{(H)}/G$ as components of the strata. ($M_{(H)}$ is the set of points of orbit type $(H)$.)*

*Proof.* If $x \in M$ and say $G_x = H$, let $h_x: V_x \to M$ be a chart about $x$. Then $h_x^{-1}(M_{(H)}) = V_x^H$. Let $W_x$ be the perpendicular space to $V_x^H$ in $V_x$. Then the class $x_*$ of $x$ in $M_{(H)}^x/G$ has a neighborhood isomorphic to $V_x^H \times c(S(W_x)/H)$, where $S(W_x)$ is the unit sphere.

Let $A \subset A'$ be closed sets of a CS set $X$ such that $A'$ is a neighborhood of $A$. Let $B \subset X$ be compact, and let $U \subset X$ be an open neighborhood of $A \cup B$.

SIEBENMANN'S DEFORMATION THEOREM. *If $h: U \to X$ is an open embedding equal to the identity inclusion $i: U \to X$ on $A' \cap U$ and $h$ is sufficiently near to $i$ (C-O topology), then there is an isotopy $h_t$, $0 \leq t \leq 1$, of $h$ through open embeddings $h_t: U \to X$ such that $h_1 = i$ on $A \cup B$ and $h_t = h$ on $A$ and outside some compact set $K$ in $U$ (independent of $t$ and $h$). Further the isotopy $h_t$ is a continuous function of $h$ for $h$ near $i$. Also $h_t = i$ when $h = i$.*

*Remark.* It follows that $h_t(U) = h(U)$, all $t$.

DEFINITION. A substratified set $Y$ of a stratified set $X$ consists of a closed subspace $Y$ equipped with the filtration $Y^{(n)} = X^{(n)} \cap Y$, such that for each $n$, $Y^{(n)} - Y^{(n-1)}$ is open (as well as closed) in $X^{(n)} - X^{(n-1)}$.

*Example.* A skeleton $X^{(n)}$ is a substratified set in $X$.

*Addendum* [5]. Let $\mathcal{S}$ be a family of substratified sets in $X$. Then if the embeddings $h: U \to X$ respect the subspaces $Y \in \mathcal{S}$ (i.e., $h(U \cap Y) \subset Y$), then the $h_t$ can be required to respect the $Y \in \mathcal{S}$.

LEMMA 1.2. *Let $X$ be a CS set and let $U \subset X$ be open. Let $h_t$, $0 \leq t \leq 1$, $h_0 = 1$, be an ambient isotopy of $U$ which respects skeleta and is the identity outside a compact set $K$. Then $h_t$ preserves strata; i.e.,*

$$h_t(U^{(n)} - U^{(n-1)}) = U^{(n)} - U^{(n-1)}.$$

*Proof.* $K \cap U^{(0)}$ is a finite set. Since $h_t: K \cap U^{(0)} \to K \cap U^{(0)}$ is an embedding, $h_t$ is a homeomorphism on $K \cap U^{(0)}$ and hence $h_t(U^{(0)}) = U^{(0)}$.

Now assume $h_t(U^{(k)} - U^{(k-1)}) = U^{(k)} - U^{(k-1)}$ for all $k < n$. Since $h_t(U^{(n)}) \subset U^{(n)}$ and $h_t$ is an embedding, $h_t(U^{(n)} - U^{(n-1)}) \subset U^{(n)} - U^{(n-1)}$. If $C$ is a component of $U^{(n)} - U^{(n-1)}$, $h_t(C) \subset C$ since $h_0(C) \subset C$. Since $(C \cup U^{(n-1)}) \cap K$ is compact, $h_t(C)$ is closed in $C$. By invariance of domain, $h_t(C)$ is open in $C$. Hence $h_t(C) = C$, and it follows that

$$h_t(U^{(n)} - U^{(n-1)}) = U^{(n)} - U^{(n-1)}.$$

COROLLARY 1. *Under the assumptions of the theorem, if the embeddings $h: U \to X$ respect strata then the $h_t$ can be required to respect strata.*

*Proof.* Apply the addendum with $\mathscr{S}$ the collection of skeleta of $X$. Then apply the lemma to $h^{-1}h_t\colon U \to U$.

COROLLARY 2. *If $X = M/G$ and the embeddings $h\colon U \to X$ respect orbit types we can require the $h_t$ to respect orbit types.*

COROLLARY 3. *Under the assumptions of the theorem, for each embedding $h$ sufficiently close to $i$ there is a homeomorphism $H$ of $X$, depending continuously on $h$, such that*

(a) $H = h$ *on* $A \cup B$,
(b) $H$ *is the identity outside* $h(K)$, $K$ *a compact set in* $U$,
(c) $H$ *is the identity if* $h = i$.

*Further, if $h$ respects strata we may require $H$ to respect strata.*

*Proof.* Let $h_t$ be as in the theorem; then $hh_1^{-1}\colon h(U) \to h(U)$ is the identity outside $h(K)$ and satisfies $hh_1^{-1} = h$ on $A \cup B$. Let $H$ be the extension $hh_1^{-1}$ to $X$ by the identity outside $h(U)$.

*Remark.* Let $K'$ be a compact neighborhood of $K$. If $h$ is sufficiently close to $i$ we can assume $h(K) \subset K'$. Thus we can assume

(b′) $H$ is the identity outside a compact set in $U$.

COROLLARY 4. *Let $I = [-1, 1]$ and let $I^k$ be the k-cube. Let $h_t\colon U \to X$, $t \in I^k$, be an isotopy through open embeddings with $h_0$ the inclusion and $h_t$ the inclusion on $A' \cap U$. Then for some $\varepsilon > 0$, there is an ambient isotopy $H_t$ of $X$, $\|t\| < \varepsilon$, such that*

(a) $H_t = h_t$ *on* $A \cup B$,
(b) $H_t$ *is the identity outside* $h_t(K)$, $K$ *a compact set in* $U$,
(c) $H_0$ *is the identity,*
(d) $H_t$ *is strata preserving if* $h_t$ *respects strata.*

Corollary 4 implies (see [5]):

ISOTOPY EXTENSION THEOREM. *Let $h_t\colon U \to X$, $t \in I^k$, be an isotopy through open embeddings, $h_t$ the inclusion on $A' \cap U$. Then there exists an ambient isotopy $H_t$ of $X$, $t \in I^k$, such that*

(a) $H_t = h_t$ *on* $A \cup B$,
(b) $H_t$ *is the identity outside* $h_t(K)$, $K$ *a compact set in* $U$,
(c) $H_0$ *is the identity if* $h = i$,
(d) $H_t$ *is strata preserving if* $h_t$ *respects strata.*

LOCAL CONTRACTIBILITY THEOREM. *Let $X$ be a CS set and $C \subset B \subset X$ compact subsets, $B$ a neighborhood of $C$. Let $\mathscr{H}_C(X)$ (resp. $\mathscr{H}_B(X)$) be the group of homeomorphisms of $X$ fixed on $X - C$ (resp. $X - B$) with the C-O* *topology. Then* $\mathscr{H}_{C'}(X)$ *is locally contractible in* $\mathscr{H}_{B'}(X)$. *The same result holds for the group of strata preserving homeomorphisms.*

*Proof.* In the deformation theorem, let $A' = X - \text{Int}(C)$, $A = X - \text{Int}(B)$ and $U = X$.

*Remark.* If $X$ is compact, then $\mathscr{H}(X)$ is locally contractible. (Take $C = B = X$.)

## 2. Palais' theorem and the $G$-deformation theorem

Let $X$ and $Y$ be $G$ spaces, $X_*$ and $Y_*$ their orbit spaces. The following results may be found in Bredon [6]:

LEMMA 2.1. *Let* $f\colon X \to Y$ *be isovariant and let* $f_*\colon X_* \to Y_*$ *be the induced map on orbit spaces. Then* $f$ *is open if and only if* $f_*$ *is open, and* $f$ *is injective if and only if* $f_*$ *is injective. In particular,* $f$ *is an embedding if and only if* $f_*$ *is an embedding.*

LEMMA 2.2. *Let* $Z$ *be any space and* $\phi\colon Z \to Y_*$ *any map. Give the pullback*

$$X = \{(z, y) \in Z \times Y \mid \phi(z) = y_*\}$$

*the* $G$ *structure* $g(z, y) = (z, gy)$. *Then*

(a) $X_*$ *is naturally identified with* $Z$,
(b) *the projection* $f\colon X \to Y$, $f(z, y) = y$, *is a* $G$ *map and*
(c) $f_* = \phi$ *under the identification of* $X_*$ *with* $Z$.

LEMMA 2.3. *If* $f\colon X \to Y$ *is isovariant, then the natural map*

$$\theta\colon X \to f_*^{\#}(Y), \quad \theta(x) = (x_*, f(x))$$

*of* $X$ *into the pullback of* $Y$ *by* $f_*$, *is a* $G$ *equivalence.*

THEOREM (Palais). *Let* $X$ *and* $Y$ *be* $G$ *spaces,* $f\colon X \to Y$ *an isovariant map and* $f_*$; $X_* \to Y_*$ *the induced map on orbit spaces, Assume every open subset of* $X_*$ *is paracompact (e.g.,* $X_*$ *metrizable). If* $F_*\colon X_* \times I \to Y_*$ *is a homotopy of* $f_*$ *that preserves orbit type, then* $F_*$ *is covered by an isovariant homotopy* $F$ *of* $f$. *Moreover, given two such lifts* $F_1$, $F_2$ *of* $F_*$, *there is a* $G$ *equivalence* $\theta$ *of* $X \times I$ *over* $X_* \times I$ *with* $\theta \mid X \times 0$ *the identity and* $F_2 = F_1\theta$.

Since the orbits are discrete if $G$ is finite, we have:

*Addendum* 1. If $G$ is finite, the lift $F$ is unique. In particular, if $A$ is an invariant subspace of $X$ and $F_*$ is constant on $A_*$, then $F$ is constant on $A$.

Applying Lemma 2.1 we have:

*Addendum* 2. $F_*$ is an isotopy through open embeddings if an only if $F$ is a $G$-isotopy through open embeddings.

Now let $X$ be a $G$ manifold, $B \subset X$ a compact invariant subspace. Let $A \subset A'$ be closed invariant subspaces with $A'$ a neighborhood of $A$. Let $U$ be an open invariant neighborhood of $A \cup B$.

$G$ DEFORMATION THEOREM. *If $h: U \to X$ is an open equivariant embedding equal to the inclusion $i$ on $A' \cap U$ and $h$ is sufficiently close to $i$, then there is a $G$ isotopy $h_t$, $0 \leq t \leq 1$, of $h$ such that $h_1 = i$ on $A \cup B$ and $h_t = h$ on $A$ and outside some compact set $K$ in $U$. Further $h_t = i$ if $h = i$ and $h_t$ depends continuously on $h$.*

*Proof.* Consider the space $Z$ of $G$-embeddings of $U$ in $X$ with the $C$-$O$ topology and trivial $G$-action, and let $F: Z \times U \to X$ be the equivariant map $F(h, x) = h(x)$. Then if $F_*: Z \times U_* \to X_*$ is the induced map, $F_*(h,\ ) = h_*$: $U_* \to X_*$. Siebenmann's theorem says there is a neighborhood $W$ of $i$ in $Z$ and an orbit type preserving homotopy $F_{*_t}: W \times U_* \to X_*$ so that $F_{*_t}(h,\ ) = h_{*_t}$ satisfies the corresponding conditions for $A_*$, $B_*$ and a compact set $K_*$ in $U_*$. Since $h_{*_t} = h_*$ outside $K_*$ we may pull back $F_{*_t}$ to $W' \times K_*$, where

$$W' = \{h \mid K;\ h \in W\},$$

$K$ the preimage of $K_*$ in $X$. Then $W' \times K_*$ is metrizable and we may apply Palais' theorem to obtain a $G$-deformation $F_t$ of $F$ with $h_t = F_t(h,\ )$ satisfying the desired conclusions.

COROLLARY 1. *Under the assumption of the $G$ deformation theorem, for each embedding $h$ sufficiently close to $i$, there is a $G$ equivalence $H$ of $X$, depending continuously on $h$, such that*

(a) *$H = h$ on $A \cup B$,*
(b) *$H$ is the identity if $h = i$,*
(c) *$H$ is the identity outside $h(K)$, $K$ a* compact *subset of $U$.*

COROLLARY 2. *Let $h_t: U \to X$, $t \in I^k$, be a $G$ isotopy with $h_0 = i$ and $h_t$ the inclusion on $A' \cap U$. Then for some $\varepsilon > 0$, there is an ambient $G$ isotopy $H_t$ of $X$, $\|t\| < \varepsilon$, such that*

(a) *$H_t = h_t$ on $A \cup B$,*
(b) *$H_t$ is the identity outside $h_t(K)$, $K$ a compact set in $U$.*
(c) *$H_0$ is the identity.*

$G$ ISOTOPY EXTENSION THEOREM [2]. *If $h_t: U \to X$, $t \in I^k$, is a $G$ isotopy with $h_t$ the inclusion $i$ on $A' \cap U$ and $h_0 = i$; then there is an ambient $G$ isotopy $H_t$ of $X$, $t \in I^k$, such that*

(a) *$H_t = h_t$ on $A \cup B$,*

(b) *$H_t$ is the identity outside $h_t(K)$, $K$ a compact set in $U$,*
(c) *$H_0$ is the identity.*

*Proof.* This follows by a standard argument from Corollary 2; or as in the proof above, by applying Palais' theorem to Siebenmann's isotopy extension theorem.

EQUIVARIANT LOCAL CONTRACTIBILITY THEOREM. *Let $X$ be a $G$ manifold and $C \subset B \subset X$ compact invariant subsets, $B$ a neighborhood of $C$. Let $\mathscr{H}_C^G(X)$ (resp. $\mathscr{H}_B^G(X)$) be the group of $G$ homeomorphisms of $X$ fixed on $X - C$ (resp. $X - B$) with the C-O topology. Then $\mathscr{H}_C^G(X)$ is locally contractible in $\mathscr{H}_B^G(X)$.*

## 3. Equivariant fibrewise deformations

Let $X$ be a locally smooth $\Gamma$ manifold, $\Gamma$ a compact Lie group. Let $p: E \to Z$ be a $G - \Gamma$ bundle [7] with fibre $X$ over the compact $G$-CW complex $Z$. In other words, $p$ is a right $\Gamma$ bundle with fibre $X$ over $Z$ and an equivariant map of left $G$-spaces, such that $g \in G$ acts on $E$ as a $\Gamma$ bundle automorphism covering the action of $g$ on $Z$, and satisfying the local triviality condition that each $z \in Z$ has a $G_z$ invariant neighborhood $U_z$ with $p^{-1}(U_z) = U_z \times X$ as a $G_z$ space over $U_z$. $G$-CW complexes are defined in [2]. Let $A \subset A'$ be closed invariant subspaces of $X$ with $A'$ a neighborhood of $A$. Let $B \subset X$ be a compact invariant subspace and $U$ an invariant neighborhood of $A \cup B$ in $X$. Let $E_A$, $E_B$, etc., be the associated $G - \Gamma$ subbundle with fibre $A$, $B$, etc.

FIBREWISE $G$ DEFORMATION THEOREM. *Let $h: E_U \to E$ be an equivariant embedding over $Z$, equal to the inclusion $i$ on $E_{A' \cap U}$. If $h$ is sufficiently close to $i$, then there is a $G$ isotopy $h_t$, $0 \le t \le 1$, of $h$ through embeddings over $Z$ such that $h_1 = i$ on $E_{A \cup B}$ and $h_t = h$ on $E_A$ and outside $E_K$, $K$ a compact set in $U$, and $h_t$ depends continuously on $h$. If $h = i$ over a subcomplex $Z_0$ of $Z$, we can assume $h_t = i$ over $Z_0$.*

*Proof.* We proceed by induction on the dimesnion of the $G$ cells of $Z$. Over a $O$-cell $G/H$, $E \mid G/H = G \times_H X$, where $H$ acts on $X$ through a homomorphism $\rho: H \to \Gamma$ (see [7]). Now $h: G \times_H U \to G \times_H X$ restricts to an $H$-embedding $h: U \to X$. By the $G$ deformation theorem there is an $H$ isotopy of $h$ restricted to $U$ which extends by equivariance to a $G$ isotopy $h_t$ of $h$ over $G/H$, $h_t = i$ on $G \times_H A$ and outside $G \times_H K$, $h_1 = i$ on $G \times_H B$ and $h_t = i$ if $h = i$.

Now assume $h_t$ has been defined over $Z^{(n-1)}$. Note that we can always replace $A$ by a closed neighborhood $A_1$ of $A$ ih $A' \cap U$ and $B$ by a compact neighborhood $B_1$ of $B$ in $U$. Then $h_1$ is the inclusion on $E_{U_1} \mid Z^{(n-1)}$, $U_1 = \text{Int}(A_1) \cup \text{Int}(B_1)$.

Let $f: G/H \times S^{n-1} \to Z^{(n-1)}$ be the attaching map of a $G$ $n$-cell and

$$\hat{f}: G/H \times D^n \to Z^{(n)}$$

its defining extension. Then $\hat{f}^*E = (G \times_H X) \times D^n$ and $h$ pulls back to a $G$ embedding $h': (G \times_H U) \times D^n \to (G \times_H X) \times D^n$ over $G/H \times D^n$. Also $h_t$ pulls back to a $G$ isotopy

$$h'_t: (G \times_H U) \times S^{n-1} \to (G \times_H X) \times S^{n-1}$$

of $h'$ over $G/H \times S^{n-1}$. This obviously extends to a $G$ isotopy, again denoted $h'_t$, of $h'$ over $G/H \times D^n$, using a product neighborhood of $S^{n-1}$ in $D^n$ to undo the isotopy over $G/H \times S^{n-1}$. Then $h'_1 \mid G \times_H U_1 \times D^n$ is a $G$ embedding over $G/H \times D^n$ which is the inclusion on $G \times_H U_1 \times S^{n-1}$. Now $k = h'_1$ restricts to an $H$ embedding $k: U_1 \times D^n \to X \times D^n$ over $D^n$. Applying the $G$ deformation theorem to the $H$ embeddings $k_d: U_1 \to X$, $d \in D$, leads to an $H$ isotopy of $k$ over $D^n$ and hence a $G$ isotopy $k_t$ of $k$ over $G/H \times D^n$ satisfying the desired conditions and with $k_t = i$ over $G/H \times S^{n-1}$. Thus reattaching the $G$ cell and using the $G$ isotopy $h'_t$ to extend $h_t$ over $Z^{(n)}$ and then $k_t$ to further deform the extended $h_1$ rel $Z^{(n-1)}$, we end up with a $G$ isotopy of $h$ over $Z^{(n)}$ satisfying the theorem.

COROLLARY 1. *The theorem holds for $Z$ a $G$ retract of a compact $G$ CW complex and $Z_0 \subset Z$ a closed invariant subset, provided we can assume $h = i$ on a neighborhood $N$ of $Z_0$.*

*Proof.* Since any compact $G$ CW complex is a $G$ ENR [8], we can assume $Z$ is equivariantly embedded in a $G$ representation space $V$ and is a $G$ retract of a neighborhood $W \subset V$, say $r: W \to Z$. Then $r^*(E)$ is a bundle over $W$ with fibre $X$ and $h$ pulls back to an embedding $r^*(h)$ of $r^*E_U$ in $r^*E$. Further $r^*(h) = i$ over $r^{-1}(N)$. Take an equivariant triangulation of $W$. Then $Z$ is contained in a finite $G$ subcomplex $L$; and if the triangulation is sufficiently fine, $Z_0 \subset L_0 \subset r^{-1}(N)$, $L_0$ a $G$ subcomplex of $L$. Restricting $r^*E$ to $L$, the theorem gives $G$ isotopies $r^*(h)_t$ satisfying the theorem for the pair $(L, L_0)$. Let $h_t$ be the restriction of $r^*(h)_t$ over $Z$. Then $h_t$ satisfies the conclusions of the theorem for $(Z, Z_0)$.

COROLLARY 2. *Let $X$ and $Y$ be $G$ manifolds and $Z$ a compact invariant subspace of $Y$. Let $A, A', B, U \subset X$ be as above. Let $h: Y \times U \to Y \times X$ be an equivariant embedding over $Y$, equal to the inclusion $i$ on $Y \times (A' \cap U)$. There is a compact neighborhood $K$ of $Z \times B$ in $Y \times U$; and if $h$ is sufficiently close to $i$, a $G$ isotopy $h_t$, $0 \leq t \leq 1$, of $h$ depending continuously on $h$, such that $h_1 = i$ on $Z \times B$, $h_t = h$ on $Y \times A$ and outside $K$ and with $h_t = i$ if $h = i$.*

*Proof.* Let $N$ be the interior of a compact neighborhood of $Z$ in $Y$. Then $N$ is a $G$ ENR and $N$ embeds equivariantly in a representation space $V$ with $N$ a $G$ retract $r: W \to N$ of an invariant neighborhood $W$. As in the proof of Corollary 1 we may find a finite $G$ triangulation $L$ of a neighborhood of $Z$ in

$W$. Apply the theorem to $r^*h$ over $L$ and obtain $r^*(h)_t$ which restricts to a $G$ isotopy $k_t$ of $h$ over $L \cap N$. By taking an invariant function $\lambda$ which is one on $Z$ and zero outside a compact neighborhood $N_1 \subset \text{Int}(L \cap N)$, we can construct a $G$ isotopy $h_t$ of $h$ which is $k_t$ over $Z$ and identically $h$ outside $N_1$; i.e.,

$$h_t(y, x) = k_{\lambda(y)t}(y, x).$$

Then $h_t$ satisfies the conclusion of Corollary 2 with $K = N_1 \times K_1$, $K_1$ a compact neighborhood of $B$ in $U$.

COROLLARY 3. *Let $X$ and $Y$ be $G$ manifolds and $C$ a compact invariant subspace of $Y \times X$. Let $U$ be an invariant neighborhood of $C$ in $Y \times X$ and $h: U \to Y \times X$ an equivariant embedding respecting the projection $\pi: Y \times X \to Y$ (i.e., $\pi h = \pi$). There is a compact neighborhood $K$ of $C$ in $Y \times X$ such that if $h$ is sufficiently close to the inclusion $i$ there is a $G$ isotopy $h_t$, $0 \le t \le 1$, satisfying*:

(a) *$h_t$ respects $\pi$;*
(b) *$h_t$ depends continuously on $h$;*
(c) *$h_1$ is the inclusion on $C$;*
(d) *$h_t = h$ outside $K$;*
(e) *$h_t = i$ if $h = i$.*

*Proof.* Cover $C$ by a finite collection of sets of the form

$$G \times_H (W_y \times V_x) \subset U,$$

$W_y$ a $G_y$ chart about $y \in Y$ and $V_x$ a $G_x$ chart about $x \in X$, $H = G_x \cap G_y$; say

$$G \times_{H_i} (W^i \times V^i), \quad i = 1, 2, \ldots, r.$$

Any such cover has an invariant shrinking, and for convenience we will assume $C \subset \bigcup_{i=1}^r G \times_{H_i} (\text{Int } D(W^i) \times \text{Int } D(V^i))$, $D(V)$ the unit disk in $V$.

We will use the following trivial observation: Call a $G$ space of the form $G \times_H S$ an *$H$ sliced* $G$ space, with $H$ slice $S$. Then:

LEMMA 3.1. *Let $G \times_{H_i} S^i$, $i = 1, 2, \ldots, r$, be a finite set of sliced subspaces of the $G$ space $X$. Then $\bigcap_{i=1}^r G \times_{H_i} S^i$ is the disjoint union of sliced subspaces of $X$ with $\bigcap_{i=1}^r g_i H_i g_i^{-1}$ slices $\bigcap_{i=1}^r g_i S^i$, $g_i \in G$, where the slices $\bigcap g_i S^i$ and $\bigcap g_i' S^i$ determine the same sliced subspace if and only if there is a $g \in G$ with $g_i' \in g g_i H_i$, $i = 1, \ldots, r$.*

For each sliced subspace in $\bigcap_{i=1}^r G \times_{G_{x_i}} D(V_{xi})$, pick a slice $\bigcap_{i=1}^r g_i D(V_{xi})$. Then

$$\left(\bigcup_{i=1}^r g_i G_{x_i} D(W_{yi})\right) \times \left(\bigcap_{i=1}^r g_i D(V_{xi})\right)$$

is a $\bigcap_{i=1}^r g_i G_{x_i} g_i^{-1}$ slice, and the corresponding sliced sets are disjoint $G$ invariant subsets of $U$. Also every $r$ intersection

$$\bigcap_{i=1}^{r} g_i(D(W_{y_i})) \times D(V_{x_i}))$$

is contained in one of these sliced sets. Now decompose the $r - 1$ intersections similarly, etc., until one comes to the $G$ cubes

$$G \times_{H_i}(D(W_{y_i}) \times D(V_{x_i})) = G \times_{G_{x_i}} (G_{x_i} D(W_{y_i}) \times D(V_{x_i}))$$

themselves. The sliced sets of $n$ intersections meet only in higher order intersection and the $n + 1$ intersection sliced sets meet each $n$ intersection

$$G \times_H \left( \bigcup_{k=1}^{n} g_{i_k} G_{x_{i_k}} D(W_y \ ) \times \bigcap_{k=1}^{n} g_{i_k} D(V_{x_{i_k}} \ ) \right)$$

in a set of the form

$$G \times_H \left( \bigcup_{k=1}^{n} g_{i_k} G_{x_{i_k}} D(W_{y_{i_k}}) \times B(i_1, \ldots, i_n) \right),$$

where

$$H = \bigcap_{k=1}^{ii} g_{i_k} G_{x_{i_k}} g_{i_k}^{-1}$$

and $B(i_1, \ldots, i_n)$ is the uniion of all $n + 1$ intersections $\bigcap_{j=1}^{n+1} g_{i_j} D(V_{x_{i_j}} \ )$ which are contained in $\bigcap_{k=1}^{n} g_{i_k} D(V_{x_{i_k}} \ )$.

Now beginning with the $r$ intersections, apply Corollary 2 with $A = \emptyset$ to $h$ restricted to

$$\left( \bigcup_{i=1}^{r} g_i G_{x_i} W_{y_i} \right) \times \left( \bigcap_{i=1}^{r} g_i V_{x_i} \right)$$

to obtain a $\bigcap_{i=1}^r g_i G_{x_i} g_i^{-1}$ isotopy $h_t$ with $h_t = h$ outside a compact set and with $h_1 = i$ on $(\bigcup_{i=1}^r g_i G_{x_i} D(W_{y_i})) \times (\bigcap_{i=1}^r g_i D(V_{x_i}))$. Extending $h_t$ by equivariance and replacing $h$ with $h_1$ where defined, we may assume $h$ is the inclusion on (a neighborhood of) the $r$ intersections. Now applying Corollary 2 inductively with $A$ (a neighborhood of) the intersection of the $n + 1$ intersections with an $n$ intersection, we may extend $h_t$ to (a neighborhood of) the $n$

intersections rel the $n+1$ intersections. Finally we obtain an extension of $h_t$ to a neighborhood of

$$\bigcup_{i=1}^{r} G \times_{H_i} D(W_{yi}) \times D(V_{xi})$$

and hence with $h_1 = i$ on $C$ and $h_t = h$ outside a compact neighborhood of $C$ in $U$.

*Remark.* The fibrewise $G$ deformation theorem and all the above corollaries can be put in the form of Corollaries 3 and 4 of Siebenmann's theorem or as a $G$ isotopy extensions theorem. In particular, we have:

COROLLARY 4. *Let $X$ and $Y$ be $G$ manifolds, $C$ a compact invariant subspace of $Y \times X$, and $U$ an invariant neighborhood of $C$. Let $h_t$: $U \to Y \times X$ be a $G$ isotopy respecting the projection $\pi$: $Y \times X \to Y$, with $h_0$ the inclusion. Then there is an $\varepsilon > 0$ and an ambient $G$ isotopy $H_t$ of $Y \times X$ over $Y$, $\|t\| < \varepsilon$, such that*

(a) *$H_t = h_t$ on $C$,*
(b) *$H_t$ is the identity outside $h_t(K)$, $K$ a compact neighborhood of $C$,*
(c) *$H_0$ is the identity.*

## 4. Theorem A and the Equivariant Submersion Theorem

*Proof of Theorem A.* Let $f$: $I^k \times N \to I^k \times Q$, $f(t, x) = (t, f_t(x))$, be a $G$ submersion. It follows from the definition that for each $x \in N$ we can choose a $G_x$ chart $h_x$: $U_x \to N$ about $x$ and a $G_y$ chart $k_y$: $W_y \to Q$ about $y = f_0(x)$, so that for $t$ near $O$ there is a $G_x$ isotopy $h_x^t$: $U_x \to N$, $h_x^0 = h_x$, with $f_t h_x^t = k_y \pi_x$, $\pi_x$: $U_x \to W_y$ a surjective $G_x$ linear map.

To simplify notation, we will identify $U_x$ with $h_x(U_x)$ and $W_y$ with $k_y(W_y)$. We will also write $U_x = V_x \oplus W_x$, where $V_x = \ker \pi_x$ and $W_x = V_x^{\perp}$, and $W_x$ is identified to $W_y$ as a $G_x$ space via $\pi_x$. Then the above conditions become:

For each $x \in N$ we can choose a $G_x$ chart $U_x$ in $N$ about $x$ so that

(a) there is a $G_y$ chart $W_y$ in $Q$ about $y = f_0(x)$ with $U_x = V_x \oplus W_x$, $W_x$ equal to $W_y$ as a $G_x$ space, and

(b) for $t$ near 0, there is a $G_x$ isotopy $h_x^t$: $U_x \to N$, $h^0$ the inclusion, with $f_t h_x^t = \pi_x$, $\pi_x$: $V_x \oplus W_x \to W_y$ the projection

Cover $B$ by a finite number of sliced open sets $G \times_{G_{x_i}} U_{x_i}$, $i = 1, \ldots, r$, $U_{xi}$ a chart about $x_i$ satisfying (a) and (b) above. Write $U_i$ for $U_{xi}$, etc. Since we can shrink the cover, we may as well assume say that the sets $G \times_{G_i}(\operatorname{Int} D(V_i) \times \operatorname{Int} D(W_i))$, $i = 1, \ldots, r$ still cover $B$.

By Corollary 2 of the $G$-deformation theorem, there is an ambient $G_1$ isotopy $H_1^t$ of $V_1 \oplus W_1$, $t$ near $O$, with $H_1^t = h_1^t$ on a compact neighborhood of $D(V_1) \times D(W_1)$, and the identity outside a larger compact set; i.e., we use the fact that for $t$ near $O$, $h_1^t(D(V_1) \times D(W_1)) \subset V_1 \oplus W_1$. Extend $H_1^t$ by equivariance to $G \times_{G_1}(V_1 \oplus W_1)$ and then by the identity to an ambient $G$ isotopy of $N$.

Assume by induction that an ambient $G$ isotopy $H_s^t$ of $N$ has been defined for $t$ near $O$ with $f_t H_s^t = f_0$ on a neighborhood $U_s$ of a compact $G$ neighborhood $C_s$ of

$$\bigcup_{i=1}^{s} G \times_{G_i} D(V_i) \times D(W_i).$$

Let $D_{s+1}$ be a compact $G_{s+1}$ neighborhood of $D(V_{s+1}) \times D(W_{s+1})$. If $t$ is close enough to $O$, $(H_s^t)^{-1} h_{s+1}^t(V_{s+1} \oplus W_{s+1})$ contains a neighborhood $U_s'$ of $C_s \cap D_{s+1}$, $U_s'$ a $G_{s+1}$ invariant subset of $U_s$. Then $(h_{s+1}^t)^{-1} H_s^t$: $U_s' \to V_{s+1} \oplus W_{s+1}$ is a $G_{s+1}$ isotopy commuting with $f_0 = \pi_{s+1}$, since $f_t h_{s+1}^t = f_0$.

By Corollary 4 of the fibrewise $G$ deformation theorem there is a $G_{s+1}$ isotopy $K_t$ of $V_{s+1} \oplus W_{s+1}$, commuting with $\pi_{s+1}$, $K_t = (h_{s+1}^t)^{-1} H_s^t$ on $C_s \cap D_{s+1}$ and the identity outside a compact neighborhood of $C_s \cap D_{s+1}$ in $U_s'$. Define $k_{s+1}^t$ as $H_s^t$ on $C_s$ and $h_{s+1}^t K_t$ on $D_{s+1}$ and by equivariance on $G \times_{G_{s+1}} D_{s+1}$. Then $k_{s+1}^t$ is a well defined $G$ isotopy of a neighborhood of

$$\bigcup_{i=1}^{s} G \times_{G_i}(D(V_1) \times D(W_i)),$$

and $f_t k_{s+1}^t = f_0$. By Corollary 2 of the $G$ deformation theorem there is an ambient $G$ isotopy $H_{s+1}^t$ of $N$ with $H_{s+1}^t = k_{s+1}^t$ on a neighborhood of

$$\bigcup_{i=1}^{s+1} G \times_{G_i}(D(V_i) \times D(W_i),$$

and thus $f_t H_{s+1}^t = f_0$ on this neighborhood.

Before we state the $G$ submersion theorem we recall:

DEFINITION 4.1. A $G$ manifold $N$ satisfies the *Bierstone condition* if for all $H \subset G$, the components of $M_{(H)}/G$ are non-closed as manifolds.

DEFINITION 4.2. Let $X$ be a $G$ space. Two $G$ spaces over $X$ with section $(E_i, p_i, s_i)$, $i = 1, 2$, are called micro $G$ equivalent if there are invariant neighborhoods $E_i^0$ of $s_i(X)$ in $E_i$, $i = 1, 2$, which are equivalent as $G$ spaces over $X$ with section; i.e., there is a $G$ equivalence $\phi$: $E_1^0 \to E_2^0$ such that $\phi s_1 = s_2$ and $p_2 \phi = p_1$.

DEFINITION 4.3. An $n$ dimensional $G$ microbundle $\mathscr{E}$ over a paracompact $G$ space $X$ is a $G$ space with section over $X$, $(E(\mathscr{E}), p_{\mathscr{E}}, s_{\mathscr{E}})$, such that $\mathscr{E}$ is locally micro $G$ equivalent to a trivial $n$ dimensional $G$ space with section; i.e., each $x \in X$ has a $G_x$ neighborhood $U_x$ such that $p^{-1}(U_x)$ is micro $G_x$ equivalent to $U_x \times V_x$, $V_x$ an $n$ dimensional $G_x$ representation space, with obvious projection and $O$-section.

*Example* 1. If $\mathscr{E}$ is a $G$ vector bundle, then the $O$-section makes it into a $G$ microbundle.

*Example* 2. If $M$ is a $G$ manifold, then the tangent microbundle $\tau = \tau M$, $E(\tau) = M \times M$, $p_\tau: M \times M \to M$ the projection onto the first factor, and $s_\tau: M \to M \times M$ the diagonal map, is locally micro $G$ trivial.

DEFINITION 4.4. If $\mathscr{E}$ and $\mathscr{N}$ are $n$ and $q$ dimensional $G$ microbundles, $n \geq q$, over $X$, then a $G$ microbundle *surjection* is a map $\phi: E(\mathscr{E})^0 \to E(\mathscr{N})^0$ of invariant neighborhoods of their sections such that:

(a) $\phi$ is a map of $G$ spaces over $X$ with section.
(b) For each $x \in X$ there is a $G_x$ neighborhood $U_x$ and local micro $G_x$ trivializations $h: U_x \times V_x \to p^{-1}(U_x)$ and $k: U_x \times W_x \to p^{-1}(U_x)$ with

$$k^{-1}\phi h = 1 \times \pi_x: U_x \times V_x \to U_x \times W_x,$$

$\pi_x$ a $G_x$ linear surjection.

Two such are identified if they agree on a neighborhood of the section.

DEFINITION 4.5. If $\mathscr{E}$ is a $G$ microbundle over $X$ and $\mathscr{N}$ a $G$ microbundle over $Y$, then a $G$ microbundle surjection over a $G$ map $f: X \to Y$ is a map

$$\phi: E(\mathscr{E})^0 \to E(\mathscr{N})^0$$

of $G$ spaces with section over $f$ of invariant neighborhoods of the sections, such that the induced map $f^*\phi: E(\mathscr{E})^0 \to E(f^*\mathscr{N})^0$ is a $G$ microbundle surjection.

DEFINITION 4.6. Let $N$ and $Q$ be $G$ manifolds. Then $R(\tau N, \tau Q)$ is the simplicial set whose $k$ simplices are $G$ microbundle surjections

$$\phi: \Delta^k \times E(\tau N)^0 \to \Delta^k \times E(\tau Q)^0$$

over $G$ maps $\bar{\phi}: \Delta^k \times N \to \Delta^k \times Q$ which commute with projection onto the $k$ simplex $\Delta^k$.

DEFINITION 4.7. Let $S(N, Q)$ be the simplicial set whose $k$ simplices are submersions

$$f: \Delta^k \times N \to \Delta^k \times Q$$

over $\Delta^k$. Then the differential $d: S(N, Q) \to R(\tau N, \tau Q)$ is given by

$$d(f)_t: E(\tau N) \to E(\tau Q),\ t \in \Delta^k, \quad d(f)_t = f_t \times f_t: N \times N \to Q \times Q.$$

G-SUBMERSION THEOREM. *If $N$ satisfies the Bierstone condition. then*

$$d: S(N, Q) \to R(\tau N, \tau Q)$$

*is a homotopy equivalence.*

The $G$ submersion theorem follows from Theorem A in the same way as the submersion theorem of Gauld [3] follows from his non-equivariant version of Theorem A, except that the induction step (over the $G$-handles) has to follow the order of induction given in [2] for the $G$-immersion theorem. We will not repeat the arguments here.

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University of Chicago
Chicago, Illinois