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SOME RESULTS ON CAUCHY SURFACE CRITERIA IN LORENTZIAN GEOMETRY

BY

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1. Introduction

Let M denote an arbitrary *space-time*, by which we mean a smooth connected time oriented Lorentzian manifold of dimension $n \geq 2$ having signature $(- + \cdots +)$. Causality conditions have played an important role in the development of the global theory of Lorentzian geometry (e.g., see Beem and Ehrlich [1]). One such condition which, for instance, ensures the geodesic connectivity of causally related points in M is *global hyperbolicity*. (For definitions and basic results in the causal theory of space-time, see, for example, Penrose [8] or Hawking and Ellis [6]. For a short exposition of the subject we recommend the excellent review article by Geroch and Horowitz [5].) A classical theorem of the causal theory due to Geroch [4] says that a space-time M is globally hyperbolic if and only if it admits a Cauchy surface. (A subset S of M is a Cauchy surface if and only if each inextendible timelike curve in M intersects S once and only once. A Cauchy surface for M is necessarily a codimension one topological submanifold of M.) In this paper we present a general result which establishes necessary and sufficient conditions on a subset S of M to be Cauchy. Its advantage over a related result of Geroch [4] (which is discussed in the next section) is that it does *not* require that S be *achronal* (i.e., that each timelike curve in M intersect S *at most* once). This general result is then used to obtain Cauchy criteria in more specific situations. In particular we obtain a technical improvement of the result of Budic et. al. [2] that a C^1 spacelike hypersurface S in a globally hyperbolic space-time is necessarily Cauchy. (Besides its use in the definition of a spacelike hypersurface, the C^1 differentiability assumption is used in their proof at one point to invoke the inverse function theorem.) Our version removes in a natural way the differentiability assumption on S (and, in particular, requires a weakening of the notion of "spacelike").

We also obtain a result concerning the topological structure of a certain class of space-times M. If S is a Cauchy surface for M then, as is well known, M must be homeomorphic to $\mathbf{R} \times S$. Here we determine the topology of those space-times M admitting a hypersurface S which is *not* Cauchy but which satisfies a certain subset of our general Cauchy criteria.

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2. The results

There are minor differences in the development of the causal theory presented in the two standard references, [6] and [8]. We will primarily adhere to the notation and terminology of the latter. (Thus, for instance, the domain of dependence is defined in terms of timelike, rather than nonspacelike, curves.)

Let M be an arbitrary space-time. A subset S of M is said to be *locally acausal* (or, in the terminology of Seifert [9], *spacelike*) if there exists an open neighborhood U of S such that S is acausal in U, i.e., such that any nonspacelike curve in U interesects S at most once. This definition of a spacelike subset generalizes the usual notion of a spacelike hypersurface, since a C^1 hypersurface in M with spacelike tangents is spacelike in the sense defined above. (A rigorous proof of this plausible result is presented in the appendix of [2].) Similarly, a subset S of M is said to be *locally achronal* (or *nontimelike*) if there exists an open neighborhood U of S such that S is achronal in U, i.e. such that any timelike curve in U intersects S at most once. All of the results presented in this paper shall refer to locally achronal S.

One of the basic techniques for establishing that a (globally) achronal subset S of M is Cauchy ($D(S) = M$) is to show that it has no Cauchy horizon ($H(S) = \emptyset$). Using this technique Geroch [4] proved the following slightly modified version of an earlier result due to Penrose [8]. *If S is a closed achronal subset of M such that each inextendible null geodesic in M intersects and then re-emerges from S then S is a Cauchy surface.* (The re-emergence condition is satisfied if, for example, S is locally acausal). We shall prove the following related result, in which the global requirement of achronality is weakened.

THEOREM 1. *Let S be a connected closed locally achronal edgeless subset of M which obeys the following conditions.*

(a) *For any point p in $J^+(S)$ (respectively, ($J^-(S)$)), any past (resp. future) inextendible null geodesic issuing from p must intersect and re-emerge from S.*

(b) *There exists a point q in M and an inextendible nonspacelike curve $\gamma: [a, b) \to M$ issuing from q such that γ intersects S for only finitely many parameter values.*

Then S is a Cauchy surface.

Remarks. (i) For any achronal subset S of M, the intersection condition of Geroch's theorem (henceforth referred to as condition (G)) easily implies (and, in fact, is equivalent to) the intersection condition (a) of Theorem 1. Hence condition (a) may replace condition (G) in Geroch's theorem. However if S is merely locally achronal these conditions are not equivalent. In fact, the statement of Theorem 1 with condition (a) replaced by condition (G) is false as the following example illustrates. Construct a cylinder "closed in time" by taking the region $|t| \leq 1$ in a 2-dimensional Minkowski space and identifying for each x the points $(x, -1)$ and $(x, 1)$. Let M be this space-time with the point $(x, t) = (0, \frac{1}{2})$ removed. Let S be the line $t = 0$. Condition (G) holds in this space-time as do all the assumptions of Theorem 1 except condition (a). (The assumption edge $S = \emptyset$ is discussed in Remark (ii)).

(ii) The usual definition of edge S assumes that S is achronal. This definition may be extended to locally achronal S in the following way. If S is achronal in U then edge $S = \{x \in U$: every neighborhood W of x in U contains points y and z and two timelike curves from y to z just one of which meets $S\}$. (This definition is easily seen to be independent of the open set U in which S is achronal.) It then follows, for example, from Proposition 5.8 in [8] that a locally achronal edgeless subset of M is a continuously embedded topological submanifold (without boundary) of codimension one in M.

Thus, in Theorem 1, the assumption that S is edgeless can be replaced by the assumption that S is a continuously embedded hypersurface without boundary (since this latter assumption, together with the assumption that S is closed, implies that S is edgeless). As some of the examples to be presented in this paper will illustrate, a locally achronal edgeless subset S needn't be closed as a subset of M (although it will be closed relative to any open set in which it is achronal).

(iii) The conditions of Theorem 1 are necessary as well as sufficient for S to be Cauchy. Furthermore, each of the conditions on S is essential in that if any one of them is omitted the theorem is false. The example in Figure 1 illustrates how Theorem 1 can fail when S is not edgeless, even though it is a submanifold (with boundary) of codimension 1. S-edge S is a submanifold of codimension 1 without boundary, but is not closed as a subset of M. The example in Figure 2 illustrates how Theorem 1 can fail when all of the conditions except the assumption that S be closed are met. In this example,

$$M = \mathbf{R}^2 - (0, 0), \quad ds^2 = -dr^2 + r^4\, d\theta^2,$$

where r and θ are polar coordinates. The null geodesics are described by the polar equations, $r^{-1} \pm \theta = \text{const}$. Each null geodesic enters from infinity along some asymptotic direction and then spirals into the origin. S consists of the null geodesic, $r = 1/\theta$, $\theta \geq \pi/2$ and a curve from $(r, \theta) = (2/\pi, \pi/2)$ which asymptotically winds around the circle $r = 1/2$.

Proof of Theorem 1. As discussed in Remark (ii), S is a continuously imbedded topological submanifold of codimension 1 in M. The local achronality of S and the time orientability of M imply that S has a future side and a past side. Indeed, let T be a smooth future pointing unit timelike vector field on M. T is tranverse to S in the sense that each integral curve of T in U (an open set in which S is achronal) intersects S exactly once. The future side of S is the side into which T along S points, and the past side of S is the side into which $-T$ points.

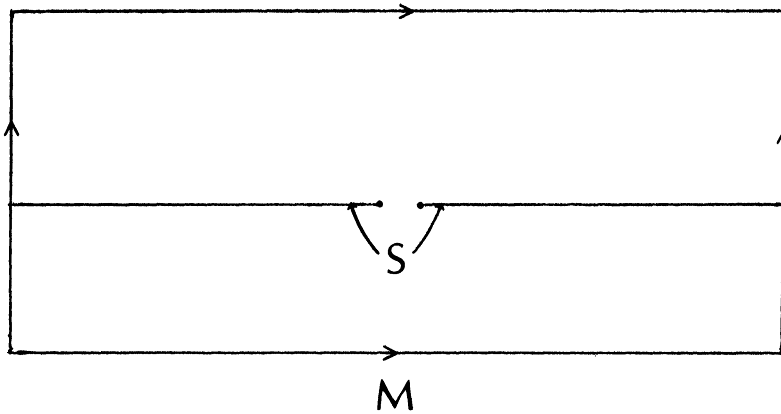

FIGURE 1
M is a flat torus obtained by taking a suitable rectangular subset of 2-dimensional Minkowski space and identifying opposite sides. S satisfies all of the conditions of Theorem 1 except the condition that it be edgeless. S is a submanifold with boundary of codimension 1; S-edge S is boundaryless, but is not closed as a subset of M.

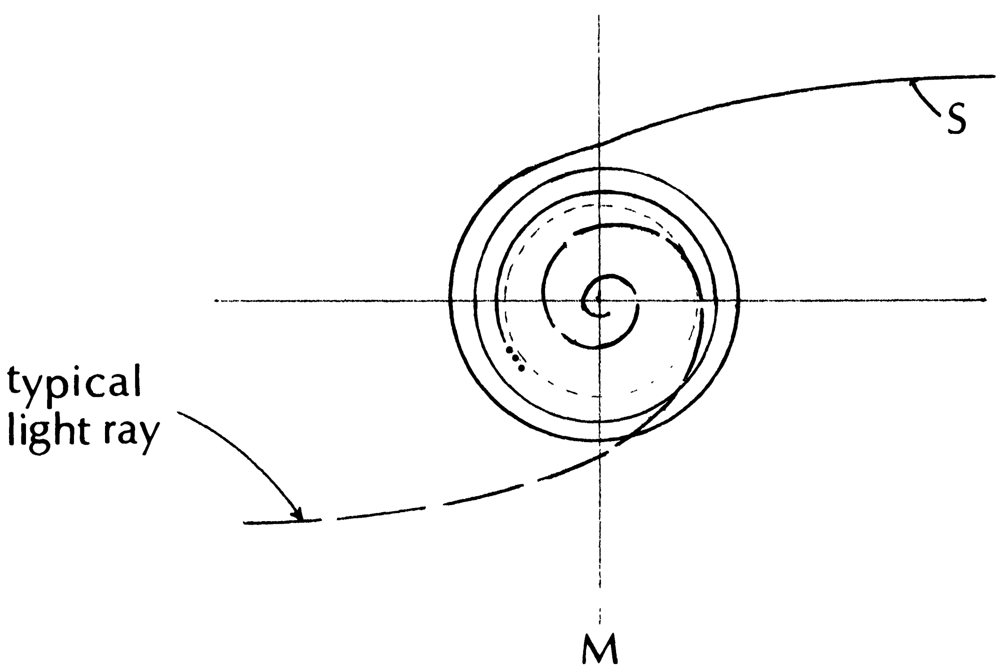

FIGURE 2
$M = \mathbf{R}^2 - (0, 0)$ with metric $ds^2 = -dr^2 + r^4\, d\theta^2$, where r and θ are polar coordinates. The null geodesics are hyperbolic spirals, described by the polar equations $r^{-1} \pm \theta = \text{const}$. S obeys all the conditions of Theorem 1 except the condition that it be closed.

This two-sidedness enables the construction of a certain covering manifold of M which we now describe. Take an infinite number of copies of M, $\{M_i: i$ an integer$\}$. Each M_i contains a copy of S which is denoted by S_i. Proceeding in a fashion analogous to the construction of a Riemann surface, make a "cut" along each S_i and identify the past edge of the cut in M_i with the future edge of the cut in M_{i+1}. Hence, M_i and M_{i+1} are joined along a copy of S, call it $\tilde{S}_i$. Let $\tilde{M} = \bigcup_{i \in \mathbf{Z}} M_i$ (with the cuts and identifications just described). If S separates M into two components, then the above construction simply yields a countable number of disconnected copies of M. Suppose, however, that S is not achronal. Then $\tilde{M}$ is connected. Define the map π: $\tilde{M} \to M$ by the requirement that $\pi|_{M_i}$ be the natural identification of points in M_i with points in M. It is easily verified that $(\tilde{M}, \pi)$ is a covering manifold of M with the property that each component $\tilde{S}_i$ of $\pi^{-1}(S)$ is homeomorphic to S (with $\pi|_{\tilde{S}_i}: \tilde{S}_i \to S$ being a homeomorphism). (The assumption that S is closed is needed here.) Lift the metric g on M via the covering map to obtain a Lorentz metric $\tilde{g}$ on $\tilde{M}$ with respect to which π is a local isometry. It follows immediately from the construction of $\tilde{M}$ that each $\tilde{S}_i$ is achronal. In fact, the covering manifold described here is isomorphic to a covering manifold introduced by Geroch [3].

We claim, assuming that the hypotheses of Theorem 1 hold, that each component $\tilde{S} = \tilde{S}_K$ of $\pi^{-1}(S)$ is a Cauchy surface in $\tilde{M}$. It suffices to show that

$$H(\tilde{S}) = H^+(\tilde{S}) \cup H^-(\tilde{S}) = \emptyset.$$

Suppose that $H^+(\tilde{S}) \neq \emptyset$. Let $\tilde{\eta}$ be a past inextendible null geodesic generator of $H^+(\tilde{S})$. (Since $\tilde{S}$ is edgeless, $\tilde{\eta}$ never leaves $H^+(\tilde{S})$ as it is extended into the past.) The future end point $\tilde{p}$ of $\tilde{\eta}$ can be chosen so that $\tilde{\eta}$ does not intersect $\tilde{S}_i$ for all $i > K$. Now, $\eta = \pi(\tilde{\eta})$ is a past inextendible null geodesic in M with future end point $p = \pi(\tilde{p}) \in J^+(S)$ (since $\tilde{p} \in J^+(\tilde{S})$). By condition (b) of Theorem 1, η intersects and re-emerges from S. Thus it follows that $\tilde{\eta}$ intersects and re-emerges from *some* component of $\pi^{-1}(S)$, which, by our choice of $\tilde{p}$, must be $\tilde{S}$. This conclusion, together with the fact that $\tilde{\eta} \subset I^+(\tilde{S})$, contradicts the achronality of $\tilde{S}$. Thus, $H^+(\tilde{S}) = \emptyset$ and, similarly, $H^-(\tilde{S}) = \emptyset$. Therefore, each component of $\pi^{-1}(S)$ is a Cauchy surface.

Now, according to condition (b) there exists a point $p \in M$ and an inextendible nonspacelike curve γ: $[0, b) \to M$ which intersects S only finitely many times. Let $\tilde{p}$ be a point in $\tilde{M}$ which covers p and let $\tilde{\gamma}$ be the unique lift of γ starting at $\tilde{p}$. Then, since each $\tilde{S}_i$ is Cauchy, $\tilde{\gamma}$ intersects all components of $\pi^{-1}(S)$ in the future (or in the past) of $\tilde{p}$. But that means that $\gamma = \pi \circ \tilde{\gamma}$ intersects S infinitely often. Thus, the assumption that S is not achronal, which allowed the construction of $\tilde{M}$, leads to a contradiction. Hence, S is achronal. The same argument used to show that $H(\tilde{S}) = \emptyset$ can be used to show that $H(S) = \emptyset$. Thus, S is a Cauchy surface. ∎

As our first corollary to Theorem 1, we obtain (a slightly generalized version of) the theorem of Budic et. al. discussed in the introduction.

COROLLARY 1. *Let S be a compact connected locally achronal edgeless subset of a globally hyperbolic space-time M. Then S is a Cauchy surface.*

Proof. It suffices to show that conditions (a) and (b) of Theorem 1 hold.

Condition (a) *holds.* Let η be any past inextendible null geodesic with future end point $p \in J^+(S)$. We first show that η intersects S. Suppose $\eta \subset J^+(S)$. Then η is contained in $J^+(S) \cap J^-(p)$ which, by the corollary on p. 207 in [6], is compact. Hence, by Proposition 6.4.7, p. 195 in [6], there must be a strong causality violation, contradicting the global hyperbolicity of M. Thus η must leave $J^+(S)$.

Let x be the first point at which η leaves $J^+(S)$. Now,

$$x \in \partial J^+(S) = J^+(S) - I^+(S)$$

(since $J^+(S)$ is closed and $\operatorname{int} J^+(S) = I^+(S)$). If $x \in S$ we are done. Suppose $x \notin S$. Then, since $x \in J^+(S)$, there is a nonspacelike curve τ from S to x. Let U be an open neighborhood of S in which S is achronal, and let $D(S, U)$ and $H(S, U)$ be the domain of dependence and Cauchy horizon, respectively, of S relative to U. Since U is strongly causal and S is compact, a null geodesic generator of $H(S, U)$ cannot be imprisoned in S. Such a generator would have to leave S at an edge point. But, by assumption, S is edgeless. It follows that $H(S, U) \cap S = \emptyset$, and, hence, $S \subset \operatorname{int} D(S, U)$. Thus, τ intersects $\operatorname{int} D(S, U)$, and it easily follows that $x \in I^+(S)$, which is a contradiction. Hence η intersects S. Furthermore η must re-emerge from S (for, otherwise it would be past imprisoned in a compact set, and, hence, there would be a strong causality violation).

The above argument and its time reverse show that condition (a) is satisfied.

Condition (b) *holds.* Let $\gamma\colon [0, b) \to M$ be any future inextendible timelike curve issuing from some point $p \in M$. Suppose γ intersects S for infinitely many parameter values $t_1 < t_2 < \cdots$. Since S is compact, the sequence of points $\{q_i\}$ $(q_i = \gamma(t_i))$ has a convergent subsequence $q_j \to q \in S$. Clearly, strong causality is violated at q. ∎

Theorem 1 may also be used to establish the following.

COROLLARY 2. *Let S_1 and S_2 be two Cauchy surfaces in a space-time M, with S_1 in the past of S_2. Let S be a connected closed locally achronal edgeless subset of M. If S lies between S_1 and S_2, i.e., if $S \subset J^+(S_1) \cap J^-(S_2)$ then S is itself a Cauchy surface.*

Remark. The assumption that S be a closed subset of M cannot be omitted as the following example illustrates. Let M be the cylinder, $x^2 + y^2 = 1$, $-\infty < t < \infty$, in Euclidean (x, y, t)-space; equipped with the locally Minkowskian metric $ds^2 = -dt^2 + d\theta^2$, where θ is the polar coordinate. Let S be the "compressed helix"

$$u \to (\cos u, \sin u, \tan^{-1} u).$$

S is bounded between the Cauchy surfaces S_1 $(t = -\pi/2)$ and S_2 $(t = \pi/2)$ but is not itself a Cauchy surface.

Proof of Corollary 2. Again apply Theorem 1.

Condition (a) *holds.* First we show that $J^+(S)$ and $J^-(S)$ are closed. (In general $J^{\pm}(S)$ are not closed even if M is globally hyperbolic and S is closed.) Suppose

$$q \in \overline{J^+(S)} - J^+(S) \subset \partial I^+(S).$$

Let η be a past inextendible null geodesic generator on the achronal boundary $\partial I^+(S)$ with future end point q. Since $J^+(S) \subset J^+(S_1)$ and $J^+(S_1)$ is closed, $q \in J^+(S_1)$. Thus η must intersect S_1 and enter into $I^-(S_1)$. Hence,

$$I^-(S_1) \cap J^+(S_1) \supset I^-(S_1) \cap I^+(S) \neq \emptyset,$$

which contradicts the achronality of S_1. Therefore $J^+(S)$ and, similarly, $J^-(S)$ are closed.

Now, let $p \in J^+(S)$ and let η be any inextendible null geodesic with future end point p. Since $J^+(S) \subset J^+(S_1)$, η must leave $J^+(S)$, for, otherwise, $\eta \subset J^+(S_1)$, which is impossible (since S_1 is Cauchy). Now one can argue just as in the proof of Corollary 1 that the point at which η leaves $J^+(S)$ is in S. Thus η intersects S, and must re-emerge (or, otherwise, $\eta \subset J^+(S_1)$).

Condition (b) *holds.* Any future inextendible timelike curve issuing from a point p in $I^+(S_2)$ cannot enter into $J^-(S_2)$ and, hence, cannot meet S. ∎

If S is a Cauchy surface for M then the topology of M is known up to the topology of S. Indeed, as is well known, M is homeomorphic to $\mathbf{R} \times S$. If a subset S of M is closed, achronal and satisfies the intersection condition (a) of Theorem 1 then S is Cauchy. Our next result describes the topology of M when M admits a connected closed locally achronal edgeless subset S which satisfies condition (a) of Theorem 1, but which is not achronal. It follows as a consequence of Theorem 1 that for such a subset S, any timelike curve issuing from a point of S reaches S again. Let T be any timelike vector field on M. Associate with T the map $\theta_T: S \to S$ defined as follows: For each $q \in S$, $\theta_T(q)$ is the first point in S reached by traveling along the integral curve of T through q. The map θ_T defines a homeomorphism of S onto itself.

As a further bit of notation, if X is any topological space and $\alpha: X \to X$ any homeomorphism, then by $[0, 1] \times X/\alpha$ we mean the space obtained from the product space $[0, 1] \times X$ by identifying the points $(0, q)$ and $(1, \alpha(q))$ for each $q \in X$.

THEOREM 2. *Let S be a connected closed locally achronal edgeless subset of M which satisfies condition* (a) *of Theorem 1. If S is not achronal then for any timelike vector field T on M, M is homeomorphic to* $[0, 1] \times S/\theta_T$.

Examples. 1. Let M be the cylinder "closed in time", constructed by taking the region $-1 \leq t \leq 1$ in 2-dimensional Minkowski space and identifying for each x the points $(x, -1)$ and $(x, 1)$. Let S be the line $t = 0$. For $T = \partial/\partial t$, $\theta_T = \text{id}$ and $[0, 1] \times S/\theta_T = [0, 1] \times \mathbf{R}/\text{id} = S^1 \times \mathbf{R}$ which is the topology of M.

A slightly less trivial example is the following.

2. Let M be the cylinder "closed in space" discussed in the remark following the statement of Corollary 2. Let S be the helix $u \to (\cos u, \sin u, u)$. For $T = \partial/\partial t$, θ_T is a translation and $[0, 1] \times S/\theta_T = [0, 1] \times \mathbf{R}/\text{trans.} =$ a cylinder.

3. Let M be the Moebius strip constructed by taking the region $-1 \leq t \leq 1$ in 2-dimensional Minkowski space and identifying the points $(x, -1)$ and $(-x, 1)$ for each x. Let S be the line $t = 0$. For $T = \partial/\partial t$, $\theta_T = -\text{id}$. This example shows that the topology of $[0, 1] \times S/\theta_T$ needn't be a product topology.

Proof of Theorem 2. Let T be any timelike vector field on M. Assume for definiteness that T is future pointing (Only monor modifications are required if T is past pointing.) Let $(\tilde{M}, \pi)$ be the covering manifold of M introduced in the proof of Theorem 1. Lift T to $\tilde{M}$ via the covering map to obtain a future pointing timelike vector field $\tilde{T}$ on $\tilde{M}$.

Consider the subset M_0 of $\tilde{M}$ with boundary consisting of the disjoint copies of S, $\tilde{S}_0$ and $\tilde{S}_1$. (Here we are making use of the notation introduced in the proof of Theorem 1). We first show that M_0 is homeomorphic to $[0, 1] \times S$. As follows from the proof of Theorem 1, $\tilde{S}_0$ and $\tilde{S}_1$ are Cauchy surfaces for $\tilde{M}$. It follows that each point M_0 is on a unique integral curve of the *past* directed vector field $-\tilde{T}$ from $\tilde{S}_1$ to $\tilde{S}_0$. Furthermore $-\tilde{T}$ can be smoothly rescaled so that to travel from $\tilde{S}_1$ to $\tilde{S}_0$ along any integral curve takes parameter time exactly one. Let γ_q: $[0, 1] \to M_0$ be the integral curve of $-\tilde{T}$ (appropriately rescaled) issuing from $q \in \tilde{S}_1$. Define the map ρ: $[0, 1] \times \tilde{S}_1 \to M_0$ by, $\rho(t, q) = \gamma_q(t)$. Let $\pi_{\tilde{S}_1}$: $\tilde{S}_1 \to S$ be the restriction of π to $\tilde{S}_1$. Then the map ψ: $[0, 1] \times S \to M_0$ defined by

$$\psi(t, q) = \rho(t, \pi_{\tilde{S}_1}^{-1}(q))$$

is the desired homeomorphism.

Let α: $\tilde{S}_1 \to \tilde{S}_0$ be the natural identification of the points of $\tilde{S}_1$ with the points of $\tilde{S}_0$. (The map α is determined by the covering map via the equation, $\pi(\alpha(\tilde{q})) = \pi(\tilde{q})$). Let M_0/α be the space obtained by identifying the boundary points of $\tilde{S}_1$ and $\tilde{S}_0$ in the natural way, i.e., by identifying $\tilde{q}$ and $\alpha(\tilde{q})$ for each $\tilde{q} \in \tilde{S}_1$. By the very construction of $\tilde{M}$, M_0/α is homeomorphic to M. Let β: $\tilde{S}_0 \to \tilde{S}_1$ be the map defined by $\beta = \rho^{-1}|_{\tilde{S}_0}$, i.e., β maps the point $\tilde{q} \in \tilde{S}_0$ to the point in $\tilde{S}_1$ reached by traveling along the integral curve of $\tilde{T}$ through q. Identifying the boundary points $\tilde{q}$ and $\alpha(\tilde{q})$ in M_0 corresponds, via the homeomorphism ψ, to identifying the boundary points $(0, \pi(\tilde{q}))$ and $(1, \pi \circ \beta \circ \alpha(\tilde{q}))$ in $[0, 1] \times S$. Carry out this latter identification to obtain the space $[0, 1] \times S/\chi$ (where $\chi = \pi \circ \beta \circ \alpha \circ \pi_{\tilde{S}_1}^{-1}$), which is homeomorphic to M_0/α, and, hence is homeomorphic to M.

One easily verifies from the properties of the covering and the definitions of the maps involved that the following diagram is commutative:

$$\begin{array}{ccccc} \tilde{S}_1 & \xrightarrow{\alpha} & \tilde{S}_0 & \xrightarrow{\beta} & \tilde{S}_1 \\ {\scriptstyle\pi}\downarrow & \swarrow{\scriptstyle\pi} & & & \downarrow{\scriptstyle\pi} \\ S & & \xrightarrow[\theta_T]{} & & S\,. \end{array}$$

Tracing through the diagram one sees that $\chi = \theta_T$, and the proof is completed. ▮

We close the paper with the following corollary, which generalizes Corollary 1.

COROLLARY 3. *Let S be a connected compact locally achronal edgeless subset of M which satisfies condition* (a) *of Theorem 1. If there are no closed timelike curves passing through S then S is a Cauchy surface.*

Proof. Suppose S is not achronal. Then according to Theorem 2, M has topology $[0, 1] \times S/\theta_T$ and, hence, is compact since S is. But compact space-times necessarily admit closed timelike curves (e.g., see [1], p. 23). Thus, there is a closed timelike curve in M, which, by assumption, does not pass through S. Consequently condition (b) of Theorem 1 is also satisfied and, hence, S is Cauchy. But this contradicts the assumption that S is not achronal. Therefore, S must be achronal and, hence, is Cauchy by Theorem 1.

Example 2 following Theorem 2 shows that Corollary 3 is false if the assumption that S is compact is dropped or replaced by the assumption that S is closed.

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