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SUPPORTS OF EXTREMAL MEASURES WITH GIVEN MARGINALS

BY

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1. Introduction

Let X and Y be countable measure spaces with positive finite measures μ and ν, respectively. Let $M(\mu, \nu)$ be the (convex) class of all positive measures on $X \times Y$ with marginals (projections) μ and ν, respectively. The extreme points of $M(\mu, \nu)$ have been characterized by Douglas [2] and Lindenstrauss [4] by the following theorem:

THEOREM 1.1 (Douglas-Lindenstrauss). *$\lambda \in M(\mu, \nu)$ is extreme if and only if*

$$\{ p + q \colon p \in L_1(\mu), q \in L_1(\nu) \}$$

is norm-dense in $L_1(\lambda)$.

Letac [3] and Denny [1] give two different characterizations of the supports of the extremes. Letac [3] proved his Theorem 4 without using the Douglas-Lindenstrauss theorem. It is a long proof using substantial machinery—he omits the 'only if' part of the Douglas-Lindenstrauss theorem from the corollary to his Theorem 4 because of the length of the proof. Denny's [1] characterization is based on an idea in Letac [3] and he does depend on the Douglas-Lindenstrauss theorem for his proof.

The problem of characterizing the supports of the extremes is essentially combinatorial rather than measure-theoretic. The purpose of this paper is to use some simple combinatorial ideas to develop several characterizations of the extremes by their supports. The proofs are elementary, geometrically intuitive, and do not use the Douglas-Lindenstrauss theorem. One result is a strengthened version of the Douglas-Lindenstrauss theorem (Theorem 2.7) in this countable case.

2. Results

We take advantage of the usual abuse of notation and write $\lambda(x, y)$, $\mu(x)$, and $\nu(y)$ for $\lambda(\{x, y\})$, $\mu(\{x\})$ and $\nu(\{y\})$, respectively.

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If λ is a measure on a countable set Z the support S of λ is defined by

$$S = \{z \in Z: \lambda(z) > 0\}.$$

Since we will be dealing with many sequences we will not indicate the ranges of the indices when they are obvious.

The notations $A + B$ and ΣA_i indicate disjoint unions of sets.

DEFINITION 1. Let λ be a measure on $X \times Y$ and let S be its support. $L = \{(x_i, y_i): 1 \leq i \leq k\}$ is called a λ-link if $L \subset S$ and, for $i \geq 1$, either $y_{2i} = y_{2i-1}$, $x_{2i+1} = x_{2i}$, $x_{2i} \neq x_{2i-1}$, and $y_{2i+1} \neq y_{2i}$, or the same conditions with x and y interchanged.

DEFINITION 2. A λ-link $L = \{(x_i, y_i): 1 \leq i \leq 2n\}$ is called a λ-loop if $x_1 = x_{2n}$ (in which case $y_1 = y_2$) or $y_1 = y_{2n}$ (in which case $x_1 = x_2$) and $x_i \neq x_j$ and $y_i \neq y_j$ for $1 \leq i \leq j - 3 < 2n - 3$.

Remark 1. Note that the λ-loop L defined above is contained in an $n \times n$ rectangle every row and column of which contains exactly two elements of L. □

DEFINITION 3. If λ is a measure on $X \times Y$ then a non-empty rectangle $A \times B$ is said to be λ-full if every row and every column of $A \times B$ contains at least one element of the support of λ.

All of our proofs depend crucially on the following technical result.

LEMMA . *Let λ be a measure on $X \times Y$ with support S. Suppose for some $N \geq 1$, $\{C_i \times R_i: 1 \leq i \leq N\}$ is a sequence of λ-full rectangles such that C_1 is a singleton, $R_{2i} \subset R_{2i-1}$, $C_{2i+1} \subset C_{2i}$, and*

$$C_{2i} \cap C_{2i-1} = C_{2i} \cap C_{2i+2} = R_{2i+1} \cap R_{2i} = R_{2i+1} \cap R_{2i-1} = \emptyset.$$

Then there exists a λ-loop if either

(i) *$R_i \cap R_j \neq \emptyset$ or $C_i \cap C_j \neq \emptyset$ for some $1 \leq i \leq j - 3 \leq N - 3$, or,*

(ii) *a row(column) of $C_i \times R_i$ contains more than one element of S for an odd(even) i.*

Proof. First suppose $C_m \cap C_n \neq \emptyset$ or $R_m \cap R_n \neq \emptyset$ for some $1 \leq m \leq n - 3 \leq N - 3$. Suppose this happens for the first time when $n = k$; i.e., $C_i \cap C_j = R_i \cap R_j = \emptyset$ for $1 \leq i \leq j - 3 < k - 3$ and $C_p \cap C_k \neq \emptyset$ (in which case k is necessarily even) or $R_p \cap R_k \neq \emptyset$ (in which case k is necessarily odd) for some $1 \leq p \leq k - 3$. We consider the case k even only; the case k odd is similar.

If $p > 1$ it is possible that $C_p \cap C_k \neq \emptyset$ for two consecutive values of p, in which case we choose p odd (the higher value).

Since $\{C_i \times R_i\}$ is λ-full and $R_{2i} \subset R_{2i-1}$, if $(x, y) \in (C_{2i} \times R_{2i}) \cap S$ then there exists

$$(x', y) \in (C_{2i-1} \times R_{2i-1}) \cap S \quad \text{for some } x' \in C_{2i-1}.$$

Similarly, $(x, y) \in (C_{2i+1} \times R_{2i+1}) \cap S$ implies there exists

$$(x, y') \in (C_{2i} \times R_{2i}) \cap S \quad \text{for some } y' \in R_{2i}.$$

Now suppose p is odd and $x_k \in C_p \cap C_k$. Then there exists a λ-link

$$(x_k, y_k), (x_{k-1}, y_{k-1}), \ldots, (x_p, y_p) \quad \text{with } (x_i, y_i) \in (C_i \times R_i) \cap S.$$

By hypothesis $x_i \neq x_j$ and $y_i \neq y_j$ for $p \leq i \leq j - 3 < k - 3$. If $x_p = x_k$ or if there exists $(x_k, y_p) \in (C_p \times R_p) \cap S$ then we clearly have a λ-loop. If not, there exists $(x'_p, y'_p) \in (C_p \times R_p) \cap S$ with $x'_p = x_k$ and y'_p (necessarily) not equal to y_p. Again from the properties of $\{C_i \times R_i\}$ there exist λ-links

$$(x_p, y_p), (x_{p-1}, y_{p-1}), \ldots, (x_1, y_1) \text{ and } (x'_p, y'_p), (x'_{p-1}, y'_{p-1}), \ldots, (x'_1, y'_1)$$

with (x_i, y_i) and (x'_i, y'_i) in $(C_i \times R_i) \cap S$, and by hypothesis $x_i(x'_i) \neq x_j(x'_j)$ and $y_i(y'_i) \neq y_j(y'_j)$ for $1 \leq i \leq j - 3 \leq p - 3$. Since C_1 is a singleton $x_1 = x'_1$. Let t be the smallest i for which $x_{p-i} = x'_{p-i}$ $(p - i$ odd) or $y_{p-i} = y'_{p-1}$ $(p - i$ even), $1 \leq i \leq p - 1$. Then, using the hypothesis on k,

$$(x_k, y_k), \ldots, (x_p, y_p), \ldots, (x_{p-t}, y_{p-t}), (x'_{p-t}, y'_{p-t}),$$
$$(x'_{p-t+1}, y'_{p-t+1}), \ldots, (x'_{p-1}, y'_{p-1})$$

is a λ-loop with $x_k = x'_p = x'_{p-1}$. If p is even we can similarly form a λ-loop from the λ-links $(x_k, y_k), \ldots, (x_1, y_1)$ and $(x'_p, y'_p), \ldots, (x'_1, y'_1)$, where $x_k = x'_p$ and $x_1 = x'_1$.

Now suppose a row(column) of $C_n \times R_n$ has at least two elements of S for some odd(even) n, $1 \leq n \leq N$, and assume n is the smallest integer for which this is true. We also assume $C_i \cap C_j = R_i \cap R_j = \emptyset$ for $1 \leq i \leq j - 3 \leq n - 3$; otherwise we have a λ-loop by (i). By hypothesis there exist (x_n, y_n) and (x'_n, y'_n) in $(C_n \times R_n) \cap S$ with $x_n \neq x'_n$ and $y_n = y'_n$ or $x_n = x'_n$ and $y_n \neq y'_n$. Then there exist λ-links

$$(x_n, y_n), \ldots, (x_1, y_1) \quad \text{and} \quad (x'_n, y'_n), \ldots, (x'_1, y'_1)$$

with (x_i, y_i) and (x'_i, y'_i) in $(C_i \times R_i) \cap S$ and $x_i(x'_i) \neq x_j(x'_j)$ and $y_i(y'_i) \neq$

$y_j(y_j')$ for $1 \le i \le j - 3 \le n - 3$. Since $x_1 = x_1'$ we can construct a λ-loop

$$(x_n, y_n), \ldots, (x_{n-t}, y_{n-t}), (x'_{n-t}, y'_{n-t}), (x'_{n-t+1}, y'_{n-t+1}), \ldots, (x'_n, y'_n)$$

in the same way as in the proof of (i) for some $1 \le t \le n - 1$. □

Our first characterization of the support of an extreme is the same as that of Letac [3] although the proof of the 'if' part is different.

THEOREM 2.1 (Letac). *Suppose $\lambda \in M(\mu, \nu)$. Then λ is extreme if and only if there exists no λ-loop.*

Proof. Suppose $\{(x_i, y_i)\colon 1 \le i \le 2n\}$ is a λ-loop. Let

$$m = \min\{\lambda(x_i, y_i)\colon 1 \le i \le 2n\}.$$

Define λ' and λ'' by

$$\lambda'(x_i, y_i) = \lambda(x_i, y_i) + (-1)^i m, \quad \lambda''(x_i, y_i) = \lambda(x_i, y_i) - (-1)^i m$$

for $1 \le i \le 2n$, and

$$\lambda'(x, y) = \lambda''(x, y) = \lambda(x, y)$$

otherwise. Then λ' and λ'' are positive measures on $X \times Y$, $\lambda' \ne \lambda$, and $\lambda = (\lambda' + \lambda'')/2$. From the definition of λ-loops both λ' and λ'' are in $M(\mu, \nu)$. Thus λ is not extreme.

Now suppose λ is not extreme. Then there exist λ' and λ'' in $M(\mu, \nu)$ such that $\lambda'' \ne \lambda$ and $\lambda = (\lambda' + \lambda'')/2$. Thus there exists $(x_1, y_1) \in X \times Y$ such that $\lambda'(x_1, y_1) - \lambda(x_1, y_1) = c > 0$. Let $C_1 = \{x_1\}$ and $R_1 = \{y_1\}$. Now for $i \ge 1$ define recursively

$$C_{2i} = \{x \in X - C_{2i-1}\colon \lambda(x, y) > \lambda'(x, y) \text{ for some } y \in R_{2i-1}\},$$

$$R_{2i} = \{y \in R_{2i-1}\colon \lambda(x, y) > \lambda'(x, y) \text{ for some } x \in C_{2i}\},$$

$$R_{2i+1} = \{y \in Y - R_{2i}\colon \lambda(x, y) < \lambda'(x, y) \text{ for some } x \in C_{2i}\},$$

and

$$C_{2i+1} = \{x \in C_{2i}\colon \lambda(x, y) < \lambda'(x, y) \text{ for some } y \in R_{2i+1}\}.$$

Since λ and λ' have the same marginals,

$$|\lambda(C_{i+1} \times R_{i+1}) - \lambda'(C_{i+1} \times R_{i+1})| \ge |\lambda(C_i \times R_i) - \lambda'(C_i \times R_i)| \ge c$$

for $i \ge 1$.

Since μ and ν are finite, $C_i \cap C_j \neq \emptyset$ or $R_i \cap R_j \neq \emptyset$ for some $1 \leq i \leq j - 3$. From their construction $\{C_i \times R_i\}$ satisfy condition (i) of the lemma and hence there exists a λ-loop. □

Remark 2. In his Theorem 2 Denny [1] generalizes his results to signed measures μ and ν. Unfortunately, using the ideas above it is easy to see that there are no extremes in $M(\mu, \nu)$ if μ and ν are nondegenerate signed measures. □

DEFINITION 4. For $\lambda \in M(\mu, \nu)$, $A \subset X$, and $B \subset Y$ let $\lambda_{A \times B}$ be the restriction of λ to $A \times B$ and let μ_A and ν_B be the corresponding marginals; i.e.,

$$\mu_A(S) = \lambda_{A \times B}(S \times B) \text{ for } S \subset A \quad \text{and} \quad \nu_B(T) = \lambda_{A \times B}(A \times T) \text{ for } T \subset B.$$

Let $M(\mu_A, \nu_B)$ be the convex class of measures on $A \times B$ with marginals μ_A and ν_B. We say λ is conditionally extreme in $A \times B$ if $\lambda_{A \times B}$ is extreme in $M(\mu_A, \nu_B)$.

The proof of the next theorem is immediate from Theorem 2.1 and is omitted.

THEOREM 2.2. *$\lambda \in M(\mu, \nu)$ is extreme if and only if λ is conditionally extreme in all finite rectangles $A \times B$.*

DEFINITION 5. Suppose λ is a measure on $X \times Y$ with support S and $F \subset X \times Y$. If a row(column) of F contains at most one element of S we say the row(column) of F is λ-erasable. F is said to be λ-erasable if by first deleting all the λ-erasable rows, then deleting all the λ-erasable columns of the remaining set, and repeating the process a finite or infinite number of times, we can delete all the rows and columns of F. We will also use expressions like "λ-erase the rows of F", whose meanings will be obvious. If a set $F \subset X \times Y$ has been λ-erased partially or completely and if $(x, y) \in F \cap S$ has been deleted by λ-erasure of the column x or the row y in some subset of F then we say (x, y) has been λ-erased in F, or simply (x, y) has been λ-erased if there is no ambiguity about the set F in question.

The next theorem possibly provides the easiest method of checking the extremality of a given $\lambda \in M(\mu, \nu)$ when X and Y are finite.

THEOREM 2.3. *Suppose $\lambda \in M(\mu, \nu)$ with support S. Then λ is extreme if and only if every finite rectangle $A \times B$ is λ-erasable, or, equivalently, for every finite rectangle $A \times B$, if $U = (A \times B) \cap S \neq \emptyset$, then at least one element of U is λ-erasable in $A \times B$.*

Proof. Suppose λ is not extreme. By Theorem 2.1 there exists a λ-loop

$$L = \{(x_i, y_i): 1 \leq i \leq 2n\}.$$

By Remark 1 the minimal rectangle containing L is not λ-erasable.

Now suppose $A \times B$ is a finite rectangle which is not λ-erasable. We first delete all the λ-erasable rows of $A \times B$, then all the λ-erasable columns of the remaining rectangle, and repeat until we have a (non-empty) rectangle $C \times D$ no row or column of which is λ-erasable. Let $U = (C \times D) \cap S$. Suppose $(x_1, y_1) \in U$. From the hypothesis that no row or column of $C \times D$ is λ-erasable we can successively find $(x_2, y_2), (x_3, y_3), \dots$ in U such that $y_{2i} = y_{2i-1}$, $x_{2i+1} = x_{2i}$, $x_{2i} \neq x_{2i-1}$, and $y_{2i+1} \neq y_{2i}$, $i \geq 1$. Since A and B are finite we will eventually have $x_i = x_j$ or $y_i = y_j$ for some $1 \leq i \leq j - 3$. By the lemma there is a λ-loop and, by Theorem 2.1, λ is not extreme. □

Example 1. Let X and Y be the positive integers. Consider the support S of a measure λ on $X \times Y$ consisting of the points $(1,1)$, $(1,2)$, and for $i \geq 2$ the points $(i, i-1)$ and $(i, i+1)$. Then it is easily checked that no element of S is λ-erasable in $X \times Y$, but every finite rectangle is. Thus $\lambda \in M(\mu, \nu)$ is extreme does not imply that every infinite rectangle is λ-erasable. □

Theorem 2.3 provides a swift proof of the Birkhoff-von Neumann theorem. Recall an $n \times n$ doubly stochastic matrix is an $n \times n$ matrix of non-negative numbers with row and column sums equal to one. An $n \times n$ permutation matrix is an $n \times n$ doubly stochastic matrix each row and column of which contains exactly one 1 and the rest 0.

COROLLARY 2.3 (Birkhoff-von Neumann). *The $n \times n$ permutation matrices are the only extremes in the $n \times n$ doubly stochastic matrices.*

Proof. Defining $X = Y = \{1, 2, \dots, n\}$ and μ and ν to be the counting measures we can identify the $n \times n$ doubly stochastic matrices with $M(\mu, \nu)$. If P is an $n \times n$ doubly stochastic matrix we write λ_P for the corresponding measure in $M(\mu, \nu)$.

If P is an $n \times n$ permutation matrix then clearly $X \times Y$ is λ_P-erasable and hence P is extreme by Theorem 2.3.

If P is an extreme $n \times n$ doubly stochastic matrix then $X \times Y$ is λ-erasable and hence P is extreme by Theorem 2.3. Since the marginals of λ_P are counting measures the first row or column λ_P-erased must contain a point (i, j) with $\lambda_P(i, j) = 1$. Then all other elements of the ith column and the jth row of P must be 0. Continuing the argument we see that every point of the support of λ_P that is λ_P-erased has λ_P measure one, and since every row and column has λ_p measure one, P must be an $n \times n$ permutation matrix. □

The next theorem is a generalization of Proposition 2 of Lindenstrauss [4].

THEOREM 2.4. *Suppose $\lambda \in M(\mu, \nu)$ with support S. Then λ is extreme if and only if every finite $m \times n$ rectangle has less than $m + n$ elements of S.*

Remark 3. The proof of Theorem 2.4 in contrast to the proof of Proposition 2 of Lindenstrauss [4] possibly best illustrates the difference between measure-theoretic and combinatorial proofs of characterization of the support of extreme measures. □

Proof of Theorem 2.4. If λ is not extreme then by Theorem 2.1 and Remark 1 there exists a λ-loop $\{(x_i, y_i): 1 \le i \le 2n\}$ contained in an $n \times n$ rectangle every row and column of which contains at least two elements of S. Thus the number of elements of S in this rectangle $\ge 2n$.

If λ is extreme and $A \times B$ is an $m \times n$ finite rectangle than $A \times B$ is λ-erasable by Theorem 2.3. During λ-erasure of $A \times B$, every time a row or column of $A \times B$ (or a subset) is deleted, at most one element of S is λ-erased. Since at most $m - 1$ rows or $n - 1$ columns can be λ-erased without λ-erasing $A \times B$ completely and since after λ-erasing $m - 1$ rows and $n - 1$ columns at most one element of $A \times B$ will be remaining, the total number of rows and columns λ-erased in λ-erasing $A \times B$ is less than or equal to $m + n - 1$. Hence the number of elements of S λ-erased is less than $m + n$. □

With Example 1 and Theorem 2.3 in mind we are now in a position to completely describe the support of an extreme in $M(\mu, \nu)$ in terms of an "orthogonal decomposition".

THEOREM 2.5. *Suppose $\lambda \in M(\mu, \nu)$ is extreme with support S. Then $S = \sum_{n \ge 0} S_n$, where:*
(1) S_0 is λ-erasable and $S_0 = [(A_0 \times Y) \cup (X \times B_0)] \cap S$ for some $A_0 \subset X$ and $B_0 \subset Y$;
(2) if $S_n \ne \emptyset$ for some $n \ge 1$ then
(i) S_n is of the form $\sum_{i \ge 1}(C_{n,i} \times R_{n,i}) \cap S$, where
(α) $C_{n,1}$ is a singleton,
(β) $R_{n,2i} = R_{n,2i-1}$, $C_{n,2i+1} = C_{n,2i}$,

$$C_{n,2i} \cap C_{n,2i-1} = R_{n,2i+1} \cap R_{n,2i} = \emptyset, \quad i \ge 1,$$

and

$$C_{n,i} \cap C_{n,j} = R_{n,i} \cap R_{n,j} = \emptyset \quad \textit{for } 1 \le i \le j - 3,$$

(γ) $C_{n,i} \times R_{n,i}$ is λ-full for $i \ge 1$, and

(δ) *for i odd(even) every row(column) of $C_{n,i} \times R_{n,i}$ has exactly one element of S and hence $C_{n,i} \times R_{n,i}$ is λ-erasable by deleting rows(columns) only, and*

(ii) *S_n is of the form $(A_n \times B_n) \cap S$, where A_n and B_n are both infinite, $A_n \times B_n$ is λ-full, and no element of S_n is λ-erasable in $A_n \times B_n$;*

(3) *$A_n \cap A_m = B_n \cap B_m = \emptyset$ for $0 \le n < m$.*

In particular, if X or Y is finite then S is λ-erasable.

Proof. Write $X = \{x_1, x_2, \dots\}$ and $Y = \{y_1, y_2, \dots\}$. Now λ-erase as much of $X \times Y$ as possible by successive row and column deletions. Let S_0 be the elements of S λ-erased and let $A_0(B_0)$ be the columns(rows) λ-erased. Then

$$S_0 = [(A_0 \times Y) \cup (X \times B_0)] \cap S.$$

Now suppose $S - S_0 \neq \emptyset$. From the definition of λ-erasure

$$S - S_0 \subset (X - A_0) \times (Y - B_0) \equiv X_1 \times Y_1$$

which is λ-full. Let $C_{1,1} = \{x_k\}$ where k is the smallest j for which

$$(x_j, y) \in (X_1 \times Y_1) \cap S \quad \text{for some } y \in Y_1.$$

Let $R_{1,1} = \{y \in Y_1 \colon (x_k, y) \in S\}$. Now for $i \ge 1$ define recursively $R_{1,2i} = R_{1,2i-1}$,

$$C_{1,2i} = \{x \in X_1 - C_{1,2i-1} \colon (x, y) \in S \text{ for some } y \in R_{1,2i}\},$$

$C_{1,2i+1} = C_{1,2i}$, and

$$R_{1,2i+1} = \{y \in Y_1 - R_{1,2i} \colon (x, y) \in S \text{ for some } x \in C_{1,2i+1}\}.$$

Define $S_1 = \Sigma_{i \ge 1}(C_{1,i} \times R_{1,i}) \cap S$. Then for $n = 1$ in part (2)(i) we note that $C_{1,1}$ is a singleton, which satisfies (α), the first string of equalities in (β) is satisfied by the construction of the $\{C_{1,i} \times R_{1,i}\}$, and the rest of ($\beta$) and ($\delta$) will be satisfied by the extremality of λ, Theorem 2.1, and the lemma, if we can prove (γ). Note that $C_{1,1} \times R_{1,1}$ is λ-full. Now suppose $C_{1,j} \times R_{1,j}$ is not λ-full for the first time for some $j > 1$. Then

$$\{C_{1,k} \times R_{1,k} \colon 1 \le k \le j - 1\}$$

is λ-full and must not satisfy either (i) or (ii) of the lemma. Hence some element of S in $C_{1,j-1} \times R_{1,j-1}$ is λ-erasable in $X \times Y$ (after S_0 has been

λ-erased) contradicting

$$(C_{1,i-1} \times R_{1,i-1}) \cap S \subset S - S_0.$$

Thus $C_{1,i} \times R_{1,i}$ is λ-full for all $i \geq 1$. This proves (2)(i) for $n = 1$. Now let $A_1 = \Sigma_{i \geq 1} C_{1,2i-1}$ and $B_1 = \Sigma_{i \geq 1} R_{1,2i}$. Then it is clear that A_1 and B_1 are both infinite, $A_1 \times B_1$ is λ-full, $S_1 = (A_1 \times B_1) \cap S$, and no element of S_1 is λ-erasable in $A_1 \times B_1$. This proves (2)(ii) for $n = 1$.

If $S - S_0 - S_1 \neq \emptyset$ then

$$S - S_0 - S_1 \subset [(X - A_0 - A_1) \cup (Y - B_0 - B_1)] \cap S,$$

since from the construction above

$$[A_1 \times (Y - B_0 - B_1)] \cap S = [(X - A_0 - A_1) \times B_1] \cap S = \emptyset.$$

Thus we can construct S_2, A_2, B_2, and $\{C_{2,i} \times R_{2,i}\}$ in the same manner as above from $(X - A_0 - A_1) \times (Y - B_0 - B_1)$. This process can be continued indefinitely or until S is exhausted, each time defining $C_{n,1} = \{x_k\}$ where k is the smallest index j for which

$$(x_j, y) \in \left[\left(X - \bigcup_{i=0}^{n-1} A_i\right) \times \left(Y - \bigcup_{i=0}^{n-1} B_i\right)\right] \cap S \quad \text{for some } y \in Y - \bigcup_{i=0}^{n-1} B_i.$$

This guarantees all elements of S in every column of $X_1 \times Y_1$ belongs to S_n for some $n \geq 1$. Thus $S = \bigcup_{n \geq 0} S_n$ and, from the construction of the $\{A_i\}$ and $\{B_i\}$, $A_n \cap A_m = B_n \cap B_m = \emptyset$ for $0 \leq n < m$, so that $S = \Sigma_{n \geq 0} S_n$.

The final remark follows from the fact that when X or Y is finite $S_n = \emptyset$ for all $n \geq 1$. □

The next characterization of the extremes in $M(\mu, \nu)$ is due to Denny [1] although our proof is entirely different. Suppose f: $E \to Y$ and g: $F \to X$ are two functions for some $E \subset X$ and $F \subset Y$.

DEFINITION 6. We say the pair (f, g) defined above is periodic if $(g \circ f)^n(x)$ is defined and is equal to x for some $x \in E$ and a positive integer n. Otherwise we say (f, g) is aperiodic.

We denote the graphs of f and g by $G(f) = \{(x, f(x)): x \in E\}$ and $G(g) = \{(g(y), y): y \in F\}$, respectively.

THEOREM 2.6 (Denny). *Suppose $\lambda \in M(\mu, \nu)$ with support S. Then λ is extreme if and only if there exists an aperiodic pair of functions (f, g) such that $S = G(f) + G(g)$.* □

Remark 4. Denny [1] proved the 'only if' part of the theorem by first proving it for all finite subsets of $X \times Y$ and then using an extension argument. This can be done easily using Theorem 2.3. However, we give an actual construction of an aperiodic (f, g) using the decomposition of S as given by Theorem 2.5. □

Proof of Theorem 2.6. Suppose λ is not extreme. Then there exists a λ-loop $\{(x_i, y_i): 1 \leq i \leq 2n\}$ by Theorem 2.1. Suppose there exists an aperiodic pair of functions (f, g) such that $S = G(f) + G(g)$. Now suppose $x_1 = x_{2n}$ and $(x_1, y_1) \in G(f)$. Since f and g are functions,

$$\begin{aligned}(x_{2n}, y_{2n}) = (x_1, y_{2n}) \in G(g), (x_{2n-1}, y_{2n-1}) \in G(f), \ldots, (x_2, y_2)\\ = (x_2, y_1) \in G(g).\end{aligned}$$

But then $(g \circ f)(x_1) = x_2, (g \circ f)^2(x_1) = x_4, \ldots, (g \circ f)^n(x_1) = x_{2n} = x_1$, contradicting the hypothesis (f, g) is aperiodic. Contradictions can be derived similarly if $(x_1, y_1) \in G(g)$ or $y_1 = y_{2n}$ in the λ-loop.

Now suppose λ is extreme. We use the decomposition of S given by Theorem 2.5 and the terminology used therein.

Consider the λ-erasure of $(A_0 \times Y) \cup (X \times B_0)$. Let

$$E_0 = \{x \in A_0\text{: for some } y \in Y, (x, y) \in S_0 \text{ was } \lambda\text{-erased by a column deletion}\}$$

and

$$F_0 = \{y \in B_0\text{: for some } x \in X, (x, y) \in S_0 \text{ was } \lambda\text{-erased by a row deletion}\}.$$

For $n \geq 1$ let $E_n = A_n - C_{n,1}$ and $F_n = B_n$. Let $E = \Sigma_{n \geq 0} E_n$ and $F = \Sigma_{n \geq 0} F_n$. Since $\{E_n\}$ and $\{F_n\}$ are pairwise disjoint collections of sets we define f and g by defining $f_n = f|_{E_n}$ and $g_n = g|_{F_n}$ for $n \geq 0$.

For $x \in E_0$ ($y \in F_0$) let $f_0(x) = y$ ($g_0(y) = x$) such that $(x, y) \in S_0$ was λ-erased by a column(row) deletion for some $y(x)$. Such a column or row can be λ-erased at most once and at most one element of S_0 (as a matter of fact, exactly one element of S_0 if $x \in E_0$ or $y \in F_0$) can be λ-erased in the process. Thus f_0 and g_0 are well defined. From the definitions of S_0, E_0 and F_0 we have

$$G(f_0) = (E_0 \times Y) \cap S, \quad G(g_0) = (X \times F_0) \cap S$$

and

$$S_0 = G(f_0) + G(g_0).$$

If $S_n \neq \emptyset$ for some $n \geq 1$ then $C_{n,i} \times R_{n,i}$ is λ-full for $i \geq 1$ and is λ-erasable by column(row) deletions only, if i is even(odd). Since $\{C_{n,2i}\}$ and

$\{R_{n,2i-1}\}$ are pairwise disjoint collections of sets we define f_n and g_n in the obvious way so that

$$G(f_n) = \Sigma_{i\geq 1}(C_{n,2i} \times R_{n,2i}) \cap S$$

and

$$G(g_n) = \Sigma_{i\geq 1}(C_{n,2i-1} \times R_{n,2i-1}) \cap S.$$

From the properties of $\{C_{n,i} \times R_{n,i}\}$ mentioned above f_n and g_n are well defined and $S_n = G(f_n) + G(g_n)$. Thus $S = \Sigma_{n\geq 0}G(f_n) + \Sigma_{n\geq 0}G(g_n) = G(f) + G(g)$.

Now suppose $x \in E_n$ for some $n \geq 1$. Then $x \in C_{n,2i}$ for some $i \geq 1$ and $f(x) \in R_{n,2i}$. Since $R_{n,2i} \cap R_{n,j} \neq \emptyset$ implies $j = 2i$ or $2i - 1$, $R_{n,2i} \cap F_m = \emptyset$ for $m \neq n$, and since a row of $R_{n,2i}$ contains exactly one element of S, $(g \circ f)(x) \in C_{n,2i-1}$. Continuing this argument we see that $(g \circ f)^i(x) \in C_{n,1}$, a singleton which is not a subset of E_n, and $(g \circ f)^j(x) \neq x$ for any $1 \leq j \leq i$. Now suppose $x \in E_0$. Consider the λ-erasure of $(A_0 \times Y) \cup (X \times B_0)$. If $f(x) \in Y - F_0$ then $(g \circ f)(x)$ is not defined. If $f(x) \in F_0$ then $((g \circ f)(x), f(x))$ must have been λ-erased after $(x, f(x))$. Continuing this argument we see that if $(g \circ f)^i(x) = x$ for some $i \geq 1$ then i must be even and

$$\left((g\circ f)^i(x), f\circ(g\circ f)^{i-1}(x)\right) \in G(g)$$

and was λ-erased after $(x, f(x)) = ((g \circ f)^i(x), f(x))$. But this is impossible since the column x of $X \times Y$ was deleted when $(x, f(x))$ was λ-erased. This proves that (f, g) is an aperiodic pair. □

The next theorem is a strengthened version of the Douglas-Lindenstrauss theorem for the discrete case.

THEOREM 2.7. *Suppose $\lambda \in M(\mu, \nu)$ with support S. Then λ is extreme if and only if for every real function h on $X \times Y$ there exist real functions p on X and q on Y such that $h = p + q$* a.e.$[\lambda]$.

Proof. Suppose λ is not extreme. By theorem 2.1 there exists a λ-loop

$$\{(x_i, y_i): 1 \leq i \leq 2n\}.$$

Let $h = \Sigma_{i=1}^n I_{\{(x_{2i}, y_{2i})\}}$. From the definition of λ-loops if p and q are any two real functions on X and Y, respectively, then

$$\sum_{i=1}^{n} [p(x_{2i-1}) + q(x_{2i-1})] = \sum_{i=1}^{n} [p(x_{2i}) + q(y_{2i})].$$

But $\Sigma_{i=1}^{n} h(x_{2i-1}, y_{2i-1}) = 0$ and $\Sigma_{i=1}^{n} h(x_{2i}, y_{2i}) = n$. Thus $h \neq p + q$ a.e.$[\lambda]$ for any pair (p, q).

Now suppose λ is extreme and h is a real function on $X \times Y$. We use an "orthogonal decomposition" of S similar to (but not necessarily the same as) the decomposition in Theorem 2.5. Well order X in some arbitrary but fixed manner. Let $C_{1,1} = \{x_1^1\}$, where x_1^1 is the smallest x for which $(x, y) \in S$ for some $y \in Y$, let $R_{1,1} = \{y \in Y: (x_1^1, y) \in S\}$, and for $i \geq 1$ define recursively

$$C_{n,2i} = \{x \in X - C_{n,2i-1}: (x, y) \in S \text{ for some } y \in R_{n,2i-1}\},$$
$$R_{n,2i} = \{y \in R_{n,2i-1}: (x, y) \in S \text{ for some } x \in C_{n,2i}\},$$
$$R_{n,2i+1} = \{y \in Y - R_{n,2i}: (x, y) \in S \text{ for some } x \in C_{n,2i}\},$$

and

$$C_{n,2i+1} = \{x \in C_{n,2i}: (x, y) \in S \text{ for some } y \in R_{n,2i+1}\}.$$

From the construction $C_{1,i} \times R_{1,i}$ is λ-full for $1 \leq i < N_1$ and empty for $i \geq N_1$ for some $2 \leq N_1 \leq \infty$, and the lemma applies to $\{C_{1,i} \times R_{1,i}: 1 \leq i \leq N_1\}$. Since λ is extreme, by Theorem 2.1 and the lemma, $\{C_{1,2i}\}$ and $\{R_{1,2i-1}\}$ are pairwise disjoint collections of sets and every row(column) of $C_{1,i} \times R_{1,i}$ contains exactly one element of S for i odd(even). Let

$$A_1 = C_{1,1} + \Sigma_{i \geq 1} C_{1,2i}, \quad B_1 = \Sigma_{i \geq 1} R_{1,2i-1} \quad \text{and} \quad S_1 = (A_1 \times B_1) \cap S.$$

Then $S_1 = \Sigma_{i \geq 1}(C_{1,i} \times R_{1,i}) \cap S$ and $S - S_1 = [(X - A_1) \times (Y - B_1)] \cap S$. Now for $n \geq 2$ successively construct S_n, A_n, and B_n from $(X - A_1) \times (Y - B_1)$ in the same manner as above, each time using $C_{n,1} = \{x_1^n\}$, where x_1^n is the smallest x in $X - \Sigma_{i=1}^{n} A_i$ for which $(x, y) \in S$ for some $y \in Y - \Sigma_{i=1}^{n-1} B_i$, so that

$$S_n = (A_n \times B_n) \cap S = \sum_{i \geq 1} (C_{n,i} \times R_{n,i}) \cap S,$$

$\{C_{n,i} \times R_{n,i}\}$ has the properties of $\{C_{1,i} \times R_{1,i}\}$ described above, $S = \Sigma_{n \geq 1} S_n$, and $\{A_n\}$ and $\{B_n\}$ are pairwise disjoint collections of sets. Then it is sufficient to define p and q by their restrictions p_n and q_n to A_n and B_n, respectively, $n \geq 1$, and defining p and q to be zero otherwise.

Fix $n \geq 1$. Let $p_n(x_1^n) = 0$ and $q_n(y) = h(x_1^n, y)$ for $y \in R_{n,1}$. Now for $x \in C_{n,2i}$ let $p_n(x) = h(x, y) - q_n(y)$, where (x, y) is the unique element of S in $C_{n,2i} \times R_{n,2i}$, and for $y \in R_{n,2i+1}$ let $q_n(y) = h(x, y) - p_n(x)$, where (x, y) is the unique element of S in $C_{n,2i+1} \times R_{n,2i+1}$. Now it is easy to check that $p_n + q_n = hI_{A_n \times B_n}$ a.e.$[\lambda]$. □

COROLLARY 2.7 (Douglas-Lindenstrauss). *$\lambda \in M(\mu, \nu)$ is extreme if and only if $W = \{p + q: p \in L_1(\mu), q \in L_1(\nu)\}$ is norm-dense in $L_1(\lambda)$.*

Proof. Suppose λ is not extreme and $h \in L_1(\lambda)$. If there exist $p_n \in L_1(\mu)$ and $q_n \in L_1(\nu)$ such that $p_n + q_n \to h$ in $\|\cdot\|_1$, the norm in $L_1(\lambda)$, then a subsequence $p_{n_k} + q_{n_k} \to h$ a.e. $[\lambda]$. But this is impossible for the h defined in the first part of the proof of Theorem 2.7. Hence W is not norm-dense in $L_1(\lambda)$.

Now suppose λ is extreme and $h \in L_1(\lambda)$. Let $A_k \times B_k$ be finite rectangles such that $A_k \times B_k \uparrow X \times Y$. As in Theorem 2.7 we can construct real functions $p_k = p_k I_{A_k}$ and $q_k = q_k I_{B_k}$ on X and Y, respectively, such that

$$p_k + q_k = hI_{A_k \times B_k} \quad \text{a.e. } [\lambda].$$

Then clearly $p_k \in L_1(\mu)$ and $q_k \in L_1(\nu)$, $k \geq 1$, and $\|hI_{A_k \times B_k} - h\|_1 \to 0$. Thus W is norm-dense in $L_1(\lambda)$. ■

The problem of going from Theorem 2.7 to Corollary 2.7 is the fact that $h \in L_1(\lambda)$ and $h = p + q$ a.e. $[\lambda]$ do not guarantee that $p \in L_1(\mu)$ and $q \in L_1(\nu)$ as the following example shows.

Example 2. Let X and Y be the positive integers. Suppose

$$\lambda(i-1,i) = 1/i^2, \qquad \lambda(i,i-1) = 1/i^3, \quad i \geq 2,$$

and $\lambda(i,j) = 0$ otherwise. Then λ is extreme by Theorem 2.3. Define h on $X \times Y$ by $h(i-1,i) = 1$, $h(i,i-1) = i$, $i \geq 2$, and $h(i,j) = 0$ otherwise. Then $h \in L_1(\lambda)$. However, $\mu(1) = 1/2^2$, $\nu(1) = 1/2^3$, and for $i \geq 2$ we have $\mu(i) = 1/(i+1)^2 + 1/i^3$ and $\nu(i) = 1/i^2 + 1/(i+1)^3$. Thus for no choice of real functions p and q on X and Y, respectively, such that $h = p + q$ a.e. $[\lambda]$ will we have both $p \in L_1(\mu)$ and $q \in L_1(\nu)$. □

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