ILLINOIS JOURNAL OF MATHEMATICS
Volume 29, Number 2, Summer 1985

SEPARABLY RELATED SETS AND THE RADON-NIKODÝM PROPERTY

BY

SIMON FITZPATRICK

1. Introduction

There is a duality between two interesting classes of Banach spaces: the Asplund spaces and the dual spaces with the Radon-Nikodým property. The Radon-Nikodým property can be defined for subsets of a Banach space and 0. Reĭnov [17] and C. Stegall [23] have defined a class of subsets called the sets which are measurable in themselves (Reĭnov) or the GSP sets (Stegall) which have some properties in common with Asplund spaces. This paper is concerned with such properties and with a generalization of the duality between these two classes of sets.

Uhl [24] proved that if $\mathscr{X}$ is a Banach space such that every separable subspace of $\mathscr{X}$ has separable dual then $\mathscr{X}^*$ has the Radon-Nikodým property. The converse was obtained by Stegall [21] using a delicate construction.

We will assume that our Banach spaces are taken over real scalars. A Banach space $\mathscr{X}$ is an Asplund space [15] provided every continuous convex function f on an open convex subset C of $\mathscr{X}$ is Fréchet differentiable on a residual set of points in C. Namioka and Phelps [15] showed that if $\mathscr{X}$ is an Asplund space then every separable subspace of $\mathscr{X}$ has separable dual. Again the converse fell to a construction by Stegall [22]. To summarize, we have the following theorem.

THEOREM 1.1. *Each of the following statements about a Banach space $\mathscr{X}$ implies the other two.*

(1) *$\mathscr{X}$ is an Asplund space.*
(2) *Every separable subspace of $\mathscr{X}$ has separable dual.*
(3) *$\mathscr{X}^*$ has the Radon-Nikodým property.*

We turn statement (2) into a definition linking sets in a dual space with sets in the predual.

DEFINITION 1.2. A subset K of a dual Banach space $\mathscr{X}^*$ is *separably related* to a subset A of $\mathscr{X}$ provided for every separable, bounded subset S of A the set K is separable for the topology of uniform convergence on S.

Received January 27, 1983.

Usually we need to assume that K is weak* closed and often that it is convex. For weak* compact *nonconvex* sets the Radon-Nikodým property is badly behaved, as examples in Section 4 will show and we use "RN sets" [18] instead.

In Section 2 we show that a weak* compact subset K of $\mathscr{X}^*$ is an RN set if and only if K is separable related to $\mathscr{X}$ and that a bounded subset A of $\mathscr{X}$ is measurable in itself if and only if $\mathscr{X}^*$ is separably related to A. Then we characterize sets which are separably related in terms of the images of the sets under certain operators being RN sets or measurable in themselves.

To start Section 3 we derive a geometric characterization of the Radon-Nidodým property for weak* closed convex sets and use this to get a similar result for pairs of separably related sets. We use that to obtain some results about differentiability of convex functions using a construction due to Kenderov [12]. As a corollary we see that a bounded set A is measurable in itself if and only if every continuous convex function f on $\mathscr{X}$ has a residual set of points of A-differentiability in the sense of Asplund and Rockafellar [1].

We thank Isaac Namioka and Robert Phelps for helpful and interesting discussions about this subject.

2. RN sets and sets which are measurable in themselves

DEFINITION 2.1. Let K be a nonempty subset of a Banach space $\mathscr{X}$. We say that K has the *Radon-Nikodým property* if for any finite positive measure space (Ω, Σ, μ) and any vector measure $m\colon \Sigma \to \mathscr{X}$ that is μ-continuous, countably additive, of finite variation, with *average range*

$$AR(m) = \left\{\mu(E)^{-1}m(E)\colon E \in \Sigma,\ \mu(E) > 0\right\}$$

contained in K, m is representable by a Bochner μ-integrable function.

If K is a weak* compact subset of $\mathscr{X}^*$ then K is an *RN set* provided every Radon probability measure on the weak* Borel subsets of K is supported almost everywhere on a countable union of strongly compact subsets of K.

The following result can be found in [23] or [18].

PROPOSITION 2.2. *The weak* closed, absolutely convex hull* $\overline{\Gamma}^*(K)$ *of a weak* compact RN set is an RN set and has the Radon-Nikodým property. A weak* compact convex set C has the Radon-Nikodým property if and only if C is an RN set.*

However, Example 4.1 will show that a weak* compact nonconvex set with the Radon-Nikodým property need not be an RN set.

A class of sets which have properties similar to those of Asplund spaces was introduced by Reĭnov [17]. In [18] he restated the definition a little differently.

DEFINITION 2.3. A bounded subset A of a Banach space $\mathscr{X}$ is *measurable in itself* provided for each finite Radon measure μ on the weak* Borel subsets of the closed unit ball $B(\mathscr{X}^*)$ of $\mathscr{X}^*$ the set of functions $\{f_x: x \in A\}$ is equimeasurable [8], where $f_x(x^*) = x^*(x)$ for each $x^* \in B(\mathscr{X}^*)$. Equimeasurability means that there are weak* Borel sets E of arbitrarily small μ-measure such that the set of functions is relatively compact in $L^\infty(\mu|_{B(\mathscr{X}^*)\setminus E})$. Thus relatively compact sets in $\mathscr{X}$ are measurable in themselves; actually, weakly compact sets are too [23].

Stegall [23] defined this class of sets slightly differently and called them the GSP sets: see the discussion in Reĭnov [18]. We will use Reĭnov's terminology, but we will use the following characterization rather than work with the definition.

THEOREM 2.4. (Reĭnov [18], Stegall [23]). *A bounded subset A of a Banach space $\mathscr{X}$ is measurable in itself if and only if there is an Asplund space $\mathscr{Z}$ and an operator T: $\mathscr{Z} \to \mathscr{X}$ such that $A \subseteq TB(\mathscr{Z})$ where $B(\mathscr{Z})$ denotes the closed unit ball of $\mathscr{Z}$.*

This result has a dual concerning RN sets.

THEOREM 2.5. (Reĭnov [18], Stegall [23]). *A weak* compact subset K of $\mathscr{X}^*$ is an RN set if and only if there is an Asplund space $\mathscr{Z}$ and an operator T: $\mathscr{X} \to \mathscr{Z}$ such that $K \subseteq T^*B(\mathscr{Z}^*)$ and the range of T is dense in $\mathscr{Z}$.*

The duality between RN sets and sets which are measurable in themselves is seen in the following theorem of Stegall [23].

THEOREM 2.6. *Each of the following statements about an operator T: $\mathscr{X} \to \mathscr{Y}$ between Banach spaces implies both the others.*

(1) *The set $TB(\mathscr{X})$ is measurable in itself.*

(2) *The set $T^*B(\mathscr{Y}^*)$ has the Radon-Nikodým property.*

(3) *There exist an Asplund space $\mathscr{Z}$ and operators T_1: $\mathscr{X} \to \mathscr{Z}$ and T_2: $\mathscr{Z} \to \mathscr{Y}$ such that $T = T_2 \circ T_1$.*

We introduce the separability conditions by the following result.

PROPOSITION 2.7 (Stegall [23]). *A weak* compact convex subset K of $\mathscr{X}^*$ has the Radon-Nikodým property if and only if for every separable Banach space $\mathscr{Y}$ and every operator T: $\mathscr{Y} \to \mathscr{X}$ the set $T^*(K)$ is norm separable in $\mathscr{Y}^*$.*

For nonconvex sets we need to apply a result of R. Bourgin [3] and Haydon [9].

THEOREM 2.8. *Let C be a weak* compact convex subset of* $\mathscr{X}^*$ *such that the extreme points of K form a norm separable set. Then K is the norm closed convex hull of its extreme points and hence is itself norm separable.*

COROLLARY 2.9. *A weak* compact subset K of* $\mathscr{X}^*$ *is an RN set if and only if for every separable Banach space* $\mathscr{Y}$ *and every operator* $T\colon \mathscr{Y} \to \mathscr{X}$ *the set* $T^*(K)$ *is norm separable.*

Proof. Let $C = \text{weak}^* \overline{\text{conv}}(K)$. If K is an RN set then Proposition 2.2 says that C has the Radon-Nikodým property. Applying Proposition 2.7 to C shows that for every separable $\mathscr{Y}$ and $T\colon \mathscr{Y} \to \mathscr{X}$ the set $T^*(C)$ is separable and hence $T^*(K)$ is separable.

Conversely if $T^*(K)$ is separable for any such operator T then $T^*(K)$ is weak* compact and Milman's theorem shows that the extreme points of weak* $\overline{\text{conv}}\, T^*(K) = T^*(C)$ are contained in $T^*(K)$ and thus form a norm separable set. By Theorem 2.8, the set $T^*(C)$ is norm separable. Finally Proposition 2.7 shows that C has the Radon-Nikodým property so that K is an RN set.

DEFINITION 2.10. Let A be a bounded subset of a Banach space $\mathscr{X}$. Define a seminorm q_A on $\mathscr{X}^*$ by $q_A(x^*) = \sup\{|x^*(x)|: x \in A\}$. This seminorm determines the (possibly non-Hausdorff) topology of uniform convergence on A. Thus a subset K of $\mathscr{X}^*$ is separably related to a subset H of $\mathscr{X}$ if and only if for each bounded separable subset A of H the set K is separable in the seminormed space $(\mathscr{X}^*, q_A)$: we write this as "(K, q_A) is separable".

The following simple result is useful.

PROPOSITION 2.11. *Suppose* $T\colon \mathscr{X} \to \mathscr{Y}$ *is an operator between Banach spaces and let A be a bounded subset of* $\mathscr{X}$. *The mapping*

$$T^*\colon \left(\mathscr{Y}^*, q_{T(A)}\right) \to \left(T^*\mathscr{Y}^*, q_A\right)$$

is an isometry between seminormed spaces.

Proof. If $y^* \in \mathscr{Y}^*$ then

$$\begin{aligned} q_{T(A)}(y^*) &= \sup\{|y^*(y)| : y \in T(A)\} \\ &= \sup\{|y^*(Tx)| : x \in A\} \\ &= \sup\{|T^*y^*(x)| : x \in A\} \\ &= q_A(T^*y^*) \end{aligned}$$

as required.

COROLLARY 2.12. *Let H be a bounded subset of $\mathscr{X}$ and K be a subset of $\mathscr{Y}^*$. If T: $\mathscr{X} \to \mathscr{Y}$ is an operator then $T^*(K)$ is separably related to H if and only if K is separably related to $T(H)$.*

Proof. Let A be a separable subset of H, so $T(A)$ is separable and if K is separably related to $T(H)$ then $(K, q_{T(A)})$ is separable. By Proposition 2.11, $(T^*(K), q_A)$ is separable.

Conversely, let $T^*(K)$ be separably related to H and let S be a separable subset of $T(H)$. For any countable dense subset L of S there is a countable subset A of H such that $L = T(A)$. Now $(T^*(K), q_A)$ is separable so $(K, q_{T(A)}) = (K, q_L)$ is separable by Proposition 2.11. However $q_L = q_S$ since L is dense in S so K is separably related to $T(H)$.

We now relate this to the Radon-Nikodým property.

THEOREM 2.13. *A weak* compact set K in $\mathscr{X}^*$ is an RN set if and only if K is separably related to $\mathscr{X}$. A weak* closed convex set C in $\mathscr{X}^*$ has the Radon-Nikodým property if and only if C is separably related to $\mathscr{X}$.*

Proof. From Proposition 2.2 and the first statement, the second statement follows easily. So let K be a weak* compact subset of $\mathscr{X}^*$ with A any bounded separable subset of $\mathscr{X}$. Take $\mathscr{Y} = \overline{\text{span}}(A)$ and T: $\mathscr{Y} \to \mathscr{X}$ to be the natural injection. By Corollary 2.9, the set $T^*(K) \subseteq \mathscr{Y}^*$ is separable, that is $(T^*(K), q_{B(\mathscr{Y})})$ is separable.

Since A is contained in a finite scalar multiple of the unit ball $B(\mathscr{Y})$ it is clear that $(T^*(K), q_A)$ is separable so by Proposition 2.11, $(K, q_{T(A)}) = (K, q_A)$ is separable.

Conversely, if K is separably related to $\mathscr{X}$ and T: $\mathscr{Y} \to \mathscr{X}$ is an operator from a separable Banach space $\mathscr{Y}$ then $S = TB(\mathscr{Y})$ is a separable bounded subset of $\mathscr{X}$ and (K, q_S) is thus separable. Therefore $(T^*(K), q_{B(\mathscr{Y})})$ is separable, that is, $T^*(K)$ is norm separable and K is an RN set by Corollary 2.9.

Next we characterize separably related sets in terms of RN sets. Firstly we note that closed absolutely convex hulls don't interfere.

LEMMA 2.14. *If K is a subset of $\mathscr{X}^*$ which is separably related to a bounded subset H of $\mathscr{X}$ then K is separably related to $\overline{\Gamma}(H)$.*

Proof. If A is a separable subset of $\overline{\Gamma}(H)$ then $L = H \cap \overline{\text{span}}(A)$ is separable and $A \subseteq \overline{\Gamma}(L)$. Therefore $q_A \leq q_L$ and since (K, q_L) is separable so is (K, q_A).

THEOREM 2.15. *Let A be a bounded subset of $\mathscr{X}$. There exist a Banach space $\mathscr{Y}$ and an operator T: $\mathscr{Y} \to \mathscr{X}$ such that $A \subseteq TB(\mathscr{Y}) \subseteq \overline{\Gamma}(A)$. A weak* compact* *subset K of $\mathscr{X}^*$ is separably related to A if and only if there exist a Banach space $\mathscr{Y}$ and an operator T: $\mathscr{Y} \to \mathscr{X}$ such that $T^*(K)$ is an RN set and $A \subseteq TB(\mathscr{Y})$.*

Proof. Let $\mathscr{Y} = l_1(A)$ and T: $\mathscr{Y} \to \mathscr{X}$ be defined by

$$T\alpha = \sum_{x \in A} \alpha(x) \cdot x, \quad (\alpha \in l_1(A)).$$

Then T is an operator with $A \subseteq TB(\mathscr{Y}) \subset \overline{\Gamma}(A)$. If K is separably related to A then K is separably related to $TB(\mathscr{Y})$ by Lemma 2.14. By Corollary 2.12, $T^*(K)$ is separably related to $B(\mathscr{Y})$ which implies that $T^*(K)$ is an RN set by Theorem 2.13.

Conversely suppose that T: $\mathscr{Y} \to \mathscr{X}$ is an operator such that $A \subseteq TB(\mathscr{Y})$ and $T^*(K)$ is an RN set. Then $T^*(K)$ is separably related to $B(\mathscr{Y})$ by Theorem 2.13 and K is separably related to $TB(\mathscr{Y})$ by Corollary 2.12. Thus K is separably related to A.

For sets measurable in themselves we get the following result.

THEOREM 2.16. *A bounded subset A of $\mathscr{X}$ is measurable in itself if and only if $\mathscr{X}^*$ is separably related to A.*

Proof. Let $\mathscr{Y}$ be a Banach space and T: $\mathscr{Y} \to \mathscr{X}$ an operator such that

$$A \subseteq TB(\mathscr{Y}) \subseteq \overline{\Gamma}(A);$$

Theorem 2.15 exhibited one. If $\mathscr{X}^*$ is separably related to A then by Lemma 2.14, $\mathscr{X}^*$ is separably related to $TB(\mathscr{Y})$ and it follows that $T^*B(\mathscr{X}^*)$ is separably related to $B(\mathscr{Y})$, or equivalently $\mathscr{Y}$, by Corollary 2.12. Therefore $T^*B(\mathscr{X}^*)$ has the Radon-Nikodým property and Theorem 2.6 shows that $TB(\mathscr{Y})$, and hence A, is measurable in itself.

Conversely, if A is measurable in itself there is an Asplund space $\mathscr{Z}$ and an operator T: $\mathscr{Z} \to \mathscr{X}$ such that $A \subseteq TB(\mathscr{Z})$ by Theorem 2.4. Since $\mathscr{Z}^*$ has the Radon-Nikodým property, $\mathscr{Z}^*$ is separably related to $B(\mathscr{Z})$ and hence $T^*(\mathscr{X}^*)$ is separably related to $B(\mathscr{Z})$. Therefore $\mathscr{X}^*$ is separably related to $TB(\mathscr{Z})$ and thus to A.

LEMMA 2.17. *If K is a weak* compact subset of $\mathscr{X}^*$ which is separably related to a bounded set A then $\overline{\Gamma}^*(K)$ is separably related to $\overline{\Gamma}(A)$.*

Proof. By Lemma 2.14 we need only show that $\overline{\Gamma}^*(K)$ is separably related to A. Let T: $\mathscr{Y} \to \mathscr{X}$ be given by Theorem 2.15 so that $A \subseteq TB(\mathscr{Y})$ and $T^*(K)$ is an RN set. By Proposition 2.2, $\overline{\Gamma}^*(T^*(K))$ is an RN set. Since $\overline{\Gamma}^*(T^*(K)) = T^*(\overline{\Gamma}^*(K))$ the latter is an RN set and Theorem 2.15 shows that $\overline{\Gamma}^*(K)$ is separably related to A.

To end this section we characterize separably related sets in terms of sets which are measurable in themselves.

THEOREM 2.18. *A weak* compact subset K of $\mathscr{X}^*$ is separably related to a bounded subset A of $\mathscr{X}$ if and only if there exist a Banach space $\mathscr{Z}$ and an operator S: $\mathscr{X} \to \mathscr{Z}$ such that $S(A)$ is measurable in itself and $K \subseteq S^*B(\mathscr{Z}^*)$.*

Proof. Let $\mathscr{Z}$ be the space of weak* continuous functions on K, equipped with the supremum norm and let S: $\mathscr{X} \to \mathscr{Z}$ be evaluation, that is, $(Sx)(x^*) = x^*(x)$ for all $x^* \in K$. Then for each $x^* \in K$, $T^*(\delta_{x^*}) = x^*$ where $\delta_{x^*} \in \mathscr{Z}^*$ is the functional given by $\delta_{x^*}(f) = f(x^*)$ for $f \in \mathscr{Z}$. The unit ball of $\mathscr{Z}^*$ is the weak* closed absolutely convex hull of $\{\delta_{x^*}\colon\ x^* \in K\}$ so we have $K \subseteq S^*B(\mathscr{Z}^*) = \overline{\Gamma}^*(K)$.

If K is separably related to A then $\overline{\Gamma}^*(K)$ is separably related to A by Lemma 2.17 so $S^*B(\mathscr{Z}^*)$ is separably related to A. Thus $B(\mathscr{Z}^*)$ is separably related to $S(A)$ and Theorem 2.16 shows that $S(A)$ is measurable in itself.

Conversely, if S: $\mathscr{X} \to \mathscr{Z}$ is an operator from $\mathscr{X}$ to a Banach space $\mathscr{Z}$ such that $S^*B(\mathscr{Z}^*)$ contains K and $S(A)$ is measurable in itself then $B(\mathscr{Z}^*)$ is separably related to $S(A)$, so that $S^*B(\mathscr{Z}^*)$ is separably related to A and the subset K must be separably related to A.

3. Geometric properties and differentiability of convex functions

In this section we will consider geometric properties of separably related sets and obtain results about differentiability properties of convex functions whose subdifferentials are constrained to lie in certain weak* closed convex sets. First we need some results for the Radon-Nikodým property.

DEFINITION 3.1. If K is a bounded subset of a Banach space $\mathscr{X}$ a *slice* of K is any set of the form

$$S(x^*, \alpha, K) = \{x \in K\colon x^*(x) > M(x^*, K) - \alpha\}$$

where $\alpha > 0$, $x^* \in \mathscr{X}^*$ and $M(x^*, K) = \sup\{x^*(y)\colon\ y \in K\}$. If $\mathscr{X} = \mathscr{Y}^*$ is a dual space then a *weak* slice* of $K \subset \mathscr{Y}^*$ is a set of the form $S(y, \alpha, K)$ where $\alpha > 0$ and $y \in \mathscr{Y} \subseteq \mathscr{X}^*$.

A bounded set $K \subset \mathscr{X}$ is *dentable* provided K has slices of arbitrarily small diameter. We say that $K \subset X$ is *strongly exposed* by a functional $x^* \in \mathscr{X}^*$ provided there is $x \in K$ (a *strongly exposed point*) such that $x^*(x) = M(x^*, K)$ and if $x_n \in K$ with $x^*(x_n) \to x^*(x)$ then $\|x_n - x\| \to 0$.

The following theorem is due to Rieffel [19], Maynard [13], Davis and Phelps [4], Huff [10], Phelps [16] and Bourgain [2]—see the book of Diestel and Uhl [6] for an entertaining discussion of these and other equivalent properties.

THEOREM 3.2. *Each of the following statements about a closed convex subset K of a Banach space $\mathcal{X}$ implies all the others.*

(1) *K has the Radon-Nikodým property.*

(2) *Every bounded subset of K is dentable.*

(3) *Every closed bounded convex subset of K is strongly exposed by some $x^* \in \mathcal{X}^*$.*

(4) *Every closed bounded convex subset of K is the closed convex hull of its strongly exposed points.*

For dual spaces one can ask whether the slices can be taken to be weak* slices.

THEOREM 3.3 (Namioka and Phelps [15]). *A Banach space $\mathcal{Z}$ is an Asplund space if and only if every bounded subset of $\mathcal{Z}^*$ has weak* slices of arbitrarily small diameter.*

We use this to get a result for weak* closed convex sets having the Radon-Nikodým property.

PROPOSITION 3.4. *A weak* closed convex subset C of a dual Banach space $\mathcal{X}^*$ has the Radon-Nikodým property if and only if every bounded subset K of C has weak* slices of arbitrarily small diameter.*

Proof. If C is weak* closed and convex with the Radon-Nikodým property then the weak* closure of any bounded subset K of C is a weak* compact RN set L. By Theorem 2.5 there exist an Asplund space $\mathcal{Z}$ and an operator T: $\mathcal{X} \to \mathcal{Z}$ with $T(\mathcal{X})$ dense in $\mathcal{Z}$ such that $L \subseteq T^*B(\mathcal{Z}^*)$. Let V be a subset of $B(\mathcal{Z}^*)$ such that $K = T^*(V)$. For each $\varepsilon > 0$, Theorem 3.3 provides a weak* slice $S(z, \alpha, V)$ of V with diameter less than ε. If $y \in Z$ and $\|y - z\| < \alpha/4$ then $S(y, \alpha/2, V) \subseteq S(z, \alpha, V)$ since $M(y, V) > M(z, V) - \alpha/4$.

However the range of T is dense in $\mathcal{Z}$ so there is $x \in \mathcal{X}$ such that $\|Tx - z\| < \alpha/4$ and hence the diameter of the slice $S = S(Tx, \alpha/2, V)$ is less than ε. Now,

$$\begin{aligned} T^*(S) &= T^*\{x^* \in V \colon x^*(Tx) > M(Tx, V) - \alpha/2\} \\ &= \{T^*x^* \in T^*V \colon (T^*x^*)(x) > M(x, T^*(V)) - \alpha/2\} \\ &= S(x, \alpha/2, T^*(V)) \\ &= S(x, \alpha/2, K). \end{aligned}$$

Since S has diameter less than ε, the weak* slice $S(x, \alpha/2, K) = T^*(S)$ has diameter less than $\varepsilon \cdot \|T\|$. Thus K has weak* slices of arbitrarily small diameter.

The converse is clear from Theorem 3.2 since any weak* slice is a slice.

Next we convert this result into a statement about separably related sets.

THEOREM 3.5. *A weak* closed convex subset C of $\mathscr{X}^*$ is separably related to a bounded subset A of $\mathscr{X}$ if and only if every bounded subset of C has weak* slices of arbitrarily small q_A-diameter.*

Proof. We may assume without loss of generality that C is weak* compact and convex. Suppose C is separably related to A. By Theorem 2.15 there is a Banach space $\mathscr{Y}$ and an operator $T\colon \mathscr{Y} \to \mathscr{X}$ such that $A \subseteq TB(\mathscr{Y})$ and $T^*(C)$ has the Radon-Nikodým property. Let K be a subset of C and $\varepsilon > 0$. Then $T^*(K)$ has a weak* slice $S(y, \alpha, T^*(K))$ of diameter less than ε. Since

$$S(y,\alpha,T^*(K)) = T^*(S(Ty,\alpha,K))$$

it follows that $S(Ty, \alpha, K)$ has $q_{TB(\mathscr{Y})}$-diameter less than ε using Proposition 2.11. Thus K has a slice $S(Ty, \alpha, K)$ of q_A-diameter less than ε.

Conversely, suppose that every bounded subset of C has weak* slices of arbitrarily small q_A-diameter and let $\mathscr{Y}$ and $T\colon \mathscr{Y} \to \mathscr{X}$ be given by Theorem 2.15 so that $A \subseteq TB(\mathscr{Y}) \subseteq \overline{\Gamma}(A)$. If V is a bounded subset of $T^*(C)$ then let $W =$ weak* $\overline{\operatorname{conv}}(V)$ and using Zorn's Lemma choose a *minimal* (under inclusion) weak* compact convex subset K of C such that $T^*(K) = W$. Let $\varepsilon > 0$ and let $S(x, \alpha, K)$ be a weak* slice of K with q_A-diameter less than ε. Then $D = K \setminus S(x, \alpha, K)$ is weak* compact and convex so that $T^*(D)$ is a weak* compact convex subset of W. By minimality of K we have $T^*(D) \neq W$ and there exists $y^* \in W \setminus T^*(D)$. Using the separation theorem we find $y \in \mathscr{Y}$ such that $y^*(y) > M(y, T^*(D))$, so y determines a slice $S(y, \beta, W)$ which does not intersect $T^*(D)$. Therefore

$$S(y,\beta,W) \subseteq T^*(S(x,\alpha,K))$$

and, since $S(x, \alpha, K)$ has q_A-diameter less than ε and T^* is an isometry from $(\mathscr{X}^*, q_A)$ into $(\mathscr{Y}^*, \text{norm})$ because $q_{TB(\mathscr{Y})} = q_A$, the weak* slice

$$S(y,\beta,V) \subseteq S(y,\beta,W) \subseteq T^*(S(x,\alpha,K))$$

has diameter less than ε. This shows that $T^*(C)$ has the Radon-Nikodým property and by Theorem 2.15 the set C is separably related to A.

COROLLARY 3.6. *A bounded subset A of $\mathscr{X}$ is measurable in itself if and only if every bounded subset of $\mathscr{X}^*$ has weak* slices of arbitrarily small q_A-diameter.*

Asplund and Rockafellar [1] introduced and studied the following generalization of Fréchet differentiability (which is the special case $A = B(\mathscr{X})$).

DEFINITION 3.7. Let A be a bounded subset of a Banach space $\mathscr{X}$. A real valued function f defined on an open subset C of $\mathscr{X}$ is said to be *A-differentiable* at the point $x \in C$ if there is $x^* \in \mathscr{X}^*$ (called an *A-gradient* of f at x) such that

$$\lim_{t \to 0+} \sup_{y \in A} \left| t^{-1}(f(x+ty) - f(x)) - x^*(y) \right| = 0.$$

Note that the set of A-gradients of f at x always has q_A-diameter 1.

To state the next theorem we need to define the subdifferential [14].

DEFINITION 3.8. Let f be a continuous convex function on the open convex subset C of $\mathscr{X}$. The *subdifferential* ∂f is defined for $x \in C$ by

$$\partial f(x) = \left\{ x^* \in \mathscr{X}^* \colon f(y) - f(x) \geq x^*(y - x) \text{ for all } y \in C \right\}.$$

Recall that a set is *residual* if its complement is of the first Baire category.

We will prove the following result as a corollary of some more general results about continuity of monotone operators using a construction due to Kenderov [12].

THEOREM 3.9. *Let A be a bounded subset of the Banach space $\mathscr{X}$ and let K be a weak* closed absolutely convex subset of $\mathscr{X}^*$. Then K is separably related to A if and only if, for every continuous convex function f on an open convex subset C of $\mathscr{X}$ such that $\partial f(x) \subseteq K$ for all $x \in C$, the set of points where f is A-differentiable is residual in C.*

DEFINITION 3.10. Let $\mathscr{P}(\mathscr{X}^*)$ denote the set of all subsets of the dual Banach space $\mathscr{X}^*$. A mapping $\mathscr{M}\colon \mathscr{X} \to \mathscr{P}(\mathscr{X}^*)$ is a *monotone operator* provided

$$x^*(x - y) \geq y^*(x - y)$$

whenever $x, y \in \mathscr{X}$ and $x^* \in \mathscr{M}(x)$ and $y^* \in \mathscr{M}(y)$. It is called *maximal monotone* if its graph

$$\{(x, x^*) \colon x \in \mathscr{X},\ x^* \in \mathscr{M}(x)\} \subseteq \mathscr{X} \times \mathscr{X}^*$$

is not properly contained in the graph of any other monotone operator on $\mathscr{X}$. We say that $\mathscr{M}$ is *locally bounded* at a point $x \in \mathscr{X}$ provided there is a neighbourhood U of x such that $\mathscr{M}(U) = \cup\{\mathscr{M}(y)\colon y \in U\}$ is a bounded subset of $\mathscr{X}^*$. The *domain* of $\mathscr{M}$ is the set

$$D(\mathscr{M}) = \{x \in \mathscr{X} \colon \mathscr{M}(x) \text{ is not the empty set}\},$$

and if A is a bounded subset of $\mathscr{X}$ we say that $\mathscr{M}$ is *A-continuous* at a point x of $D(\mathscr{M})$ provided $q_A(x_n^* - x^*) \to 0$ whenever $x_n \to x$, $x_n^* \in \mathscr{M}(x_n)$ and $x^* \in \mathscr{M}(x)$, or equivalently, for each $\varepsilon > 0$ there exists a neighbourhood V of x such that q_A-diam $\mathscr{M}(V) \leq \varepsilon$.

We need some technical results.

LEMMA 3.11 (Rockafellar [20]). *Let $\mathscr{M}$ be a monotone operator with $D(\mathscr{M})$ containing the open set C. Then $\mathscr{M}$ is locally bounded at each point of C.*

LEMMA 3.12 (Kenderov [11]). *If $\mathscr{M}$ is a maximal monotone operator on $\mathscr{X}$ with $D(\mathscr{M})$ containing an open set C then $\mathscr{M}$ is norm-weak* upper semicontinuous at each point of C; that is, for each $x \in C$ and each weak* neighbourhood V of 0 in $\mathscr{X}^*$ there is a (norm) neighbourhood U of x such that*

$$\mathscr{M}(U) \subseteq \mathscr{M}(x) + V.$$

For each $x \in C$ the set $\mathscr{M}(x)$ is weak compact and convex.*

LEMMA 3.13. *Suppose $\mathscr{M}$ is a monotone operator on $\mathscr{X}$ with $D(\mathscr{M})$ containing an open set C and let K be a weak* closed convex subset of $\mathscr{X}^*$ such that $\mathscr{M}(x) \cap K$ is nonempty for all x in a dense subset S of C. Then $\mathscr{M}(x) \subseteq K$ for all $x \in C$.*

Proof. Without loss of generality we may assume that $\mathscr{M}$ is maximal monotone. Let $z \in C$ and choose $x_n \in S$ with $x_n \to z$. If $x_n^* \in K \cap \mathscr{M}(x_n)$ for each n then by local boundedness of $\mathscr{M}$ at z (Lemma 3.11) and weak* closedness of K, there is a weak* convergent subnet of (x_n^*) converging weak* to $z^* \in K$. Now $z^* \in \mathscr{M}(z)$ by Lemma 3.12. Thus $\mathscr{M}(z) \cap K$ is nonempty for each $z \in C$.

Now suppose $x \in C$ and $x^* \in \mathscr{M}(x)$. If $x^* \notin K$ then the separation theorem gives $y \in \mathscr{X}$ such that $x^*(y) > \sup\{v^*(y): v^* \in K\}$. For $t > 0$ small enough, the point $z = x + ty$ is in C so we can find $z^* \in K \cap \mathscr{M}(z)$. Then

$$z^*(z - x) \geq x^*(z - x)$$

by monotonicity so that

$$z^*(y) \geq x^*(y) > \sup\{v^*(y): v^* \in K\} \geqq z^*(y).$$

This contradiction shows that $\mathscr{M}(x) \subseteq K$.

THEOREM 3.14. *Suppose $\mathscr{M}$ is a monotone operator on $\mathscr{X}$ and the weak* closed convex subset K of $\mathscr{X}^*$ is separably related to the bounded subset A of $\mathscr{X}$.*

If C is an open set contained in $D(\mathcal{M})$ such that $\mathcal{M}(x) \cap K$ is nonempty for all x in a dense subset of C then the set of points of A-continuity of $\mathcal{M}$ is residual in C.

Proof. Without loss of generality we may assume that $\mathcal{M}$ is maximal monotone so that by Lemma 3.13, $\mathcal{M}(x) \subseteq K$ for all $x \in C$. Define open subsets V_n of C by

$$V_n = \{y \in C\text{: there is a neighbourhood } U \text{ of } y \text{ such that the } q_A\text{-diameter of } \mathcal{M}(U) \text{ is less than } n^{-1}\},$$

for $n = 1, 2, 3, \dots$ and let $G = \bigcap_n V_n$. Clearly G is the set of points where $\mathcal{M}$ is A-continuous and each V_n is open, so by the Baire category theorem we need only show that each V_n is dense in C.

Suppose that $x \in C$ and $\varepsilon > 0$. By Lemma 3.11 there is an open subset C_1 of C, containing x, such that $\mathcal{M}(C_1)$ is a bounded subset of K. Let W be an open subset of C_1 containing x with diameter less than ε. The bounded subset $\mathcal{M}(W)$ of K has a weak* slice $S = S(z, \alpha, \mathcal{M}(W))$ of q_A-diameter less than n^{-1} by Theorem 3.5. Take any $v^* \in S$ and let $v \in W$ be such that $v^* \in \mathcal{M}(v)$. For sufficiently small $t > 0$ the point $y = v + tz$ is in W and if $y^* \in \mathcal{M}(y)$ then we have $y^*(y - v) \geq v^*(y - v)$ and hence $y^*(z) \geq v^*(z)$. It follows that $\mathcal{M}(y) \subseteq S$. Since there is $\beta \in \mathbf{R}$ such that

$$S = \{x^* \in \mathcal{M}(W)\colon x^*(z) > \beta\},$$

by Lemma 3.12 we can find an open neighbourhood U of y such that $U \subseteq W$ and $\mathcal{M}(U)$ is contained in the weak* open set $\{x^* \in \mathcal{X}^*\colon x^*(z) > \beta\}$. Since $\mathcal{M}(U) \subseteq \mathcal{M}(W)$ we have $\mathcal{M}(U) \subseteq S$ which has q_A-diameter less than n^{-1}. Thus $y \in V_n$ and $\|y - x\| < \varepsilon$ by our restriction on the diameter of W. As required, we have shown that V_n is dense in C.

To use this to obtain results about convex functions we note that if f is a continuous convex function on an open set C then ∂f is monotone. Minty [14] showed that $C \subseteq D(\partial f)$ and that ∂f is maximal monotone if one defines it correctly for points outside of C. For differentiability we need the following result which is implicit in [1].

PROPOSITION 3.15. *Suppose that f is a continuous convex function on an open convex subset C of a Banach space $\mathcal{X}$ and that A is a bounded subset of $\mathcal{X}$.*

(a) *If ∂f is A-continuous at a point $x \in C$ then f is A-differentiable at x.*

(b) *If $A = -A$ and f is A-differentiable at $x \in C$ then each member of $\partial f(x)$ is an A-gradient of f at x and the q_A-diameter of $\partial f(x)$ is zero.*

Proof. (a) Let $x^* \in \partial f(x)$, $y \in A$ and $t > 0$. If $x + ty \in C$ then

$$x^*(ty) \leq f(x + ty) - f(x) \leq y^*(ty) \quad \text{for any } y^* \in \partial f(x + ty),$$

so that

$$0 \le t^{-1}(f(x+ty)-f(x)) - x^*(y) \le y^*(y) - x^*(y) \le q_A(y^* - x^*).$$

Now the A-continuity of ∂f at x implies that

$$\lim_{t\to 0+} \sup\{q_A(y^* - x^*):\ y^* \in \partial f(x+ty),\ y \in A\} = 0;$$

therefore

$$\lim_{t\to 0+} \sup_{y\in A} |t^{-1}(f(x+ty)-f(x)) - x^*(y)| = 0.$$

(b) Let $x^* \in \partial f(x)$ and let y^* be any A-gradient of f at x. For $t > 0$ we have $x^*(y) \le t^{-1}(f(x+ty)-f(x))$ so taking the limit as $t \to 0+$ we see that $x^*(y) \le y^*(y)$ for all $y \in A$. If $A = -A$ then $x^*(y) = y^*(y)$ for all $y \in A$ and hence x^* is also an A-gradient of f at x.

COROLLARY 3.16. *Let f be a continuous convex function on an open convex subset C of a Banach space $\mathscr{X}$. If the weak* closed convex subset K of $\mathscr{X}^*$ is separably related to the bounded subset A of $\mathscr{X}$ and if $\partial f(x) \cap K$ is nonempty for all x in a dense subset of C then f is A-differentiable at a residual set of points in C.*

Proof. Theorem 3.14 shows that the set of points where ∂f is A-continuous is residual in C and Proposition 3.15 shows that f is A-differentiable at each of those points.

For the converse we consider a particular type of convex function.

DEFINITION 3.17. Let W be a weak* compact convex subset of $\mathscr{X}^*$. The *gauge function* for W is the function g_W: $\mathscr{X} \to \mathbf{R}$ where

$$g_W(x) = M(x, W) = \sup\{x^*(x):\ x^* \in W\} \quad \text{for all } x \in \mathscr{X}.$$

This is clearly a continuous convex function defined on all of $\mathscr{X}$. It is easily seen (see Giles' book [7]) that $\partial g_W(x) = \{x^* \in W:\ x^*(x) = g_W(x)\}$. The following proposition is similar to a result in [1].

PROPOSITION 3.18. *Suppose $A \subset \mathscr{X}$ is bounded, $A = -A$ and W is a weak* compact convex subset of $\mathscr{X}^*$. If g_W is A-differentiable at a point $c \in \mathscr{X}$ then x determines weak* slices of W of arbitrarily small q_A-diameter.*

Proof. Let x^* be an A-gradient for g_W at x. By 3.15(b) we may take x^* in $\partial g_W(x)$. If x does not determine weak* slices of W with arbitrarily small

q_A-diameter we can find $\varepsilon > 0$ and points $x_n^* \in W$ with $x_n^*(x) \to M(x, W)$ such that $q_A(x_n^* - x^*) > \varepsilon$. Since $A = -A$ there are $y_n \in A$ such that

$$x_n^*(y_n) - x^*(y_n) > \varepsilon \quad \text{for each } n.$$

Since $x_n^* \in W$ we have $x_n^* \leq g_W$ and hence $x_n^*(x + y) \leq g_W(x + y)$ for all $y \in \mathscr{X}$. Therefore

$$x_n^*(y) \leqq g_W(x + y) - g_W(x) + \beta_n$$

where

$$\beta_n = g_W(x) - x_n^*(x) = M(x, W) - x_n^*(x) \to 0.$$

There exists $\delta > 0$ such that $g_W(x + ty) - g_W(x) - x^*(ty) \leqq \varepsilon t/2$ whenever $0 < t \leq \delta$ and $y \in A$, from the definition of an A-gradient. Thus setting $t = \delta$ we have

$$\begin{aligned} \varepsilon\delta < x_n^*(\delta y_n) - x^*(\delta y_n) &\leq g_W(x + \delta y_n) - g_W(x) + \beta_n - x^*(\delta y_n) \\ &\leq BF\beta_n + \varepsilon\delta/2 \end{aligned}$$

which is impossible since $\beta_n \to 0$.

COROLLARY 3.19. *If A is a bounded subset of $\mathscr{X}$ with $A = -A$ and K is a weak* closed convex subset of $\mathscr{X}^*$ such that every continuous convex function f: $\mathscr{X} \to \mathbf{R}$ with $\partial f(x) \subseteq K$ for all $x \in \mathscr{X}$ has a point of A-differentiability, then K is separably related to A.*

Proof. Let V be a bounded subset of K and let $W = \text{weak}^* \overline{\text{conv}}(V)$. Then $f = g_W$ has $\partial f(x) \subseteq W \subseteq K$ for all $x \in \mathscr{X}$. Thus there is a point $x \in \mathscr{X}$ at which g_W is A-differentiable. By Proposition 3.18, x determines weak* slices of W of arbitrarily small q_A-diameter. Therefore x determines weak* slices of V of arbitrarily small q_A-diameter and K is separably related to A by Theorem 3.5.

COROLLARY 3.20. *If K is a weak* closed convex subset of $\mathscr{X}^*$ then each of the following statements implies both the others.*

(1) *K has the Radon-Nikodým property.*

(2) *K is separably related to $\mathscr{X}$.*

(3) *Every weak* compact convex subset of K is strongly exposed by some point of $\mathscr{X}$.*

Proof. Proposition 3.4 shows that (3) implies (1). If K is separably related to $\mathscr{X}$ then for any weak* compact convex subset W of K the gauge function

g_W is $B(\mathscr{X})$-differentiable at some point $x \in \mathscr{X}$ by Corollary 3.16 and then Corollary 3.18 shows that W is strongly exposed by x.

The following result completes the proof of Theorem 3.9.

PROPOSITION 3.21. *Suppose that A is a bounded subset of $\mathscr{X}$ and K is a weak* closed convex subset of $\mathscr{X}^*$ such that every continuous convex function f on $\mathscr{X}$ with $\partial f(x) \subseteq K \cup -K$ for all $x \in \mathscr{X}$ is A-differentiable at a residual set of points in $\mathscr{X}$. Then K is separably related to A.*

Proof. We note that K is separably related to A if K is separably related to each countable subset of A, so we may assume that $A = \{y_n: n = 1, 2, 3, \dots\}$ is itself countable. The set $H = \overline{\Gamma}\{n^{-1}y_n: n = 1, 2, 3, \dots\}$ is compact since A is bounded so H is measurable in itself.

Let f be any continuous convex function on $\mathscr{X}$ with $\partial f(x) \subseteq K$ for all $x \in \mathscr{X}$ and define a continuous convex function g on $\mathscr{X}$ by $g(x) = f(-x)$ for all $x \in \mathscr{X}$. Note that $\partial g(x) \subseteq -K$ for all $x \in \mathscr{X}$. Then there are residual subsets G_1, G_2 and G_3 of $\mathscr{X}$ such that f and g are A-differentiable at each point of G_1 and G_2 respectively while by Corollary 3.16 f is H-differentiable at each point of G_3 since $\mathscr{X}^*$ and H are separably related. Let G be the residual subset $G_1 \cap (-G_2) \cap G_3$ of $\mathscr{X}$ and let $x \in G$. Thus there are elements x^*, y^* and z^* of $\mathscr{X}^*$ such that

(a) $$\lim_{t \to 0+} \sup_n \left| t^{-1}(f(x + ty_n) - f(x)) - x^*(y_n) \right| = 0,$$

(b) $$\lim_{t \to 0+} \sup_n \left| t^{-1}(f(x - ty_n) - f(x)) - y^*(y_n) \right| = 0$$

since $g(-x + ty_n) = f(x - ty_n)$ and

(c) $$\lim_{t \to 0+} \sup_{h \in H} \left| t^{-1}(f(x + th) - f(x)) - z^*(h) \right| = 0.$$

Formula (c) implies that $z^*(h) = \lim_{t \to 0+} t^{-1}(f(x + th) - f(x))$ for each $h \in A \cup -A$ so that (a) and (b) hold with z^* replacing x^* and $(-y^*)$ respectively. Therefore

$$\lim_{t \to 0+} \sup_{h \in A \cup -A} \left| t^{-1}(f(x + th) - f(x)) - z^*(h) \right| = 0.$$

That is, f is $(A \cup -A)$-differentiable on the residual subset G of $\mathscr{X}$. By Corollary 3.19, K is separably related to $A \cup -A$ hence to A.

We summarize some of the results of this section in the following two theorems.

THEOREM 3.22. *Each of the following statements about a weak* closed convex subset K of $\mathscr{X}^*$ and a bounded subset A of $\mathscr{X}$ implies all the others.*

(1) *K is separably related to A.*

(2) *Every bounded subset of K has weak* slices of arbitrarily small q_A-diameter.*

(3) *Every continuous convex function f on $\mathscr{X}$ such that $\partial f(x) \subseteq \Gamma(K)$ for all $x \in \mathscr{X}$ is A-differentiable on a residual subset of $\mathscr{X}$.*

(4) *Every continuous convex function f on an open convex subset C of $\mathscr{X}$ such that $\partial f(x) \cap K$ is nonempty for a dense set of x in C is $\Gamma(A)$-differentiable on a residual subset of C.*

(5) *Every monotone operator $\mathscr{M}$ on $\mathscr{X}$ such that $D(\mathscr{M})$ contains an open set C with $\mathscr{M}(x) \subseteq K$ for all $x \in C$ is A-continuous on a residual subset of C.*

THEOREM 3.23. (a) *A weak* closed convex subset K of $\mathscr{X}^*$ has the Radon-Nikodým property if and only if every continuous convex function f on $\mathscr{X}$ with $\partial f(x) \subseteq K$ for all $x \in \mathscr{X}$ is Fréchet differentiable at a residual set of points in $\mathscr{X}$.*

(b) *A bounded subset $A \subset \mathscr{X}$ is measurable in itself if and only if every continuous convex function f on $\mathscr{X}$ is A-differentiable at a residual set of points in $\mathscr{X}$.*

4. Examples

The following construction is useful for generating counterexamples.

Example 4.1. A weak* compact nonconvex set K with the Radon-Nikodým property need not be an RN set.

Let V be a compact Hausdorff topological space and let

$$K = \{\delta_v : v \in V\} \subset C(V)^*$$

where δ_v denotes the evaluation functional $\delta_v(f) = f(v)$ ($v \in V$, $f \in C(V)$). Then K is a weak* compact subset of $C(V)^*$ and the norm-closed linear span of K is isometric to $l_1(V)$ which has the Radon-Nikodým property (see [5]). Thus K has the Radon-Nikodým property. However if $V = [0,1]$ is the unit interval with its usual topology then $C(V)$ is separable and K is not norm separable so that K is not an RN set by Theorem 2.13.

Example 4.2. A norm-closed nonconvex set $S \subset \mathscr{X}$ with the Radon-Nikodým property such that its convex hull fails the Radon-Nikodým property.

Let $\mathscr{X}$ be any Banach space with strictly convex norm but not having the Radon-Nikodým property. See Diestel's lecture notes [5] for examples. Let

$$S = \{x \in \mathscr{X}: \|x\| = 1\}$$

be the unit sphere. Since conv(S) = $B(\mathscr{X})$ we see that conv(S) does not have the Radon-Nikodým property (else $\mathscr{X}$ would). However, suppose (Ω, Σ, μ) is a finite positive measure space and $m: \Sigma \to \mathscr{X}$ is a μ-continuous vector measure with average range contained in S. If E_1 and E_2 are disjoint sets in Σ with positive μ-measure then

$$\frac{m(E_1 \cup E_2)}{\mu(E_1 \cup E_2)} = \frac{m(E_1) + m(E_2)}{\mu(E_1) + \mu(E_2)}$$
$$= \left(\frac{\mu(E_1)}{\mu(E_1) + \mu(E_2)}\right)\frac{m(E_1)}{\mu(E_1)} + \left(\frac{\mu(E_2)}{\mu(E_1) + \mu(E_2)}\right)\frac{m(E_2)}{\mu(E_2)}$$

and since

$$\frac{m(E_1)}{\mu(E_1)}, \quad \frac{m(E_2)}{\mu(E_2)} \quad \text{and} \quad \frac{m(E_1 \cup E_2)}{\mu(E_1 \cup E_2)}$$

are all in S, strict convexity implies that they are all equal. It follows that, fixing a set E_1 of positive measure, $m(E) = [m(E_1)/\mu(E_1)]\mu(E)$ for any $E \in \Sigma$. Therefore S has the Radon-Nikodým property.

PROPOSITION 4.3. *Let K be a weak* compact RN set in $\mathscr{X}^*$ and let T: $\mathscr{Y} \to \mathscr{X}$ be an operator. Then T^*K is an RN set.*

Proof. Since K is separably related to $TB(\mathscr{Y})$ the weak* compact set $T^*(K)$ is separably related to $B(\mathscr{Y})$ and $T^*(K)$ is an RN set (Theorem 2.13, Corollary 2.12).

In [23, Corollary 1.11] it was incorrectly stated that Proposition 4.3 was true with nonconvex sets with the Radon-Nikodým property replacing RN sets.

Example 4.4. A weak* compact set K with the Radon-Nikodým property and an operator T: $\mathscr{Y} \to \mathscr{X}$ such that $T^*(K)$ does not have the Radon-Nikodým property.

In Example 4.1 choose V to be the unit ball of $l_\infty = l_1^*$ with its weak* topology. Let T: $l_1 \to C(V)$ be defined by $T\alpha = \sum_{n=1}^{\infty} \alpha_n f_n$ where $f_n(v)$ is the nth coordinate of $v \in l_\infty$. Then $T^*(\delta_v) = v \in B(l_\infty)$ for each $\delta_v \in K$ and $T^*(K) = B(l_\infty)$ which does not have the Radon-Nikodým property since

$l_\infty = l_1^*$ is not separable while l_1 is separable. However K has the Radon-Nikodým property.

Finally a suggestion of the need for weak* compactness.

Example 4.5. A norm-closed convex subset C of $\mathscr{X}^*$ which has the Radon-Nikodým property but is not separably related to $\mathscr{X}$.

Let $\mathscr{X} = C[0,1]$ which is separable and let K be as in Example 4.1 If $C = \overline{\text{conv}}(K)$ then C has the Radon-Nikodým property but is nonseparable.

REFERENCES

1. E. ASPLUND and R.T. ROCKAFELLAR, *Gradients of convex functions*, Trans. Amer. Math. Soc., vol. 139 (1969), pp. 443–467.
2. J. BOURGAIN, *Dentability and the Bishop-Phelps property*, Israel J. Math., vol. 28 (1977), pp. 265–271.
3. RICHARD D. BOURGIN, *Weak* compact convex sets with separable extremal subsets have the Radon-Nikodým property*, Proc. Amer. Math. Soc., vol. 69 (1978), pp. 81–84.
4. W.J. DAVIS and R.R. PHELPS, *The Radon-Nikodým property and dentable sets in Banach spaces*, Proc. Amer. Math. Soc., vol. 45 (1974), pp. 119–122.
5. J. DIESTEL, *Geometry of Banach spaces—selected topics*, Springer-Verlag, New York, 1975.
6. J. DIESTEL and J.J. UHL, JR. *Vector measures*, Mathematical Surveys No. 15, Amer. Math. Soc., Providence, R.I., 1977.
7. J.R. GILES, *Convex analysis with application in differentiation of convex functions*, Pitnam, London, 1982.
8. A. GROTHENDIECK, *Produits tensoriels et espaces nucléairs*, Mem. Amer. Math. Soc., No. 16, Amer. Math. Soc., Providence, R.I., 1955.
9. R. HAYDON, *An extreme point criterion for separability of a dual space and a new proof of a theorem of Corson*, Quart. J. Math. Oxford Ser. 2, vol. 11 (1976), pp. 379–385.
10. R.E. HUFF, *Dentability and the Radon-Nikodým property*, Duke Math. J., vol. 41 (1974), pp. 111–114.
11. P.S. KENDEROV, *The set-valued nonlinear monotone mappings are almost everywhere single-valued*, C.R. Acad. Bulgare Sci., vol. 27 (1974), pp. 1173–1175.
12. ______, *Monotone operators in Asplund spaces*, C.R. Acad. Bulgare Sci., vol. 30 (1977), pp. 963–964.
13. H.B. MAYNARD, *A geometrical characterization of Banach spaces having the Radon-Nikodým property*, Trans. Amer. Math. Soc., vol. 185 (1973), pp. 493–500.
14. GEORGE J. MINTY, *On the monotonicity of the gradient of a convex function*, Pacific J. Math., vol. 14 (1964), pp. 243–247.
15. I. NAMIOKA and R.R. PHELPS, *Banach spaces which are Asplund spaces*, Duke Math. J., vol. 42 (1975), pp. 735–750.
16. R.R. PHELPS, *Dentability and extreme points*, J. Functional Analysis, vol. 16 (1974), pp. 78–90.
17. O.I. REINOV, *Operators of type RN in Banach spaces*, Soviet. Math. Dokl., vol. 16 (1975), pp. 119–123.
18. ______, *On a class of Hausdorff compacts and GSG Banach spaces*, Studia Math., vol. 71 (1981), pp. 113–126.
19. M.A. RIEFFEL, *Dentable subsets of Banach spaces with applications to a Radon-Nikodým theorem*, Proc. Conference Functional Analysis, Thompson Book Co., Washington, D.C. 1967, pp. 71–77.

20. R.T. ROCKAFELLAR, *Local boundedness of nonlinear monotone operators*, Michigan Math. J., vol. 16 (1969), pp. 397–407.
21. CHARLES STEGALL, *The Radon-Nikodým property in conjugate Banach spaces*, Trans. Amer. Math. Soc., vol. 206 (1975), pp. 213–223.
22. ______, *The duality between Asplund spaces and spaces with the Radon-Nikodým property*, Israel J. Math., vol. 29 (1978), pp. 408–412.
23. ______, *The Radon-Nikodým property in conjugate Banach spaces. II*, Trans. Amer. Math. Soc., vol. 264 (1981), pp. 507–519.
24. J.J. UHL, JR. *A note on the Radon-Nikodým property for Banach spaces*, Rev. Roumaine Math. Pures Appl., vol. 17 (1972), pp. 113–115.

UNIVERSITY OF AUCKLAND
AUKLAND, NEW ZEALAND