ILLINOIS JOURNAL OF MATHEMATICS
Volume 29, Number 2, Summer 1985

L-SUBALGEBRAS OF MEASURES RELATED TO DISSOCIATE SETS

BY

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0. Introduction

This paper deals with the structure of the algebra $M(G)$ of regular Borel measures on the LCA group G. The notion of independent power measures (exemplified by relation (0.2) below) is due to Williamson [6]. The structure results we obtain generalize results obtained first by Hewitt and Kakutani [3] and Simon [4], [5] and extended by others. A recent generalization of the asymmetry result of [6] is in [1]. See [2, Chapters 6 and 8] for mathematical and historical details. The methods of the present paper differ from those of [2, Chapter 6] in that here is found a much greater emphasis on product measures. The novelty of the proofs of the present paper lies in the explicit use of the mappings S_m and in the permutation argument used in Section 2 to prove Theorem 0.1.

A subset E of the LCA group G is *dissociate* if for all $n \geq 1$, all distinct $y_1, \ldots, y_n \in E$ and all integers $m_1, \ldots, m_n$ with $|m_j| \leq 2$, $\Sigma m_j y_j = 0$ if and only if $m_1 y_1 = \cdots = m_n y_n = 0$. The set E is *independent* if for all $n \geq 1$, all distinct $y_1, \ldots, y_n \in E$ and all integers $m_1, \ldots, m_n$, $\Sigma m_j y_j = 0$ if and only if $m_1 y_1 = \cdots = m_n y_n = 0$. We let $Gp'(E)$ be the set of all sums $\Sigma m_j y_j$, where $|m_j| \leq 1$ and the y_j are distinct elements of E. The property of a dissociate set E that is salient for the present paper is that elements of $Gp'(E)$ have unique representations. That uniqueness of representation is used to extend well-known facts about L-algebras generated by measures on dissociate sets.

We now develop the notation needed to express concisely and precisely our results. G will always denote a non-discrete LCA group and E will denote a Borel subset of G that is dissociate. For each integer $m > 0$, $E(m)$ will denote the set of all sums of the form $\pm y_1 \pm \cdots \pm y_m$, where the y_j are distinct elements of E, $E^{(m)}$ will denote the analogous subset of a product set,

$$_{(m)} = \{(\pm y_1, \ldots, \pm y_m) \colon y_j \in E,\ y_j \neq y_k,\ 1 \leq j \neq k \leq m\},$$

and $Gp'(E) = \cup E(m)$. One fact about dissociate sets E that we shall use

Received January 7, 1983.
[1]Partially supported by a grant from the National Science Foundation.

often is this: for each subset $F \subset E$ and each integer $m \geq 1$,

$$E(m) = (E \setminus F)(m) \cup \bigcup z + E(m-r), \tag{0.1}$$

where the second union is taken over $0 \leq r \leq m$ and $z \in F(m)$, where $F(0) = E(0) = \{0\}$. S_m will denote the mapping induced on measures by the mapping

$$(y_1, \ldots, y_m) \mapsto +y_1 + \cdots +y_m.$$

Let $I^{(m)}$ denote the set of continuous measures on $E^{(m)}$ that are invariant under permutation of variables. We let I_m denote the continuous measures on $E(m)$. Lemma 1.2 asserts that S_m is an isometry from $I^{(m)}$ to I_m. A measure ω in $I^{(m)}$ is *admissible* if μ is singular to all measures of the form $\mu \times \nu$, where $\omega \in M(E^{(p)})$, $\nu \in M(E^{(q)})$ and $p + q = m$. The point of admissibility is this: when S_m is applied to the admissible measures in $I^{(m)}$, we obtain (Lemma 1.2) exactly the measures on $E(m)$ that are singular to all translates of measures on $E(k)$ for all $k < m$ and that are also singular to all convolution products of measures $\mu \in M_c(E(p))$, $\nu \in M_c(E(q))$, $p + q = m$. We shall also call such measures on $E(m)$ *admissible*. Our main result is the following theorem.

THEOREM 0.1. *Let E be a dissociate Borel subset of the LCA group G. Let $\{\mu_j\}_1^n$ be a set of pairwise mutually singular, continuous measures each one of which belongs to one of the sets I_q and is admissible. For each pair $(m(1), \ldots, m(n))$, $(p(1), \ldots, p(n))$ of n-tuples of non-negative integers and each $y \in G$, we define the measures λ and ρ by*

$$\lambda = \delta_y * \mu_1^{m(1)} * \cdots * \mu_n^{m(n)} \quad \textit{and} \quad \rho = \mu_1^{p(1)} * \cdots * \mu_n^{p(n)}.$$

Then

$$\lambda \perp \rho \tag{0.2}$$

unless $y = 0$ and $(m(1), \ldots, m(n)) = (p(1), \ldots, p(n))$.

The following corollary is typical of the kind of result that can be obtained easily once Theorem 0.1 is established. See, for example, [2, 6.2–6.3] for details, standard results and standard arguments.

COROLLARY 0.2. *Let E be as in 0.1. Then the L-subalgebra of measures $N(E)$ generated by the union of $M_c(Gp'(E))$ and $M_d(G)$ is an algebra whose maximal ideal space and Šilov boundary are equal and both isomorphic to the product of the unit ball of the dual of $M_c(E)$ with the Bohr compactification of the dual group of G.*

The methods used to prove Theorem 0.1 and Corollary 0.2 can be adapted to prove the next result.

THEOREM 0.3. *Let E be an independent Borel subset of the LCA group G. Then $N(E)$, the L-subalgebra of measures generated by $M_c(Gp(E)) \cup M_d(G)$, is an algebra whose maximal ideal space and Šilov boundary are equal and both are isomorphic to the product of the unit ball of the dual of $M_c(E)$ with the Bohr compactification of the dual group of G.*

This paper is organized as follows: preliminary results in Section 1, Proof of Theorem 0.1 in Section 2, Sketch of the proof of Theorem 0.3 in Section 3, examples in Section 4, and an open question in Section 5. It is a pleasure to thank Michel Talagrande for suggesting the example that appears at the end of Section 4 and which provided some of the stimulus for this paper.

1. Preliminary Results

We begin with the following lemma.

LEMMA 1.1. *Let E be a dissociate Borel subset of the LCA group G.*

(i) *For each $y \in Gp'(E) + Gp'(E)$, there exists a finite set F such that $y \notin Gp'(E \setminus F) + Gp'(E \setminus F)$.*

(ii) *For each countable subset F of E, and $n \geq 1$, if μ has zero measure on each set of the form $y + E(k)$, $y \in G$, $k < n$, then μ is concentrated on $(E \setminus F)(n)$.*

Proof. (i) Immediate from the definition of "dissociate".

(ii) This is immediate from (0.1).

The relationship of absolute continuity applies to $I^{(m)}$ and induces an L-space structure on $I^{(m)}$ that respects the invariance, under permutations of coordinates of elements of $I^{(m)}$.

LEMMA 1.2. *Let E be a dissociate Borel subset of the LCA group G and $n > 0$. Then*

(i) *S_n is an L-space isometry of $I^{(n)}$ onto $M_c(E(n))$; and*

(ii) *if $\mu \in I^{(n)}$ then μ is admissible if and only if $S_n\mu$ is admissible.*

Proof. (i) Because E is dissociate, two elements of $E^{(n)}$ map onto the same element of $E(n)$ if and only if the two elements are the same except for the order of the coordinates. Let E/n denote the quotient space of $E^{(n)}$ under that equivalence relation. Then E/n and $E(n)$ are homeomorphic. Furthermore, the mapping of measures induced by $E^{(n)} \to E/n$ is an L-space isomorphism of $I^{(n)}$ onto $M_c(E/n)$. Part (i) of the lemma now follows easily.

(ii) Suppose that $S_m\mu$ is not admissible. We then have two cases.

Case I. There exist measures $\nu \in M(E(p))$ and $\omega \in M(E(q))$, where $p \geq 1$ and $p + q = m$, such that $\nu * \omega$ is not mutually singular with respect to $S_m\mu$. For a permutation σ of $\{1, \dots, m\}$, let $\check{\sigma}$ denote the resulting mapping of $M(E^{(m)})$ induced by the permutation σ of coordinates. If μ were singular with respect to $S_p^{-1}\nu \times S_q^{-1}\omega$, then μ would be singular with respect to the average of the measures $\check{\sigma}S_p^{-1}\nu \times S_q^{-1}\omega$ as σ ranges over all permutations of $\{1, \dots, m\}$, since μ is unchanged by $\check{\sigma}$ and $\check{\sigma}$ is an isomorphism of $M(E^{(m)})$. Since the extension of S_m to all of $M(E^{(m)})$ (obviously) maps cross products to convolution products, the singularity of $\nu * \omega$ and $S_m\mu$ would follow. That shows that in Case I, if $S_m\mu$ is not admissible, then neither is μ.

Case II. $S_m\mu$ is not mutually singular with respect to a translate $\delta_y * \nu$, where $\nu \in M(E(p))$, $0 \leq p < m$ and $y \in G$. We may assume that p is the smallest integer such that m, μ, ν and y exist with that property and that m is minimal with respect to p.

With those assumptions, we shall reduce to the situation of Case I.

By 1.1(i), there exists a finite set F such that $y \notin Gp'(E \setminus F) + Gp'(E \setminus F)$. In particular,

$$(y + (E \setminus F)(p)) \cap (E \setminus F)(m) = \emptyset. \tag{1.1}$$

Of course, E being dissociate implies that (0.1) holds for the pair E and F. The minimality of p implies that we may assume that ν is concentrated on $(E \setminus F)(p)$. By (1.1) and the non-singularity of $\delta_y * \nu$ and $S_m\mu$, we may assume μ has zero mass on $(E \setminus F)(m)$. By (0.1), $S_m\mu$ has mass on a set of the form $z + E(m - r)$, where $z \in F(r)$ and $r > 0$. Therefore $S_m\mu$ is not singular with respect to the convolution product $\delta_z * (\delta_{-z} * S_m\mu)$. That completes the reduction to Case I.

That proves that the non-admissibility of $S_m\mu$ implies the non-admissibility of μ. The opposite implication is obvious. That ends the proof of 1.2(ii) and the proof of Lemma 1.2.

LEMMA 1.3. *Let E be a dissociate Borel subset of the LCA group G. Let $\{\mu_1, \dots, \mu_n\}$ be a family of continuous measures such that for each j, there exists $m(j) > 0$ such that μ_j is an admissible element of $I_{m(j)}$. Then*

(i) *$\mu = \mu_1 * \cdots * \mu_n$ is concentrated on $E(\Sigma m(j))$; and*

(ii) *μ has zero mass on each set $y + E(q)$, for $y \in G$, $q < \Sigma m_j$.*

Proof. (i) Set $m = \Sigma\, m(j)$. Then μ is concentrated on $E + \cdots + E$ (m times). Since the $S_{m(j)}^{-1}(\mu_j)$ are continuous, the Fubini theorem implies that $S_m^{-1}\mu$ is concentrated on $E^{(n)}$ and (i) follows.

(ii) Consider sets of the form $y + E(q)$. If $q = 0$, then there is nothing to prove, since the measure μ_j are all continuous. Let p be the smallest integer such that there exists $y \in G$ with $\mu(y + E(p)) \neq 0$.

Case I. $y = 0$. Then $E(p) \cap E(m) = \emptyset$ if $0 \leq p < m$, since E is dissociate. Since μ is concentrated on $E(m)$, $\mu(E(p)) = 0$.

Case II. $y \neq 0$. By Lemma 1.1(i), there exists a finite set $F \subset E$ such that $y \notin Gp'(E \setminus F) + Gp'(E \setminus F)$. Since each μ_j is admissible, each μ_j is concentrated on $(E \setminus F)(m_j)$. By 1.1(ii) and the first part of the present lemma, μ is concentrated on $(E \setminus F)(m)$. Therefore $\mu(y + (E \setminus F)(p)) = 0$, since $y + (E \setminus F)(p)$ and $(E \setminus F)(m)$ are disjoint. Since p is minimal, μ has zero mass on the other sets in the union (0.1) for $E(p)$. That ends the proof of Lemma 1.3.

COROLLARY 1.4. *Let E be a dissociate Borel subset of the LCA group G. Let m and n be distinct positive integers and let $\mu \in I_m$ and $\nu \in I_n$ be admissible measures. Then μ and ν are mutually singular.*

Proof. $E(p) \cap E(m) = \emptyset$ if $0 \leq p < m$, since E is dissociate. The conclusion is now immediate.

LEMMA 1.5. *Let E be a dissociate Borel subset of the LCA group G. Let $\mu \in I^{(m)}$. Then μ is singular with respect to every measure of the form $\nu \times \rho$, $\nu \in I^{(n)}$, $\rho \in I^{(r)}$, $m = n + r$, if and only if $S_m\mu$ is singular with respect to every convolution product $\nu * \rho$, where $\nu \in I_n$, $\rho \in I_r$.*

Proof. The lemma is immediate from 1.2(i) and the observation that

$$S_m(\nu \times \rho) = S_n\nu * S_r\rho.$$

2. Proof of Theorem 0.1

Case I. $y = 0$. We may assume that each μ_j is a probability measure. For each j, let $q(j)$ be such that $\mu_j \in I_{q(j)}$. For each j, let $\mu'_j = S_{q(j)}^{-1}\mu_j$. Since $\mu_j \in I_{m(j)}$ for all j, Lemma 1.3 implies that $\lambda \in I_m$ and $\rho \in I_r$, where $m = \Sigma m(j)q(j)$ and $r = \Sigma p(j)q(j)$. If $m \neq r$, then (0.2) follows from Corollary 1.4. We may therefore assume that $m = r$.

Set $\lambda' = \mu'_1 \times \cdots \times \mu'_n$ (μ'_j appears $m(j)$ times). Then λ' is not in $I^{(m)}$, but λ'' is, where λ'' is the average of the $m!$ measures obtained from λ' by permuting the coordinates. Define ρ' and ρ'' analogously. Of course, $S_m\lambda'' = S_m\lambda' = \lambda$. It follows from 1.1(i) that (0.2) holds if and only if λ'' and ρ'' are mutually singular. Of course, that occurs if and only if λ'' and ρ' are mutually singular.

It is convenient here to recall that λ' and ρ' are the restrictions of product measures to $E^{(m)}$. We may therefore work in the product space $E' = E \times \cdots \times E$ (m times). Let a permutation σ of the m coordinates of E' be given, and let $\check{\sigma}$ be the mapping of measures induced by σ. It will suffice to show that, for all σ,

$$\check{\sigma}\lambda' \perp \rho'. \tag{2.1}$$

For $1 \leq j \leq n$, $1 \leq k \leq m(j)$, and $1 \leq h \leq p(j)$, we make the following definitions.

$$\begin{aligned}
a_{j,k} &= \sum_{r=1}^{j-1} q(r)m(r) + (k-1)q(j) \\
b_{j,k} &= a_{j,k} + q(j) \\
c_{j,h} &= \sum_{r=1}^{j-1} q(r)p(r) + (h-1)q(j) \\
d_{j,h} &= c_{j,h} + q(j) \\
A(j,k) &= \{i\colon a_{jk} < i \leq b_{jk}\} \\
C(j,h) &= \{i\colon c_{jh} < i \leq d_{jh}\};
\end{aligned}$$

that is, $A(j,k)$ is the set of integers labeling the coordinates involved in the k-th appearance of μ_j in the product λ' and $C(j,h)$ is the set of integers labeling the coordinates involved in the h-th appearance of μ_j in the product ρ'.

Fix σ. Suppose that $\check{\sigma}\lambda'$ and ρ' are not mutually singular. We consider several subcases.

Case I(a). For every j and every k, there exists h such that $\sigma A(j,k) = C(j,h)$. Then $m(j) = p(j)$ and the hypothesis on the $m(j)$ and $p(j)$ is contradicted.

Case I(b). For some j, k, there exist g, h such that $j \neq g$ and $\sigma A(j,k) = C(g,h)$. Then $\check{\sigma}\lambda'$ and ρ' are mutually singular since they are singular in the $C(g,h)$-factor.

If the conditions of both Case I(a) and Case I(b) are unfulfilled then the conditions of the final Case I(c) immediately following are fulfilled.

Case I(c). There exist j, k and g, h such that

$$\sigma A(j,k) \cap C(g,h) \neq \emptyset \quad \text{and} \quad \sigma A(j,k) \neq C(g,h).$$

That implies that the h-th occurance of μ_g in ρ' is not mutually singular with a certain product measure, since we can write $\check{\sigma}\lambda'$ as a product measure with the break occurring among the coordinates that are in $C(g,h)$. Therefore μ_g is not admissible, a contradiction. That ends the proof of Theorem 0.1 in the case that $y = 0$.

Case II. $y \neq 0$. Let F be given by Lemma 1.1(i). When we replace E with $E' = E \setminus F$, we do not change the class of admissible measures, by 1.1(ii). Therefore, we may assume that $y \notin Gp'(E) + Gp'(E)$. But λ is concentrated on $y + (E \setminus F)$, ρ is concentrated on $(E \setminus F)(m)$ and those sets are disjoint. Therefore (0.2) holds in Case II. That ends the proof of Theorem 0.1.

3. Sketch of the proof of Theorem 0.3

Let E be an independent subset of the LCA group G. We define three sets:
$E' = \{ky : y \in E, k \in Z\} \setminus \{0\}$;
$E^{(n)\prime}$ is the set of $(m_1 y_1, \ldots, m_n y_n)$, where the $y_1, \ldots, y_n$ are distinct elements of E and $0 < m_j <$ order $(y_j)\}$ whenever y_j has finite order;
$E(n)' = \{\Sigma m_j y_j : (m_1 y_1, \ldots, m_n y_n) \in E^{(n)\prime}\}$.
We let $I^{(n)}$, be the set of measures on $E^{(n)}$, that are invariant under permutation of coordinates, while $I(n)'$ is the set of measures on $E(n)'$ that are images (induced by sum-of-coordinates) of measures in $I^{(n)\prime}$. It is easy to see that the analogues of Lemmas 1.1–1.5 hold for those spaces and spaces of measures (the general changes are a matter of adding primes and replacing occurrences of "dissociate" with "independent"; in addition "finite" in 1.1.(i) must be replaced by "countable"; that replacement causes no difficulties). The version of Theorem 0.1 for "admissible" measures needed for Theorem 0.3 is then proved (again there are no significant changes that need to be made to the proof of 0.1). Theorem 0.3 will then follow easily.

4. Examples

We show here that there are many measures on $E(2)$ that are not products or translates of measures on E. The methods can easily be extended to $E(n)$, $n \geq 2$.

We begin first with "arcs". Then we show that there are other measures of the type sought that are not concentrated on arcs.

An *arc* in $E(2)$ is a set of the form

$$\{y + f(y) \colon y \in E\}$$

where f is a homeomorphism of E onto itself with the property that $f(y) \neq y$ for all $y \in E$. The proof of the following proposition is straightforward.

PROPOSITION 4.1. *Let E be a dissociate subset of the LCA group G. Let μ be a continuous measure concentrated on an arc in $E(2)$. Then μ is an admissible element of I_2.*

The continuous measures on arcs and the products of measures from E do not generate all measures on $E(2)$ as the following example shows. For simplicity, we work in $E^{(2)}$, rather than in $E(2)$. An application of 1.2(i) completes the proof.

Let E be a compact perfect metrizable dissociate subset of the LCA group G. We can represent E as a product $E_1 \times E_2$ of two compact perfect metrizable spaces. Then

$$E \times E = E_1 \times E_2 \times E_3 \times E_4,$$

where $E_3 = E_1$ and $E_4 = E_2$. Let $f\colon E_2 \to E_3$ be a homeomorphism. Let μ_1 be a continuous probability measure on E_1, let μ_2 be a continuous measure on the graph (arc) of f, and let μ_4 be a continuous measure on E_4. Set $\mu = \mu_1 \times \mu_2 \times \mu_4$. Then the Fubini theorem shows that μ is singular with respect to every product of measures on $E \times E$, while simple calculations show that μ is singular with respect to every measure on every graph in $E \times E$.

We cannot hope to extend Theorem 0.1 to all (appropriate sets of) measures on the group generated by a dissociate set: just let E_1 be a set of type K_5 in the compact metrizable group all of whose elements have order 5 and let f be a continuous mapping of E_1 onto the circle group. Then $E = \{(y, f(y))\colon y \in E_1\}$ is dissociate, but $5E = E + \cdots + E$ (5 times) contains a copy of the circle group, so the analogue of Theorem 0.1 is false for $5E$, as is the analogue of Theorem 0.3 for $Gp(E)$.

5. An open question

For a set E of the group G, we denote by $Gp''(E)$ the set of all sums $\Sigma m_j y_j$, where the y_j are distinct elements of E, and $|m_j| \leq 2$.

A set E is called *almost dissociate* if whenever $y \in G$, $y \neq 0$, there is a countable subset F of E such that $y \notin Gp''(E \setminus F)$. Lemma 1.1 asserts that every dissociate set is almost dissociate. Algebraically scattered sets (see [2, 6.2]) are (trivially) also almost dissociate. It is not hard to show that the analogue of [2, 6.2.1] holds for measures on almost dissociate sets.

Can Theorem 0.1 be extended to the space of all continuous measures on $Gp'(E)$ when E is almost dissociate?

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