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THE SETS OF FIXED RADIAL LIMIT VALUE FOR INNER FUNCTIONS

BY

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1. Introduction

Let $\mathscr{M}$ be the class of all nonconstant meromorphic functions on $\Delta = \{|z| < 1\}$ and let $\hat{\mathbf{C}}$ denote the extended plane $\mathbf{C} \cup \{\infty\}$. Given a subclass Φ of $\mathscr{M}$ and $\alpha \in \hat{\mathbf{C}}$, there is associated a class $\mathscr{Z}_\alpha(\Phi)$ of subsets of the unit circumference $C = \{|z| = 1\}$ which arise as the collection of preimages of α under the radial limit functions of the members of Φ. Thus $E \in \mathscr{Z}_\alpha(\Phi)$ if and only if

$$E = \{\eta \in C: f^*(\eta) = \alpha\}$$

for some $f \in \Phi$, where $f^*(\eta) = \lim_{r \to 1} f(r\eta)$ at each η in C for which the limit exists (finite or infinite).

Let $\mathscr{I}$, $\mathscr{B}$, and $\mathscr{S}$ be the classes of nonconstant inner functions, Blaschke products, and singular inner functions respectively (cf. §4 for a review of the definitions and basic facts concerning $\mathscr{I}$, $\mathscr{B}$, and $\mathscr{S}$). For Φ a subclass of $\mathscr{M}$, let Φ_ρ denote the class of functions in Φ having radial limits at each point of C. Our main concern in this paper is the study of $\mathscr{Z}_\alpha(\Phi)$ when $\Phi = \mathscr{I}$, $\mathscr{B}$, $\mathscr{S}$, $\mathscr{I}_\rho$ and $\mathscr{B}_\rho$. Results are also obtained for related classes of functions.

In §2 we establish notation and state the main results. In §3, some topological results are proved. Explicit constructions of inner functions with radial limit functions that take on a specified value in prescribed subsets of C are given in §4. Finally, applications for singular monotone functions are taken up in §5.

Most of the results of this paper appear in some form in the author's doctoral dissertation. The dissertation was directed by Professor Maurice Heins at the University of Maryland. At this time, the author wishes to express his appreciation to Professor Heins for his help and to the University of Maryland for its support. In addition, the author wishes to thank Professor A. H. Stone for providing him with the proof given at the end of §3 to show that the last inclusion in (3.9) is proper, and Professor G. Piranian for proving the first assertion of Theorem 2.5 based on a technique of C. Belna.

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2. Main results

Before stating our main results, we establish some notation and state several background theorems.

Let $\mathcal{M}$, $\mathcal{I}$, $\mathcal{B}$, and $\mathcal{S}$ be defined as in §1. Denote by $\mathcal{M}\mathcal{I}$ the class of "meromorphic inner" functions, that is, the class of f in $\mathcal{M}$ satisfying $\lim_{r\to 1}|f(r\eta)| = 1$ a.e. in C. For Φ a subclass of $\mathcal{M}$, let Φ_ρ be as stated in §1. Also, let Φ_ρ^0 be the subclass of functions in Φ_ρ having radial limits of modulus 1 or 0 at each point of C.

Let $\mathcal{F}$ (resp. $\mathcal{G}$) denote the class of all closed (resp. open) sets in C. We adjoin the superscript '0' to the notation for a class of sets to denote the class consisting of all those sets in the original class which have linear measure 0. Also, the subscript σ (or δ) is adjoined to the symbol for a class to represent the class whose elements are unions (or intersections) of countably many elements in the original class. Note that there is no ambiguity in writing $\mathcal{F}_\sigma{}^0$ since $(\mathcal{F}_\sigma)^0 = (\mathcal{F}^0)_\sigma$. Generally, if $\mathcal{X}$ and $\mathcal{Y}$ are two classes of subsets of C, then we use not only the conventional notation

$$\mathcal{X} \cap \mathcal{Y} = \{W\colon W \in \mathcal{X},\ W \in \mathcal{Y}\},$$

but also the *wedge* notation

$$\mathcal{X} \wedge \mathcal{Y} = \{X \cap Y\colon X \in \mathcal{X},\ Y \in \mathcal{Y}\}.$$

Recall that a set E is nowhere dense if $\operatorname{int} \bar{E} = \emptyset$, where $\bar{E}$ denotes the closure of E and 'int' the interior operation, and of first category (resp. second category) if E is (resp. is not) a countable union of nowhere dense sets. In the sequel, the symbol '$\subset$' will be used to denote proper inclusion.

The most general radial limit results concerning the classes $\mathcal{M}\mathcal{I}$ and $\mathcal{M}\mathcal{I}_\rho$ are given in the following theorem. Assertion (1) of this theorem is due to Cargo [5; Theorem 5] and assertion (2) was proved by Bagemihl and Seidel [1; p. 1070] for $f \in \mathcal{B}_\rho$ with the present form a special case of [3; Cor. 2.3].

THEOREM 2.1. *Let $E \subseteq C$ and let $\alpha \in \hat{\mathbf{C}}$.*

(1) *If $f \in \mathcal{M}\mathcal{I}$ and $f^*(\eta) = \alpha$, $\eta \in E$, then E is of first category* (Cargo).

(2) *If $f \in \mathcal{M}\mathcal{I}_\rho$, then f is analytic (and of modulus 1) in an open dense subset of C. Thus if $f^*(\eta) = \alpha$, $\eta \in E$, then E is nowhere dense.*

A result of Bagemihl and Seidel [2] asserts that if E is of first category, then there exists a nonconstant analytic function g on Δ such that $g^*(\eta) = 0$, $\eta \in E$. Since any first-category set is contained in a first-category set of full measure, it follows that if $\alpha \in C$, then there exists $f \in \mathcal{M}\mathcal{I}$ such that $f^*(\eta) = \alpha$, $\eta \in E$. Putting this together with Theorem 2.1 (1) we have the following.

THEOREM 2.2. *Let $E \subseteq C$ and $\alpha \in C$. Then E is of first category if and only if there exists $f \in \mathcal{MI}$ such that $f^*(\eta) = \alpha$, $\eta \in E$.*

Theorems 2.1 and 2.2 provide information concerning the size of the sets in $\mathscr{Z}_\alpha(\mathcal{MI})$ and $\mathscr{Z}_\alpha(\mathcal{MI}_\rho)$, but do not give the precise structure of these sets. In this paper, we are interested in obtaining characterizations of the following type [14; pp. 8–11].

THEOREM 2.3 (Lohwater and Piranian). *The following equalities hold*:

$$\mathscr{Z}\left(\mathscr{I}_\rho^0\right) = \mathscr{Z}\left(\mathscr{S}_\rho^0 \cup \{1\}\right) = \mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta. \tag{2.1}$$

Here and in the sequel, the subscript '0' is sometimes dropped from the notation $\mathscr{Z}_0(\Phi)$.

In the next theorem, we summarize the main results of this paper.

THEOREM 2.4. *Let $\alpha \in \Delta$. The following equalities and inclusions are valid*:

$$\mathscr{Z}\left(\mathscr{B}_\rho^0\right) = \mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta, \tag{2.2}$$

$$\mathscr{Z}_\alpha\left(\mathscr{B}_\rho\right) = \mathscr{Z}_\alpha\left(\mathscr{I}_\rho\right) = \left(\mathscr{F}_\sigma^0\right)_\delta \cap \mathscr{G}_\delta, \tag{2.3}$$

and

$$\begin{matrix} \mathscr{F}_\sigma^0 \wedge \mathscr{G}_\delta \subseteq \mathscr{Z}_\alpha(\mathscr{B}) \subseteq \\ \mathscr{F}_\sigma^0 \subseteq \mathscr{Z}(\mathscr{S}) \subseteq \end{matrix} \mathscr{Z}_\alpha(\mathscr{I}) \subseteq \left(\mathscr{F}_\sigma^0\right)_\delta. \tag{2.4}$$

Suppose now that $\alpha \in C$ and $E \subseteq C$. By Theorem 2.1 and the Riesz uniqueness theorem, if there exists $f \in \mathscr{I}$ (resp. $\mathscr{I}_\rho$) such that $f^*(\eta) = \alpha$, $\eta \in E$, then E is of first category (resp. nowhere dense) and measure 0. It is not known whether the converses hold. However, the results in the following theorem are proved in §4 based on a technique of C. Belna communicated to the author by G. Piranian.

THEOREM 2.5. *Let $\alpha \in C$. If $E \in \mathscr{F}_\sigma^0$ then there exists $f \in \mathscr{I}$ such that $f^*(\eta) = \alpha$, $\eta \in E$. Furthermore,*

$$\mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta \setminus \{\emptyset\} \subseteq \mathscr{Z}_\alpha\left(\mathscr{B}_\rho^0\right). \tag{2.5}$$

3. Topological results

In this section we prove three theorems. The first, Theorem 3.1, determines the combined topological and measure-theoretic structure of the sets in

$\mathscr{Z}_\alpha(\mathscr{M}\mathscr{I})$, $\mathscr{Z}_\alpha(\mathscr{M}\mathscr{I}_\rho)$, and $\mathscr{Z}(\mathscr{M}\mathscr{I}_\rho^{\,0})$, for $\alpha \in \hat{\mathbf{C}} \setminus C$. Theorem 3.2 is a structure theorem for first-category subsets of C. A corollary of this theorem is used in the constructions of §4. Finally, Theorem 3.3 gives the inclusion relations which hold among some of the classes of sets that arise in this paper.

We start with a proposition. Its first two assertions follow from remarks made by Piranian in MR52#11049.

PROPOSITION 3.1. *If $f \in \mathscr{M}$, then*

$$E(s) = \{\eta \in C\colon |f(r\eta)| \le s,\ t \le r < 1,\ \textit{for some } t \in (0,1)\} \tag{3.1}$$

is an $\mathscr{F}_\sigma$ set for each $s \in [0, +\infty)$. If $f \in \mathscr{M}\mathscr{I}$, then

$$E(s) \in \mathscr{F}_\sigma^{\,0}, \quad s \in (0,1). \tag{3.2}$$

In this case $\{\eta \in C\colon f^*(\eta) \in \hat{\mathbf{C}} \setminus C\}$ is contained in an $\mathscr{F}_\sigma^{\,0}$ set.

Proof. Let $s \in [0, +\infty)$. By the continuity of $|f|$, the set

$$F(t) = \{\eta \in C\colon |f(r\eta)| \le s,\ t \le r < 1\}$$

is closed for each $t \in (0,1)$. Since $E(s) = \cup_1^\infty F(1 - 1/n)$, we have exhibited $E(s)$ as an $\mathscr{F}_\sigma$ set.

If $f \in \mathscr{M}\mathscr{I}$, then $E(s)$ is of measure 0 for each $s \in (0,1)$ since

$$\lim_{r \to 1} |f(r\eta)| = 1 \quad \text{almost everywhere.}$$

The second assertion follows.

For the last assertion, observe that

$$\{\eta \in C\colon f^*(\eta) \in \Delta\} \subseteq \bigcup_1^\infty E(1 - 1/n)$$

and that $\cup_1^\infty E(1 - 1/n)$ is an $\mathscr{F}_\sigma^{\,0}$ set. Applying a similar argument to $g = 1/f$ to get a corresponding statement for

$$\{\eta \in C\colon f^*(\eta) \in \hat{\mathbf{C}} \setminus \bar{\Delta}\},$$

we arrive at the desired conclusion.

COROLLARY 3.1. *If $f \in \mathscr{M}\mathscr{I}$, then $\{f^* = 0\}$ is an $(\mathscr{F}_\sigma^{\,0})_\delta$ set.*

Proof. This follows immediately from (3.2) and the observation that

$$\{f^* = 0\} = \bigcap_1^\infty E(1/n).$$

COROLLARY 3.2. *If $f \in \mathscr{M}$, then*

$$Z_T(f) = \left\{\eta \in C\colon \liminf_{r \to 1} |f(r\eta)| \le T\right\} \tag{3.3}$$

is a $\mathscr{G}_\delta$ set for each $T \in [0, +\infty)$.

Proof. It suffices to prove the corollary for $T \in (0, +\infty)$ since

$$Z_0(f) = \bigcap_1^\infty Z_{1/n}(f).$$

Let $T \in (0, +\infty)$. For convenience, we show that $Z_T(1/f)$ is a $\mathscr{G}_\delta$ set. (The same argument applies equally well with f replacing $1/f$.) Observe that

$$Z_T(1/f) = \left\{\eta \in C\colon \limsup_{r \to 1} |f(r\eta)| \ge 1/T\right\}. \tag{3.4}$$

Thus

$$Z_T(1/f) = C \setminus \bigcup_N^\infty E[(1/T) - (1/n)],$$

where N is a positive integer large enough to that $1/T > 1/N$. By Proposition 3.1, each set $E[(1/T) - (1/n)]$ is an $\mathscr{F}_\sigma$ set, and the corollary follows.

COROLLARY 3.3. *Let $f \in \mathscr{M}\mathscr{I}_\rho$. If $\varepsilon \in (0,1)$, then*

$$H(\varepsilon) = \{\eta \in C\colon f^*(\eta) \in \hat{\mathbf{C}} \setminus \{1 - \varepsilon < |z| < 1 + \varepsilon\}\} \tag{3.5}$$

is an $(\mathscr{F}_\sigma^{\,0})_\delta \cap \mathscr{G}_\delta$ set. If $\alpha \in \hat{\mathbf{C}}$, then $\{f^* = \alpha\}$ is a $\mathscr{G}_\delta$ set.

Proof. That $H(\varepsilon)$ is contained in an $\mathscr{F}_\sigma^{\,0}$ set follows from the third assertion of Proposition 3.1. From Corollary 3.2, and the fact that $f \in \mathscr{M}\mathscr{I}_\rho$ implies $\{\eta \in C\colon |f^*(\eta)| \le 1 - \varepsilon\} = Z_{1-\varepsilon}(f)$, we conclude that

$$\{\eta \in C\colon |f^*(\eta)| \le 1 - \varepsilon\}$$

is a $\mathscr{G}_\delta$ set. A similar argument applied to $g = 1/f$ shows that

$$\{\eta \in C\colon 1 + \varepsilon \le |f^*(\eta)| \le +\infty\}$$

is also a $\mathscr{G}_\delta$ set. Thus $H(\varepsilon)$ is a $\mathscr{G}_\delta$ set contained in an $\mathscr{F}_\sigma^{\,0}$ set, or equivalently, an $(\mathscr{F}_\sigma^{\,0})_\delta \cap \mathscr{G}_\delta$ set.

For the second assertion, if $\alpha \ne \infty$, then $Z_0(f - \alpha) = \{f^* = \alpha\}$ and Corollary 3.2 applies. If $\alpha = \infty$, the result follows on considering $1/f$.

Using Corollaries 3.1 and 3.2, we prove the following.

THEOREM 3.1. *We have the inclusions*

$$\mathscr{Z}_\alpha(\mathscr{M}\mathscr{I}) \subseteq (\mathscr{F}_\sigma^{\,0})_\delta, \tag{3.6}$$

$$\mathscr{Z}_\alpha(\mathscr{M}\mathscr{I}_\rho) \subseteq (\mathscr{F}_\sigma^{\,0})_\delta \cap \mathscr{G}_\delta, \tag{3.7}$$

for each $\alpha \in \hat{\mathbf{C}} \setminus C$, *and*

$$\mathscr{Z}(\mathscr{M}\mathscr{I}_\rho^{\,0}) \subseteq \mathscr{F}_\sigma^{\,0} \cap \mathscr{G}_\delta. \tag{3.8}$$

Proof. Inclusion (3.6). For $\alpha = 0$, the inclusion follows from Corollary 3.1. If $\alpha \in \Delta$, the result is a consequence of the special case $\alpha = 0$ and the observation that if $f \in \mathscr{M}\mathscr{I}$ and $E = \{f^* = \alpha\}$, then $g = L_\alpha \circ f \in \mathscr{M}\mathscr{I}$ and $E = \{g^* = 0\}$, where L_α is the Möbius transformation given by

$$L_\alpha(z) = (\alpha - z)/(1 - \bar{\alpha}z), \quad z \in \hat{\mathbf{C}}.$$

The case when $\alpha \in \hat{\mathbf{C}} \setminus \overline{\Delta}$ is reduced to the one just treated on considering $g = 1/f$ and noting that $1/\alpha \in \Delta$ and $\{f^* = \alpha\} = \{g^* = 1/\alpha\}$.

Inclusion (3.7). For $\alpha = 0$, this is a consequence of Corollary 3.1 and Corollary 3.2 when $T = 0$ since $\{f^* = 0\} = Z_0(f)$ for $f \in \mathscr{M}\mathscr{I}_\rho$. The cases when $\alpha \in \Delta$ and $\alpha \in \hat{\mathbf{C}} \setminus \overline{\Delta}$ are treated as for the inclusion (3.6).

Inclusion (3.8). If $f \in \mathscr{M}\mathscr{I}_\rho^{\,0}$, then $\{f^* = 0\}$ is a $\mathscr{G}_\delta$ set of measure 0 by (3.7) in the case $\alpha = 0$. By Corollary 3.2 with f replaced by $g = 1/f$ and $T > 1$, the set

$$W = \left\{\eta \in C\colon \limsup_{r \to 1} |f(r\eta)| \geq 1/T\right\}$$

is also a $\mathscr{G}_\delta$. It follows from the definition of $\mathscr{M}\mathscr{I}_\rho^{\,0}$ that

$$W = \{\eta \in C\colon |f^*(\eta)| = 1\}.$$

Thus $\{f^* = 0\} = C \setminus W$ is an $\mathscr{F}_\sigma$ set.
THEOREM 3.1 is established.

For the next theorem as well as later results, we make a convention concerning the use of the words 'right' and 'left' in relation to subsets of the unit circumference C. Let S_k, $k = 1, 2, 3$, be disjoint subsets of C. We shall say that S_1 is to the right of S_2 which is to the right of S_3 if there exists some point

$\zeta \in C \setminus \cup_1^3 S_k$ and a continuous determination arg of the argument in $C \setminus \{\zeta\}$ such that $\arg \eta_1 < \arg \eta_2 < \arg \eta_3$ whenever $\eta_k \in S_k$, $k = 1, 2, 3$. If A is an arc in C with proper closure (i.e., A is connected, int $A \neq 0$, and $\bar{A} \neq C$) then η is the "right hand" endpoint of A and ξ is the "left hand" endpoint of A if $\{\eta\}$ is to the right of int A which is to the right of $\{\xi\}$ and η, $\xi \in \bar{A}$.

THEOREM 3.2. *Let $E \subseteq C$. Then E is of first category if and only if there exists a sequence $(F_j)_1^\infty$ of mutually disjoint closed nowhere dense sets such that $E \subseteq \cup_1^\infty F_j$. If E is an $\mathscr{F}_\sigma$ set, then the preceding assertion remains valid when '$\subseteq$' is replaced by '$=$'.*

Proof. Since every first-category set is contained in an $\mathscr{F}_\sigma$ of first category, it suffices to prove the theorem under the conditions of the last assertion. Sufficiency is trivial so we shall prove only necessity.

By assumption, $E = \cup_1^\infty K_j$ where each K_j is a closed nowhere dense set. Since $E = \cup_1^\infty (K_j \setminus \cup_{m=1}^{j-1} K_m)$ and $(K_j \setminus \cup_{m=1}^{j-1} K_m)_1^\infty$ is a countable sequence of disjoint differences of closed nowhere dense sets, it suffices to prove the following assertion: *If A and B are closed nowhere dense sets, then $B \setminus A$ is a countable union of disjoint closed nowhere dense sets.* Without loss of generality, assume that A has at least two points. (Otherwise, add two points from $C \setminus B$ to A.) Then $C \setminus A$ is the union of a countable collection of mutually disjoint open arcs each having proper closure, so the assertion follows once it is shown that *$B \cap I$ is a countable union of disjoint closed nowhere dense sets when I is an open arc with proper closure*. The case when $B \cap I$ is finite does not call for attention. Let $\mathbf{Z}$ denote the set of integers. Let η denote the right hand endpoint and ξ the left hand endpoint of I. Choose $\{p_k\}_{k \in \mathbf{Z}}$ to be a subset of $I \setminus B$ so that $\{\eta\}$ is to the right of $\{p_{k+1}\}$ which is to the right of $\{p_k\}$ which is to the right of $\{\xi\}$ for each $k \in \mathbf{Z}$, with $\lim_{k \to \infty} p_k = \eta$ and $\lim_{k \to -\infty} p_k = \xi$. Let E_k be the intersection of B with the closed arc with left hand endpoint equal to p_k and right hand endpoint equal to p_{k+1} for each $k \in \mathbf{Z}$. Then $\cup_{k \in \mathbf{Z}} E_k = B \cap I$ and $\{E_k\}_{k \in \mathbf{Z}}$ is a countable collection of mutually disjoint closed nowhere dense sets. This completes the proof.

COROLLARY 3.4. *If $E \in \mathscr{F}_\sigma^0$, then $E = \cup_1^\infty F_j$ where $(F_j)_1^\infty$ is a sequence of mutually disjoint $\mathscr{F}^0$ sets.*

Before stating the final theorem of this section, we prove a lemma. The proof of the lemma is based on the Baire category theorem which, we recall, can be formulated as follows. *If X is a locally compact Hausdorff space, then a countable intersection of dense open sets is again dense.* We turn now to the lemma.

LEMMA 3.1. *Let X be a locally compact Hausdorff space. If E is a first-category $\mathscr{G}_\delta$ subset of X, then E is nowhere dense.*

Proof. Suppose not. Then there exists a nonempty open set U in X such that E is dense in U. Since U is itself a locally compact Hausdorff space and $E \cap U$ is a dense $\mathscr{G}_\delta$ subset, the Baire category theorem implies that $E \cap U$ is of second category in U. But $E \cap U$ is of first category in U since E is of first category in X and U is an open subset of X. Contradiction. The lemma is thereby established.

In the following theorem, the inclusion relations that hold among some of the classes of sets in this paper are given. Recall that '$\subset$' denotes proper inclusion.

THEOREM 3.3. *We have*

$$\mathscr{F}^0 \subset \mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta \begin{array}{l} \subset \left(\mathscr{F}_\sigma^0\right)_\delta \cap \mathscr{G}_\delta \subset \\ \subset \left(\mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta\right) \wedge \mathscr{F}_\sigma \subset \mathscr{F}_\sigma^0 \subset \end{array} \mathscr{F}_\sigma^0 \wedge \mathscr{G}_\delta \subset \left(\mathscr{F}_\sigma^0\right)_\delta. \tag{3.9}$$

However, no inclusion relations hold between $(\mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta) \wedge \mathscr{F}_\sigma$ *and* $(\mathscr{F}_\sigma^0)_\delta \cap \mathscr{G}_\delta$ *or* $\mathscr{F}_\sigma^0$ *and* $(\mathscr{F}_\sigma^0)_\delta \cap \mathscr{G}_\delta$. *Furthermore, all these classes are classes of first-category sets and all except* $\mathscr{F}_\sigma^0$, $\mathscr{F}_\sigma^0 \wedge \mathscr{G}_\delta$, *and* $(\mathscr{F}_\sigma^0)_\delta$ *are classes of nowhere dense sets.*

Proof. That the weak inclusions hold in (3.9) where the strong inclusions '$\subset$' are given presents no difficulty and the proofs are omitted.

For convenience, we start by proving the last assertion of the theorem. Since $\mathscr{F}_\sigma^0$ is a class of first-category sets and every $(\mathscr{F}_\sigma^0)_\delta$ set is contained in some $\mathscr{F}_\sigma^0$ set, it follows from the weak inclusions in (3.9) that all the classes given contain only first-category sets. The second part of the assertion follows from what was just proved, Lemma 3.1, and the weak inclusions in (3.9). In particular, note that every $(\mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta) \wedge \mathscr{F}_\sigma$ set is contained in some $\mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta$ set and that $\mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta$ and $(\mathscr{F}_\sigma^0)_\delta \cap \mathscr{G}_\delta$ are classes of $\mathscr{G}_\delta$ sets.

From the last assertion of the theorem and the fact that $\mathscr{F}_\sigma^0$ and $\mathscr{F}_\sigma^0 \wedge \mathscr{G}_\delta$ contain dense sets (such as countable dense sets), we conclude that the proper inclusions

$$\left(\mathscr{F}_\sigma^0\right)_\delta \cap \mathscr{G}_\delta \subset \mathscr{F}_\sigma^0 \wedge \mathscr{G}_\delta \quad \text{and} \quad \left(\mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta\right) \wedge \mathscr{F}_\sigma \subset \mathscr{F}_\sigma^0$$

hold.

We proceed now to give examples to verify that the remaining inclusions given in (3.9) are proper and that the assertion immediately following (3.9) is valid. For the remainder of the proof, *let K be a Cantor set of linear measure* 0 *contained in C and E the collection of endpoints of the component arcs of $C \setminus K$.*

$\mathscr{F}^0 \subset \mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta$. Let $\eta \in K$. Then $K \setminus \{\eta\} \notin \mathscr{F}^0$ since $K \setminus \{\eta\}$ is not closed. However, K and $C \setminus \{\eta\}$ are both $\mathscr{F}_\sigma$ as well as $\mathscr{G}_\delta$ sets so the same is true of $K \setminus \{\eta\} = K \cap (C \setminus \{\eta\})$. Since K is of measure 0 so is $K \setminus \{\eta\}$, and it follows that $K \setminus \{\eta\} \in \mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta$ as required.

For most of the remaining verifications, either E or $K \setminus E$ is the example used. The following lemma will be useful.

LEMMA 3.2. *We have $K \setminus E \in \mathscr{G}_\delta \setminus \mathscr{F}_\sigma$ and $E \in \mathscr{F}_\sigma^0 \setminus \mathscr{G}_\delta$.*

Proof of Lemma 3.2. Since E is a countable set, $E \in \mathscr{F}_\sigma^0$. Thus

$$K \setminus E = K \cap (C \setminus E)$$

is the intersection of two $\mathscr{G}_\delta$ sets so that $K \setminus E \in \mathscr{G}_\delta$. Since $K \setminus E$ is dense in K and K is a compact Hausdorff space, Lemma 3.1 implies that $K \setminus E$ is of second category in K. On noting that the density of E in K insures that any $\mathscr{F}_\sigma$ set contained in $K \setminus E$ is of first category in K, we conclude that $K \setminus E \notin \mathscr{F}_\sigma$. It also follows that $E \notin \mathscr{G}_\delta$ since otherwise $K \setminus E = K \cap (C \setminus E)$ is the intersection of a closed set with an $\mathscr{F}_\sigma$ set which implies $K \setminus E \in \mathscr{F}_\sigma$ contrary to what was just proved. This completes the proof of Lemma 3.2.

$\mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta \subset (\mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta) \wedge \mathscr{F}_\sigma$. By Lemma 3.2, $E \notin \mathscr{G}_\delta$ so that $E \notin \mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta$. On the other hand, $E = K \cap E \in (\mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta) \wedge \mathscr{F}_\sigma$.

$\mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta \subset (\mathscr{F}_\sigma^0)_\delta \cap \mathscr{G}_\delta$. From Lemma 3.2 we have $K \setminus E \in \mathscr{G}_\delta \setminus \mathscr{F}_\sigma$. Thus

$$K \setminus E \notin \mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta$$

but $K \setminus E \in \mathscr{G}_\delta$. On observing that a $\mathscr{G}_\delta$ set contained in a closed set of measure 0 is an $(\mathscr{F}_\sigma^0)_\delta$ set, we conclude that

$$K \setminus E \in (\mathscr{F}_\sigma^0)_\delta \cap \mathscr{G}_\delta.$$

$\mathscr{F}_\sigma^0 \subset \mathscr{F}_\sigma^0 \wedge \mathscr{G}_\delta$. Since $(\mathscr{F}_\sigma^0)_\delta \cap \mathscr{G}_\delta \subset \mathscr{F}_\sigma^0 \wedge \mathscr{G}_\delta$, it follows from the preceding paragraph that $K \setminus E \in \mathscr{F}_\sigma^0 \wedge \mathscr{G}_\delta$. However, Lemma 3.2 asserts that $K \setminus E \notin \mathscr{F}_\sigma$ so in particular $K \setminus E \notin \mathscr{F}_\sigma^0$.

Before verifying that the last inclusion of (3.9) is proper, we prove the assertion immediately following (3.9). It has already been shown that $E \in (\mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta) \wedge \mathscr{F}_\sigma$. Lemma 3.2 asserts that $E \notin \mathscr{G}_\delta$ so that $E \notin (\mathscr{F}_\sigma^0)_\delta \cap \mathscr{G}_\delta$. Hence

$$(\mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta) \wedge \mathscr{F}_\sigma \subsetneq (\mathscr{F}_\sigma^0)_\delta \cap \mathscr{G}_\delta.$$

On the other hand, we have seen that $K \setminus E \in (\mathscr{F}_\sigma^0)_\delta \cap \mathscr{G}_\delta$ but that $K \setminus E \notin \mathscr{F}_\sigma^0$. Therefore, $(\mathscr{F}_\sigma^0)_\delta \cap \mathscr{G}_\delta \subsetneq \mathscr{F}_\sigma^0$. Since

$$(\mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta) \wedge \mathscr{F}_\sigma \subset \mathscr{F}_\sigma^0,$$

it also follows that $\mathscr{F}_\sigma^0 \subsetneq (\mathscr{F}_\sigma^0)_\delta \cap \mathscr{G}_\delta$ and

$$\left(\mathscr{F}_\sigma^0\right)_\delta \cap \mathscr{G}_\delta \subsetneq \left(\mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta\right) \wedge \mathscr{F}_\sigma.$$

We shall now sketch a proof that the last inclusion of (3.9) is proper. The author is indebted to Professor A.H. Stone for providing essentially this proof in a written communication. The details of the following argument can be supplied using standard facts from topology which can be found, for example, in [18].

$\mathscr{F}_\sigma^0 \wedge \mathscr{G}_\delta \subset (\mathscr{F}_\sigma^0)_\delta$. There is a standard example [12; p. 278] of a subset H of the irrationals $\mathbf{P}$ such that H is an $\mathscr{F}_{\sigma\delta}$ but not a $\mathscr{G}_{\delta\sigma}$ subset of $\mathbf{P}$. There exists a homeomorphism φ of $\mathbf{P}$ onto a $\mathscr{G}_\delta$ subset of C which is contained in K. Thus $\varphi(H) \in \mathscr{F}_{\sigma\delta} \wedge \mathscr{G}_\delta = \mathscr{F}_{\sigma\delta}$. Since $\varphi(H) \subseteq K$ and K is of measure 0, it follows that $\varphi(H) \in (\mathscr{F}_\sigma^0)_\delta$. However, $\varphi(H) \notin \mathscr{F}_\sigma^0 \wedge \mathscr{G}_\delta$ since $\mathscr{F}_\sigma^0 \wedge \mathscr{G}_\delta \subseteq \mathscr{G}_{\delta\sigma}$. Indeed, if $\varphi(H) \in \mathscr{G}_{\delta\sigma}$, then H is a $\mathscr{G}_{\delta\sigma}$ subset of $\mathbf{P}$ contrary to assumption.

Theorem 3.3 is established.

4. Constructions with inner functions

With Theorem 3.1 of §3 proved, we proceed in this section to give constructions of inner functions which complete the proof of Theorem 2.4.

We start with a review of the definitions and basic facts concerning the classes $\mathscr{I}$, $\mathscr{B}$, and $\mathscr{S}$ referred to in §§1 and 2. The classes $\mathscr{I}$, $\mathscr{B}$, and $\mathscr{S}$ are the classes of *nonconstant* inner functions, Blaschke products, and singular inner functions respectively. By definition, f is an *inner function* if f is an analytic mapping of Δ into its closure $\bar{\Delta}$ such that $|f^*(\eta)| = 1$ for almost all η in C. A *Blaschke product* B is a function of the form

$$\eta z^m \prod_1^n (\bar{a}_k/|a_k|) L_{a_k} \quad \text{or} \quad \eta z^m \prod_1^\infty (\bar{a}_k/|a_k|) L_{a_k},$$

where $|\eta| = 1$, m and n are nonnegative integers, $(a_k)_1^\infty$ is a sequence taking values in $\Delta \setminus \{0\}$ with $\Sigma_1^\infty(1 - |a_k|) < +\infty$, and

$$L_a(z) = (a - z)/(1 - \bar{a}z), \quad a \in \Delta,\ z \in \hat{\mathbf{C}}.$$

(The convention $\prod_1^0 L_{a_k} = 1$ is used.) We say that B is *normalized* if $\eta = 1$ in a representation in one of the above forms. A *singular inner function* is a function of the form

$$S_\mu(z) = \exp\left\{ -\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\, d\mu(t) \right\}, \quad z \in \hat{\mathbf{C}} \setminus C, \tag{4.1}$$

where the *singular generating function* μ is a monotone nondecreasing function defined on the real numbers $\mathbf{R}$ satisfying $\mu' = 0$ a.e.,

$$\mu(t + 2\pi) = \mu(t) + \mu(2\pi),$$

and the normalization

$$2\mu(t) = \mu(t^+) + \mu(t^-), \quad t \in \mathbf{R}.$$

For convenience, we shall also allow unimodular multiples of S_μ as singular inner functions.

Recall that a Blaschke product is uniformly product convergent on compact subsets of the complement with respect to $\hat{\mathbf{C}}$ of the cluster set of its zeros [11; p. 227] and that Blaschke products restricted to Δ are inner functions [8; Theorem 2.4]. Recall also the following facts concerning singular inner functions. The function S_μ (defined in (4.1)) is analytic at each point of $\hat{\mathbf{C}} \setminus C$ [11; pp. 131–133] and

$$|S_\mu| = \exp(-u),$$

where

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi}[(1 - |z|^2)/|e^{it} - z|^2]\, d\mu(t), \quad z \in \mathbf{C} \setminus C.$$

The restriction of u to Δ is a nonnegative harmonic function for which the Fatou radial limit theorem implies $u^*(e^{it}) = \mu'(t)$ at each point t in $\mathbf{R}$, where the symmetric derivative

$$\mu'(t) = \lim_{h \to 0}\, [\mu(t + h) - \mu(t - h)]/(2h)$$

exists (finite or infinite) [8; Theorem 1.2]. Furthermore, a singular inner function restricted to Δ is in fact an inner function. In addition, a theorem of de la Vallée Poussin [17; p. 128] implies that a singular generating function μ has derivative μ' equal to $+\infty$ at some point in every neighborhood of t if and only if μ is not locally constant at t, whence S_μ is analytic at e^{it} if and only if S_μ has no radial limit value of 0 in some neighborhood of e^{it} if and only if μ is locally constant at t, for each $t \in \mathbf{R}$. In the sequel, we shall always view Blaschke products and singular inner functions as functions on the unit disk Δ.

All of the constructions of this section are dependent on the following result of Lohwater and Piranian [14; pp. 8–11].

THEOREM 4.1. *If E is a nonempty $\mathscr{F}_\sigma^{\,0} \cap \mathscr{G}_\delta$ set, then there exists a singular generating function μ such that $S_\mu \in \mathscr{S}_\rho^{\,0}$,*

$$\{S_\mu^* = 0\} = E, \tag{4.2}$$

and the full derivative

$$\mu'(t) = \begin{cases} +\infty, & e^{it} \in E, \\ 0, & e^{it} \in C \setminus E. \end{cases} \tag{4.3}$$

Though Lohwater and Piranian were only concerned with one-sided derivatives, it can be verified that the normalization $\mu(t^+) + \mu(t^-) = 2\mu(t)$ in their construction insures that the full derivative of μ exists at each point.

Two results of Frostman will also be used repeatedly in the constructions of this section. For convenience, we state them here as Theorems 4.2 and 4.3.

THEOREM 4.2 [9; pp. 107–109]. *An inner function is a Blaschke product if it admits no radial limit values of zero.*

Before stating Theorem 4.3, we introduce a convention and a definition, both stated for an infinite Blaschke product B, the corresponding agreements being understood, mutatis mutandis, when B is a finite Blaschke product.

Convention. The zeros of B will always be given by a sequence $(a_k)_1^\infty$ with the enumeration according to multiplicity.

DEFINITION 4.1. The Frostman function for B, denoted φ_B, is the map given by

$$\eta \mapsto \sum_1^\infty \frac{1 - |a_k|}{|\eta - a_k|}, \quad \eta \in C. \tag{4.4}$$

We now state the second theorem of Frostman.

THEOREM 4.3 [10; pp. 170–172]. *Let $\eta \in C$. Then $|b^*(\eta)| = 1$ for each subproduct b of the Blaschke product B if and only if $\varphi_B(\eta) < +\infty$.*

Our first goal will be to prove Lemma 4.2 below, which gives a necessary and sufficient condition for a Blaschke product B to be analytic at η in terms of the behavior of the Frostman function φ_B near η, $\eta \in C$. We first prove an elementary inequality, the statement of which appears without proof in [10; p. 171].

LEMMA 4.1. *If $a \in \Delta$ and $0 \le r \le 1$, then $\frac{1}{2}|1 - a| < |1 - ar|$.*

Proof. Observe first that

$$|1 - ar| \ge |1 - a| - |a - ar| = |1 - a| - |a|(1 - r).$$

Since $1 - r \leq |1 - ar|$, we conclude that

$$|1 - ar|(1 + |a|) \geq |1 - a|.$$

The lemma follows on noting that $|a| < 1$.

We now proceed to the statement and proof of Lemma 4.2. Note that this lemma is stated for an infinite Blaschke product with the corresponding facts for a finite Blaschke product being either trivial or empty.

LEMMA 4.2. *Let $\eta \in C$ and B be an infinite Blaschke product. Then B is analytic at η if and only if $\varphi_B(\eta) < +\infty$ and φ_B is continuous at η. Furthermore, if B is analytic at each point of a closed subset K of C, then the series* (4.4) *converges uniformly to φ_B on K (and consequently φ_B is finite-valued and continuous at each point of K).*

Proof. For convenience, we prove the last assertion first. To that end, recall that the set of analyticity of B in C is the complement in C of the cluster set of the zeros of B. Thus the zeros of B are bounded away from K. The assertion follows on estimating the terms of the series (4.4). Note that the zeros of B satisfy the Blaschke condition $\Sigma(1 - |a_k|) < +\infty$.

We turn now to the proof of the first assertion. Since B is analytic in some closed neighborhood of η, the *necessity* follows from the result of the preceding paragraph.

Consider *sufficiency*. Since $\varphi_B(\eta) < +\infty$, there exists a positive integer N such that $\Sigma_N^\infty(1 - |a_k|)/|\eta - a_k| < 1/4$. By the continuity of φ_B and hence of $\Sigma_N^\infty(1 - |a_k|)/|\zeta - a_k|$ at η, there exists an open arc A in C containing η such that

$$\sum_N^\infty \frac{1 - |a_k|}{|\zeta - a_k|} < \frac{1}{4}, \quad \zeta \in A. \tag{4.5}$$

Now for $\zeta \in A$, $k \geq N$, and $0 \leq r \leq 1$, we have $a_k \neq 0$ and

$$\begin{aligned}\left|1 - \frac{\bar{a}_k}{|a_k|}\frac{a_k - r\zeta}{1 - \bar{a}_k r\zeta}\right| &= \frac{|1 - \bar{a}_k r\zeta - |a_k| + (\bar{a}_k/|a_k|)r\zeta|}{|1 - \bar{a}_k r\zeta|} \\ &= \frac{(1 - |a_k|)[1 + (\bar{a}_k/|a_k|)r\zeta]}{|1 - \bar{a}_k r\zeta|} \\ &< \frac{2(1 - |a_k|)}{(1/2)|1 - \bar{a}_k\zeta|} \\ &< 4\left(\frac{1}{4}\right) \\ &= 1,\end{aligned}$$

use being made of Lemma 4.1 and (4.5). It follows that $\{a_k\}_N^\infty$ has no points in common with the open sector determined by A and so B can only have finitely many zeros in it. Hence $\{a_k\}_1^\infty$ does not cluster at η so that B is analytic at η. This completes the proof.

Our first main result of this section is Theorem 4.4. We start with a lemma which employs Lemma 4.2 and Theorems 4.1–4.3 in its proof.

LEMMA 4.3. *Let $E \in \mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta$, K a compact subset of $C \setminus E$, and $\varepsilon > 0$. Let $T \in (0,1)$. Then there exists a Blaschke product B for which*

$$|B^*(\eta)| = T, \quad \eta \in E, \tag{4.6}$$

$$\varphi_B(\eta) < +\infty, \quad \eta \in C \setminus E, \tag{4.7}$$

$$\varphi_B(\eta) < \varepsilon, \quad \eta \in K, \tag{4.8}$$

and φ_B is continuous at each point of $C \setminus \overline{E}$.

Proof. If $E = \emptyset$, then $B \equiv 1$ serves. Suppose that E is a nonempty $\mathscr{F}^0$ set. By Theorem 4.1 there exists $S \in \mathscr{S}_\rho^0$ such that $E = \{S^* = 0\}$. Theorem 4.2 implies that the inner function $b = L_T \circ S$ is a Blaschke product. Note that b is analytic at each point of $C \setminus E$ since S is, and that $b^*(\eta) = T$, $\eta \in E$. By Lemma 4.2, the Frostman function φ_b is finite-valued and continuous at each point of $C \setminus E$. Furthermore, the second assertion of the same lemma insures that by suppressing the factors associated with finitely many of the zeros of b if necessary, we can arrive at a Blaschke product B for which (4.8) holds. Then φ_B is still finite-valued and continuous at each point of $C \setminus E$. Though we can no longer assert that $B^*(\eta) = T$, $\eta \in E$, equation (4.6) remains valid. This completes the proof in this case.

Assume now that E is not an $\mathscr{F}^0$ set. By Corollary 3.4, we have $E = \cup_1^\infty F_k$ where $(F_k)_1^\infty$ is a sequence of *mutually disjoint nonempty* $\mathscr{F}^0$ sets. We claim now that $C \setminus E = \cup_1^\infty F_k^*$, where $(F_k^*)_1^\infty$ is a monotone nondescending sequence of closed subsets of C with the property that $K \subseteq F_1^*$ and $C \setminus \overline{E} = \cup_1^\infty \operatorname{int} F_k^*$. In fact, since E is a $\mathscr{G}_\delta$, it follows that $C \setminus E$ is an $\mathscr{F}_\sigma$ and there exists a sequence of closed sets $(K_j)_1^\infty$ such that $\cup_1^\infty K_j = C \setminus E$. Furthermore, since $C \setminus \overline{E}$ is an open subset of C, we have $C \setminus \overline{E} = \cup_1^\infty A_k$, where $(A_k)_1^\infty$ is a sequence of mutually disjoint open arcs (some, or possibly all of which may be empty). Now for each k, there exists a sequence $(A_{kj})_{j=1}^\infty$ of open subarcs of A_k, such that $A_k = \cup_{j=1}^\infty A_{kj}$ and $\overline{A}_{kj} \subseteq A_k$ for each j. Define

$$F_n^* = \left(\bigcup_{k=1}^n \bigcup_{j=1}^n \overline{A}_{kj} \right) \cup \left(\bigcup_1^n K_j \right) \cup K$$

for each positive integer n. From the defining properties of the A_{kj} and the K_j, we see that the sequence $(F_k^*)_1^\infty$ is as required.

Since the lemma has been proved for the case when E is an $\mathscr{F}^0$ set, there exists for each positive integer k, a Blaschke product B_k such that B_k fulfills the conditions of the lemma with F_k replacing E, $F_k^* \cup (\cup_1^{k-1} F_j)$ replacing K, and $\varepsilon/2^k$ replacing ε. We assume as we may, that each B_k is normalized. Let $B = \prod_1^\infty B_k$. We claim that *B is a convergent Blaschke product with the required properties.*

It is an immediate consequence of the definitions that if B is convergent, then

$$\varphi_B = \sum_1^\infty \varphi_{B_k}. \tag{4.9}$$

Furthermore, if the right hand side of (4.9) converges (to a finite value) for some $\eta \in C$, then the Blaschke condition is satisfied for the zeros of B and hence B is a convergent Blaschke product. Therefore, since $C \setminus E \neq \emptyset$, it suffices to show that

$$\sum_1^\infty \varphi_{B_k}(\eta) < +\infty, \quad \eta \in C \setminus E$$

in order to conclude that B is a convergent Blaschke product for which (4.7) holds.

Let $\eta \in C \setminus E$. Since $(F_k^*)_1^\infty$ is a monotone nondescending sequence with $\cup_1^\infty F_k^* = C \setminus E$, there exists a positive integer N for which $\eta \in F_k^*$ whenever $k > N$. Thus

$$\sum_{N+1}^\infty \varphi_{B_k}(\eta) < \sum_{N+1}^\infty \varepsilon/2^k = \varepsilon/2^N < +\infty.$$

Furthermore, since $\eta \in C \setminus E \subset C \setminus F_k$, we have $\varphi_{B_k}(\eta) < +\infty$ for each k. Thus $\sum_1^N \varphi_{B_k}(\eta) < +\infty$. We conclude that $\sum_1^\infty \varphi_{B_k}(\eta) < +\infty$ and hence B is a convergent Blaschke product and (4.7) is verified.

To verify (4.8) recall that $K \subseteq F_k^*$ for each k. Thus for $\eta \in K$ we have $\varphi_B(\eta) = \sum_1^\infty \varphi_{B_k}(\eta) < \sum_1^\infty \varepsilon/2^k = \varepsilon$ as required.

The continuity of φ_B at each point of $C \setminus \overline{E}$ is seen as follows. If K_1 is a compact subset of $C \setminus \overline{E}$, then there exists a positive integer N such that $k > N$ implies $K_1 \subset \operatorname{int} F_k^*$ since $\cup_1^\infty \operatorname{int} F_k^* = C \setminus \overline{E}$. Thus for $k > N$ we have $\varphi_{B_k}(\eta) < \varepsilon/2^k$, $\eta \in K_1$. Since $\sum_{N+1}^\infty \varepsilon/2^k = \varepsilon/2^N < +\infty$ and since φ_{B_k} is by assumption finite-valued and continuous at each point of $C \setminus F_k \supset K_1$ for each k, we conclude that the right hand side of (4.9) converges uniformly to a continuous function on K_1. Since K_1 is an arbitrary compact subset of $C \setminus \overline{E}$, it follows that φ_B is finite-valued and continuous at each point of $C \setminus \overline{E}$ as required.

It remains to verify (4.6). Let $\eta \in E$. Then there exists a unique positive integer k such that $\eta \in F_k$. When $j < k$ we have $\varphi_{B_j}(\eta) < +\infty$ since $\eta \in F_k \subset C \setminus F_j$. On the other hand $\varphi_{B_j}(\eta) < \varepsilon/2^j$ when $j > k$. Thus

$$\begin{aligned}(4.10)\qquad \varphi_{\Pi_{j\neq k}B_j}(\eta) &= \sum_{j<k} \varphi_{B_j}(\eta) + \sum_{j>k} \varphi_{B_j}(\eta)\\ &< \sum_{j<k} \varphi_{B_j}(\eta) + \sum_{j>k} \varepsilon/2^j\\ &< +\infty.\end{aligned}$$

Therefore, by Theorem 4.3 we conclude that $|(\Pi_{j\neq k}B_j)^*(\eta)| = 1$, and hence

$$|B^*(\eta)| = |B_k^*(\eta)| \left| \left(\prod_{j\neq k} B_j \right)^*(\eta) \right| = |B_k^*(\eta)| = T.$$

Since η in E was arbitrary, (4.6) follows. This completes the proof.

We now prove a proposition which has Theorem 4.4 as an easy consequence.

PROPOSITION 4.1. *Let $E \in (\mathscr{F}_\sigma^0)_\delta \cap \mathscr{G}_\delta$ and $W \in \mathscr{F}_\sigma^0$ such that $E \subseteq W$. Let K be a compact subset of $C \setminus W$ and $\varepsilon > 0$. Let $T \in (0,1)$. Then there exists a Blaschke product B such that*

$$(4.11)\qquad B^*(\eta) = 0, \quad \eta \in E,$$

$$(4.12)\qquad |B^*(\eta)| \in \{T^k\}_0^\infty, \quad \eta \in W \setminus E,$$

$$(4.13)\qquad \varphi_B(\eta) < +\infty, \quad \eta \in C \setminus W,$$

$$(4.14)\qquad \varphi_B(\eta) < \varepsilon, \quad \eta \in K,$$

and φ_B is continuous at each point of $C \setminus \overline{W}$.

Proof. Since $W \in \mathscr{F}_\sigma^0$, there exists a monotone nondescending sequence $(F_k)_1^\infty$ of $\mathscr{F}^0$ sets such that $\cup_1^\infty F_k = W$. Since $E \in \mathscr{G}_\delta$, there exists a monotone nonascending sequence $(G_k)_1^\infty$ of open sets such that $\cap_1^\infty G_k = E$. We assume also that

$$K \subseteq C \setminus G_1 \quad \text{and} \quad C \setminus \overline{W} \subset \bigcup_1^\infty \operatorname{int}(C \setminus G_k).$$

(These assumptions can be satisfied in essentially the same way as the corresponding assumptions concerning $(F_k^*)_1^\infty$ in the proof of Lemma 4.3 can.)

Since $E_k \equiv F_k \cap G_k \in \mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta$ and $C \setminus G_k \subseteq C \setminus E_k$, there exists a normalized Blaschke product B_k which is in the same relation to E_k, $C \setminus G_k$,

$\varepsilon/2^k$, and T as B is to E, K, ε, and T in Lemma 4.3 for each positive integer k. Let $B = \prod_1^\infty B_k$. We claim that B is a Blaschke product as required.

That B is convergent Blaschke product satisfying (4.13) and (4.14) with φ_B continuous at each point of $C \setminus \overline{W}$ is verified similarly to the verifications of the corresponding facts in Lemma 4.3. Since $|B_k^*(\eta)| = T\ (<1)$, $\eta \in E_k$ for each k, and for $\eta \in E$ there exists a positive integer N such that $\eta \in E_k$ for all $k \geq N$, we conclude that (4.11) holds.

It remains to check that (4.12) is valid. Let $\eta \in W \setminus E$. Let k (resp. j) be the first positive integer such that $\eta \in F_k$ (resp. $\eta \notin G_j$). We claim that

$$|B^*(\eta)| = \begin{cases} T^{j-k}, & j > k, \\ 1, & \text{otherwise}. \end{cases} \tag{4.15}$$

Suppose that $j > k$. If $j > m \geq k$ we have by assumption $|B_m^*(\eta)| = T$ since $\eta \in E_m$. Thus $|(\prod_k^{j-1} B_m)^*(\eta)| = T^{j-k}$. If $k > m \geq 1$ then $\varphi_{B_m}(\eta) < +\infty$ since $\eta \in C \setminus E_m$. Hence

$$\varphi_{\prod_{m<k} B_m}(\eta) = \sum_1^{k-1} \varphi_{B_m}(\eta) < +\infty$$

and Theorem 4.3 implies that $|(\prod_1^{k-1} B_m)^*(\eta)| = 1$. If $m \geq j$ then $\varphi_{B_m}(\eta) < \varepsilon/2^m$ since $\eta \in C \setminus G_m$. Therefore

$$\varphi_{\prod_{m \geq j} B_m}(\eta) = \sum_j^\infty \varphi_{B_m}(\eta) < \varepsilon/2^{j-1}.$$

Again it follows from Theorem 4.3 that $|(\prod_j^\infty B_m)^*(\eta)| = 1$. We conclude that

$$|B^*(\eta)| = \left|\left(\prod_1^{k-1} B_m\right)^*(\eta)\right| \left|\left(\prod_k^{j-1} B_m\right)^*(\eta)\right| \left|\left(\prod_j^{\infty} B_m\right)^*(\eta)\right| = T^{j-k}.$$

When $k \geq j$, the case $j > m \geq k$ does not come into play and the remaining part of the proof is similar. This completes the verification of the claim and (4.12) follows.

Proposition 4.1 is established.

We turn now to Theorem 4.4.

THEOREM 4.4. *If $E \in (\mathscr{F}_\sigma^0)_\delta \cap \mathscr{G}_\delta$ and $\alpha \in \Delta$, then there exists $B \in \mathscr{B}_\rho$ such that $E = \{B^* = \alpha\}$. Furthermore, if $W \in \mathscr{F}_\sigma^0$ with $E \subseteq W$, then B can be chosen so that it has radial limits of modulus 1 at each point of $C \setminus W$ and so that B is analytic at each point of $C \setminus \overline{W}$. When $\alpha = 0$, the additional condition that all of the subproducts of B have radial limits of modulus 1 at each point of $C \setminus W$ can be satisfied.*

Theorem 4.4 taken together with Theorem 3.1 proves (2.2) and (2.3) of Theorem 2.4. In fact the inclusions '$\subseteq$' follow from Theorem 3.1. The reverse inclusion in (2.3) is provided by the first assertion of Theorem 4.4. The reverse inclusion in (2.2) follows from the second assertion of Theorem 4.4 with $E \in \mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta$ and $W = E$.

Proof. Let $W \in \mathscr{F}_\sigma^0$ such that $E \subseteq W$. Such a set W exists since $E \in (\mathscr{F}_\sigma^0)_\delta$.

If $E = \emptyset$, then $B(z) = z$ is as required. For the remainder of the proof we assume that $E \neq \emptyset$.

Suppose that $\alpha = 0$. For E and W as above and arbitrary admissible K, T, and ε, let B be a Blaschke product as in Proposition 4.1. We claim that B is as required. Since $E \neq \emptyset$, we have B is nonconstant. The last assertion of the theorem follows from (4.13) and Theorem 4.3 and this assertion taken together with (4.12) and (4.11) insures that $E = \{B^* = 0\}$. The analyticity of B at each point of $C \setminus \overline{W}$ follows from the continuity of φ_B at each of these points and Lemma 4.2. This completes the proof in this case.

Suppose now that $\alpha \in \Delta \setminus \{0\}$. Let B be as in the preceding paragraph with the added requirement that $T \in (0,1)$ is chosen so that $|\alpha| \notin \{T^k\}_1^\infty$. Then by Theorem 4.2, the function $L_\alpha \circ B$ is a Blaschke product. That $L_\alpha \circ B$ has the required properties follows directly from the properties of B and L_α. This completes the proof of Theorem 4.4.

Theorem 4.4 can be applied to the problem of determining the sets where the radial limits of an inner function can fail to exist. In [14; pp. 14–15], Lohwater and Piranian proved the following.

THEOREM 4.5. *If $A \in \mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta$, then there exist $S \in \mathscr{S}$ and a countable set $E \subset C \setminus A$ such that*

$$(4.16) \qquad \liminf_{r \to 1} |S(r\eta)| = 0, \ \limsup_{r \to 1} |S(r\eta)| = 1, \quad \eta \in A,$$

$$(4.17) \qquad S^*(\eta) = 0, \quad \eta \in E,$$

and

$$(4.18) \qquad |S^*(\eta)| = 1, \quad \eta \in C \setminus (AUE).$$

Using Theorems 4.4 and 4.5, we prove the following.

COROLLARY 4.1. *If $E \in (\mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta) \wedge \mathscr{F}_\sigma$, then there exists an inner function f such that the subset of C where f fails to have a radial limit is precisely E.*

Note that Theorem 3.3 asserts that the proper inclusion $\mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta \subset (\mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta) \wedge \mathscr{F}_\sigma$ holds.

Proof. Let $A \in \mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta$ and $W \in \mathscr{F}_\sigma$ such that $E = A \cap W$. Let S be as in Theorem 4.5 relative to A and let $B \in \mathscr{B}_\rho$ such that

$$\{B^* = 0\} = A \cap (C \setminus W)\left[\in \left(\mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta\right) \wedge \mathscr{G}_\delta \subseteq \left(\mathscr{F}_\sigma^0\right)_\delta \cap \mathscr{G}_\delta\right]$$

as allowed by Theorem 4.4. Then $f = BS$ is seen to have the required property on noting that $C \setminus E = (C \setminus A) \cup [A \cap (C \setminus W)]$. This completes the proof of Corollary 4.1.

We turn now to Theorem 4.6 which has the inclusion $\mathscr{F}_\sigma^0 \subseteq \mathscr{Z}(\mathscr{S})$ of (2.4) as a consequence. For the statement and proof of Theorem 4.6, notation and terminology given in the paragraph containing (4.1) are used.

THEOREM 4.6. *If E is an $\mathscr{F}_\sigma^0$ set, then there exists a singular generating function μ such that*

$$\mu'(t) = +\infty, \quad e^{it} \in E, \tag{4.19}$$

$$\left\{S_\mu^* = 0\right\} = E, \tag{4.20}$$

and

$$\limsup_{r \to 1} |S_\mu(r\eta)| = 1, \quad \eta \in C \setminus E. \tag{4.21}$$

Proof. If $E = \emptyset$, take $\mu \equiv 0$ so that $S_\mu \equiv 1$. If E is a nonempty $\mathscr{F}^0$ set, then Theorem 4.1 is directly applicable.

Assume that $E \notin \mathscr{F}^0$. By Corollary 3.4, we have $E = \cup_1^\infty F_k$, where $(F_k)_1^\infty$ is a sequence of *mutually disjoint nonempty* $\mathscr{F}^0$ sets. We shall prove the existence of a sequence $(\mu_k)_1^\infty$ of singular generating functions so that

$$\mu = \sum_1^\infty \mu_k \tag{4.22}$$

is a singular generating function for which (4.19), (4.20), and (4.21) are satisfied.

We proceed by induction. Let $(s_k)_1^\infty$ be a strictly increasing sequence with $0 < s_k < 1$ for each k and $\prod_1^\infty s_k > 0$.

$n = 1$. By Theorem 4.1 there exists a singular generating function μ_1 such that the full derivative

$$\mu_1'(t) = \begin{cases} +\infty, & e^{it} \in F_1, \\ 0, & \text{otherwise.} \end{cases} \tag{4.23}$$

It is assumed that $\exp[-\mu_1(2\pi)] < s_1$. (Otherwise replace μ_1 by $\alpha\mu_1$ for α a suitably large positive constant.)

Inductive hypothesis. Suppose that $n \geq 1$ and the singular generating function μ_k has full derivative

$$\mu_k'(t) = \begin{cases} +\infty, & e^{it} \in F_k, \\ 0, & \text{otherwise}, \end{cases} \tag{4.24}$$

with

$$\left|\prod_1^n S_{\mu_j}(z)\right| \geq \prod_k^n s_j, \quad z \in \mathscr{L}_k = \left\{\left|\prod_1^k S_{\mu_j}(w)\right| = s_k\right\}, \tag{4.25}$$

for $k = 1, \ldots, n$, and

$$\mu_k(2\pi) < 1/2^{k+1}, \quad k = 2, \ldots, n. \tag{4.26}$$

$n + 1$. By Theorem 4.1 there exists a singular generating function ν such that (4.23) holds with μ_1 replaced by ν and F_1 replaced by F_{n+1}. Since S_{μ_k} is analytic and of modulus 1 at each point of $C \setminus F_k$ for $k = 1, \ldots, n$ and since F_{n+1} and $\cup_1^n F_k$ are disjoint closed subsets of C, we have $\overline{\cup_1^n \mathscr{L}_k} \subset \overline{\Delta} \setminus F_{n+1}$. Furthermore, since the singular inner functions $S_{\alpha\nu}$, $0 < \alpha \leq 1$, converge uniformly to the constant function 1 on compact subsets of $\overline{\Delta} \setminus F_{n+1}$ as the parameter α approaches 0, we can choose $\alpha > 0$ sufficiently small so that with $\mu_{n+1} = \alpha\nu$, the inequalities

$$|S_{\mu_{n+1}}(z)| \geq s_{n+1}, \quad z \in \bigcup_1^n \mathscr{L}_k, \tag{4.27}$$

and

$$\mu_{n+1}(2\pi) < 1/2^{n+1} \tag{4.28}$$

are valid.

We claim that the inductive hypothesis is satisfied with '$n + 1$' replacing 'n'. From the choice of μ_{n+1} and the inductive hypothesis, it is evident that (4.24) and (4.26) hold as required. We turn now to (4.25). From the inductive hypothesis and (4.27), we have

$$\begin{aligned} \left|\prod_1^{n+1} S_{\mu_j}(z)\right| &= \left|\prod_1^n S_{\mu_j}(z)\right| |S_{\mu_{n+1}}(z)| \\ &\geq \left(\prod_k^n s_j\right) s_{n+1} = \prod_k^{n+1} s_j, \quad z \in \mathscr{L}_k, \end{aligned} \tag{4.29}$$

for $k = 1, \ldots, n$. When $k = n + 1$ (and 'n' is replaced by '$n + 1$'), the inequality (4.25) is trivial. Thus our claim is verified.

On noting that the inductive hypothesis is satisfied when $n = 1$, the induction is completed.

Select (by the axiom of choice) a sequence $(\mu_k)_1^\infty$ of singular generating functions such that the initial section $(\mu_k)_1^n$ satisfies the inductive hypothesis for each positive integer n. By the conditions satisfied by each μ_k and the fact that (4.26) holds for $n \geq 2$, the series in (4.22) converges (uniformly on compact subsets of $\mathbf{R}$) and μ is a monotone nondecreasing function such that

$$\mu(t + 2\pi) = \mu(t) + \mu(2\pi) \quad \text{and} \quad 2\mu(t) = \mu(t^+) + \mu(t^-)$$

for $t \in \mathbf{R}$ with (4.19) satisfied. By a classical theorem of Fubini, $\mu' = 0$ a.e. since $\mu_k' = 0$ a.e. for each k, and it follows that μ is a singular generating function.

It remains to show that S_μ satisfies (4.20) and (4.21). If $t \in \mathbf{R}$ such that $e^{it} \in E$, then (4.19) and the Fatou radial limit theorem imply $S_\mu^*(e^{it}) = 0$. Thus the proof will be complete once (4.21) is verified. Referring to the definition of $\mathscr{L}_n$ (cf. (4.25)), we see that

$$C \setminus \bigcup_1^n F_k \subseteq sg(\mathscr{L}_n)$$

where $sg(z) = z/|z|$, $z \in \hat{\mathbf{C}} \setminus \{0, \infty\}$, since

$$\prod_1^n S_{\mu_k}(0) \leq S_{\mu_1}(0) = \exp[-\mu_1(2\pi)] < s_1 \leq s_n$$

and since $\prod_1^n S_{\mu_k}$ is analytic and of modulus 1 at each point of $C \setminus \cup_1^n F_k$ for each n. Hence a radius from 0 (which is not in $\mathscr{L}_n$) to a point of $C \setminus \cup_1^n F_k$ cuts $\mathscr{L}_n$ for each n. By the validity of (4.25) for all positive integers n, we have for each n that

$$|S_\mu(z)| = \left| \prod_1^\infty S_{\mu_k}(z) \right| \geq \prod_n^\infty s_k, \quad z \in \mathscr{L}_n. \tag{4.30}$$

Since $\prod_n^\infty s_k \to 1$ as $n \to \infty$, it follows that (4.21) holds. This completes the proof of Theorem 4.6.

Our next goal is to prove Theorem 4.7 which leads to the inclusion $\mathscr{F}_\sigma^0 \wedge \mathscr{G}_\delta \subseteq \mathscr{Z}_\alpha(\mathscr{B})$, $\alpha \in \Delta$, of (2.4).

Proposition 4.2. *Suppose that $E = W \cap H$, where $W \in \mathscr{F}_\sigma^0$ and $H \in \mathscr{G}_\delta$. Furthermore, suppose that $E^* \in \mathscr{F}_\sigma$ with $E^* \subseteq C \setminus W$, K is a compact subset of*

E^*, and $\varepsilon > 0$. Let $T \in (0,1)$. Then there exists a Blaschke product B such that

$$B^*(\eta) = 0, \quad \eta \in E, \tag{4.31}$$

$$|B^*(\eta)| \in \{T^k\}_0^\infty, \quad \eta \in W \setminus E, \tag{4.32}$$

$$\limsup_{r \to 1} |B(r\eta)| = 1, \quad \eta \in C \setminus W, \tag{4.33}$$

$$\varphi_B(\eta) < +\infty, \quad \eta \in E^*, \tag{4.34}$$

$$\varphi_B(\eta) < \varepsilon, \quad \eta \in K, \tag{4.35}$$

and φ_B is continuous at each point of $C \setminus \overline{W}$.

Proof. If $E \in (\mathscr{F}_\sigma^0)_\delta \cap \mathscr{G}_\delta$, then Proposition 4.1 is directly applicable. We remark only that (4.33) is guaranteed by (4.13) and Theorem 4.3.

Assume now that $E \notin (\mathscr{F}_\sigma^0)_\delta \cap \mathscr{G}_\delta$. Then $W \notin \mathscr{F}^0$ and Corollary 3.4 implies that $W = \cup_1^\infty W_k$ where $(W_k)_1^\infty$ is a sequence of mutually disjoint nonempty $\mathscr{F}^0$ sets. We assume, as we may, that $W_k \cap E \neq \emptyset$ for each k. Since W is of measure 0, we can suppose (by simply adding a point of $C \setminus W$ to K and E^* if necessary), that $K \neq \emptyset$. Furthermore, since $C \setminus \overline{W}$ is an $\mathscr{F}_\sigma$ set contained in $C \setminus W$, it can be assumed (by replacing E^* with $E^* \cup (C \setminus \overline{W})$ if necessary), that $C \setminus \overline{W} \subseteq E^*$. From these assumptions and the fact that $K \subseteq E^* \in \mathscr{F}_\sigma$, there exists a monotone nondescending sequence of closed sets $(F_k^*)_1^\infty$ such that $K \subseteq F_1^*$, $\cup_1^\infty F_k^* = E^*$, and $C \setminus \overline{W} \subseteq \cup_1^\infty \operatorname{int} F_k^*$ (cf. the proof of Lemma 4.3).

Since $E_k \equiv E \cap W_k \in (\mathscr{F}_\sigma^0)_\delta \cap \mathscr{G}_\delta$ and $W_k \in \mathscr{F}^0 \subset \mathscr{F}_\sigma^0$ with $\emptyset \neq E_k \subseteq W_k$, there exists a sequence of normalized infinite Blaschke products $(b_k)_1^\infty$ such that b_k satisfies the conditions of Proposition 4.1 with E_k, W_k, $(\cup_1^{k-1} W_j) \cup F_k^*$, and $\varepsilon/2^k$ replacing E, W, K, and ε respectively for each k. We shall show that B can be taken as $\prod_1^\infty B_k$, where for each k the function B_k is formed from b_k by possibly suppressing the factors corresponding to finitely many zeros.

Let $(s_k)_1^\infty$ be an increasing sequence satisfying $|b_1(0)| < s_k < 1$ for each k and $0 < \prod_1^\infty s_k$. We define $(B_n)_1^\infty$ by induction from $(b_n)_1^\infty$.

$n = 1$. Let $B_1 = b_1$ and define $\mathscr{L}_1 = \{|B_1(w)| = s_1\}$.

Inductive hypothesis. Suppose that $n \geq 1$ and there exists a finite sequence of Blaschke products $(B_k)_1^n$ such that b_k/B_k is a finite Blaschke product and

$$\left|\prod_1^n B_j(z)\right| \geq \prod_k^n s_j, \quad z \in \mathscr{L}_k = \left\{\left|\prod_1^k B_j(w)\right| = s_k\right\}, \tag{4.36}$$

for $k = 1, \ldots, n$.

$n + 1$. Observe first that b_k is analytic and of modulus 1 at each point of $C \setminus W_k$ (by the continuity of φ_{b_k} at each point of $C \setminus W_k$ and Lemma 4.2), $1 \le k \le n$. Hence the same is true for B_k, $1 \le k \le n$. Thus

$$\overline{\bigcup_1^n \mathscr{L}_k} \subset \Delta \cup \left(\bigcup_1^n W_k \right).$$

Since $(\bigcup_1^n W_k) \cap W_{n+1} = \emptyset$, the inclusion $\overline{\bigcup_1^n \mathscr{L}_k} \subset \bar{\Delta} \setminus W_{n+1}$ holds. Since b_{n+1} is analytic at each point of $\bar{\Delta} \setminus W_{n+1}$, it follows that b_{n+1} is uniformly product convergent on the compact set $\overline{\bigcup_1^n \mathscr{L}_k}$. We can therefore define B_{n+1} by suppressing the factors corresponding to finitely many zeros of b_{n+1} if necessary so that

$$|B_{n+1}(z)| \ge s_{n+1}, \quad z \in \bigcup_1^n \mathscr{L}_k. \tag{4.37}$$

Since the inductive hypothesis clearly holds when $n = 1$, it remains to check that the inductive hypothesis holds with '$n + 1$' replacing 'n'. With the definition given above for B_{n+1}, it is only necessary to check (4.36) (with '$n + 1$' replacing 'n').

If $k = n + 1$, then the inequality is trivial. Suppose now that $k \in \{1, \dots, n\}$. Then by the inductive hypothesis and (4.37) we have

$$\begin{aligned} \left| \prod_1^{n+1} B_j(z) \right| &= \left| \prod_1^n B_j(z) \right| |B_{n+1}(z)| \\ &\ge \left(\prod_k^n s_j \right) s_{n+1} = \prod_k^{n+1} s_j, \quad z \in \mathscr{L}_k. \end{aligned} \tag{4.38}$$

This completes the induction.

A sequence $(B_k)_1^\infty$ of normalized infinite Blaschke products can now be selected (by the axiom of choice) so that B_n is a subproduct of the Blaschke product b_n and the initial section $(B_k)_1^n$ satisfies (4.36) for each positive integer n. We claim that $B = \prod_1^\infty B_k$ is as required. That B is a convergent Blaschke product satisfying (4.34) and (4.35) with φ_B continuous at each point of $C \setminus \overline{W}$ is proved as in the proof of Lemma 4.3.

Since $B_k^*(\eta) = 0$, $\eta \in E_k$ for each k, and since $E = \bigcup_1^\infty E_k$, it follows that (4.31) holds.

We verify that (4.32) holds as follows. If $\eta \in W \setminus E$, then $\eta \in W_n \setminus E_n$ for some positive integer n since $W \setminus E = \bigcup_1^\infty (W_k \setminus E_k)$. Now

$$\left| \left(\prod_{k \ne n} B_k \right)^* (\eta) \right| = 1$$

since

$$\varphi_{\prod_{k \neq n} B_k}(\eta) = \sum_{k \neq n} \varphi_{B_k}(\eta) < \sum_{k<n} \varphi_{B_k}(\eta) + \varepsilon/2^n < +\infty$$

(using Theorem 4.3). Therefore, by assumption on B_n we have

$$|B^*(\eta)| = |B_n^*(\eta)| \left| \Big(\prod_{k \neq n} B_k \Big)^* (\eta) \right| = |B_n^*(\eta)| \in \{T^k\}_0^\infty$$

as required.

It remains only to check (4.33). Now

$$C \setminus \bigcup_1^n W_k \subseteq sg(\mathscr{L}_n)$$

(where $sg(z) = z/|z|$, $z \in \hat{\mathbf{C}} \setminus \{0, \infty\}$) since

$$\left| \prod_1^n B_k(0) \right| < |B_1(0)| = |b_1(0)| < s_1 \leq s_n$$

and $\prod_1^n B_k$ is analytic and of modulus 1 at each point of $C \setminus \bigcup_1^n W_k$. Hence a radius from 0 (which is not contained in $\mathscr{L}_n$) to a point of $C \setminus \bigcup_1^n W_k$ cuts $\mathscr{L}_n$ for each n. By the validity of (4.36) for all n, we have

$$|B(z)| = \left| \prod_1^\infty B_k(z) \right| \geq \prod_n^\infty s_k, \quad z \in \mathscr{L}_n. \tag{4.39}$$

Since $\prod_n^\infty s_k \to 1$ as $n \to \infty$, we conclude that (4.33) holds. This completes the proof of Proposition 4.2.

The next theorem is an easy consequence of Proposition 4.2.

THEOREM 4.7. *If $E \in \mathscr{F}_\sigma^0 \wedge \mathscr{G}_\delta$ and $\alpha \in \Delta$, then there exists $B \in \mathscr{B}$ such that $E = \{B^* = \alpha\}$. Furthermore, if $E = W \cap H$ where $W \in \mathscr{F}_\sigma^0$ and $H \in \mathscr{G}_\delta$, and $E^* \in \mathscr{F}_\sigma$ with $E^* \subseteq C \setminus W$, then B may be chosen in such a way that* (4.33) *holds, B is analytic at each point of $C \setminus \overline{W}$, and B has radial limits of modulus* 1 *at each point of E^*. When $\alpha = 0$, the additional condition that all of the subproducts of B have radial limits of modulus* 1 *at each point of E^* can be satisfied.*

The first assertion of Theorem 4.7 proves the inclusion

$$\mathscr{F}_\sigma^0 \wedge \mathscr{G}_\delta \subseteq \mathscr{Z}_\alpha(\mathscr{B}), \quad \alpha \in \Delta,$$

of (2.4).

Proof. Let $W \in \mathscr{F}_\sigma^0$ and $H \in \mathscr{G}_\delta$ such that $E = W \cap H$.

If $E = \emptyset$, then $B(z) = z$ is as required. For the remainder of the proof it is assumed that $E \neq \emptyset$.

Suppose that $\alpha = 0$. For E, W, and H as above and arbitrary admissible K, T, and ε, let B be a Blaschke product as in Proposition 4.2. That $E = \{B^* = 0\}$ is a direct consequence of (4.31), (4.32), and (4.33). The analyticity of B at each point of $C \setminus \overline{W}$ follows from the continuity of φ_B at each point of this set and Lemma 4.2. The assertion concerning E^* is a consequence of (4.34) and Theorem 4.3. The proof is completed for this case.

Suppose now that $\alpha \in \Delta \setminus \{0\}$. Let B be a Blaschke product as in the preceding paragraph with the added requirement that $T \in (0, 1)$ is chosen so that $|\alpha| \notin \{T^k\}_1^\infty$. Then by Theorem 4.2, the function $L_\alpha \circ B$ is a Blaschke product, where

$$L_\alpha(z) = (\alpha - z)/(1 - \bar{\alpha} z), \quad z \in \hat{\mathbf{C}}.$$

That $L_\alpha \circ B$ has the required properties follows from the properties of B and L_α.

Theorem 4.7 is established.

The next result shows that the inclusion (3.6) is best possible using purely topological and measure-theoretic considerations.

THEOREM 4.8. *If $E \in (\mathscr{F}_\sigma^0)_\delta$ and $\alpha \in \hat{\mathbf{C}} \setminus C$, then there exists a continuous function g: $\Delta \to \hat{\mathbf{C}}$ with $\lim_{r \to 1} |g(r\eta)| = 1$ a.e. such that*

$$\left\{\eta \in C \colon \lim_{r \to 1} g(r\eta) = \alpha\right\} = E.$$

Proof. By assumption $E = \bigcap_1^\infty E_k$ where each $E_k \in \mathscr{F}_\sigma^0$. Theorem 4.6 implies that there exists for each k, a singular generating function μ_k such that $\{S_{\mu_k}^* = 0\} = E_k$.

If $\alpha = 0$, let $g = \sum_1^\infty (1/2^k)|S_{\mu_k}|$. Since each singular inner function S_{μ_k} maps Δ into $\overline{\Delta}$, we have the uniform convergence and hence continuity of g. Using the fact that $|S_{\mu_k}|$ has radial limits equal to 1 on a set of full measure and the specific construction of g, we conclude that

$$\lim_{r \to 1} g(r\eta) = \sum_1^\infty (1/2^k)|S_{\mu_k}^*(\eta)| = \sum_1^\infty 1/2^k = 1, \tag{4.40}$$

for almost all η in C. Furthermore,

$$\left\{\lim_{r \to 1} g(r\eta) = 0\right\} = \bigcap_1^\infty \left\{S_{\mu_k}^* = 0\right\} = \bigcap_1^\infty E_k = E.$$

Thus g has the required properties, completing the proof in this case.

If $\alpha \in \hat{\mathrm{C}} \setminus (\mathrm{C} \cup \{0\})$ and g is as in the preceding paragraph, the function $L_\alpha \circ g$ or $1/(L_{1/\alpha} \circ g)$ serves according as $\alpha \in \Delta \setminus \{0\}$ or $\alpha \in \hat{\mathrm{C}} \setminus \bar{\Delta}$ respectively (the convention $1/\infty = 0$ being understood). The proof of Theorem 4.8 is thereby completed.

We remark that by Theorem 4.7 Blaschke products could have been used in the proof instead of singular inner functions.

The final theorem of this section, Theorem 4.9, implies Theorem 2.5. We first prove a lemma which is essentially a result of C. Belna communicated to the author by G. Piranian.

LEMMA 4.4. *If $S \in \mathscr{S}$, then there exists $f \in \mathscr{I}$ such that*

$$\{S^* = 0\} \subseteq \{f^* = 1\}, \tag{4.41}$$

$$\{0 < |S^*| < 1\} \subseteq \{0 \le |f^*| < 1\}, \tag{4.42}$$

and

$$\{|S^*| = 1\} \subseteq \{|f^*| = 1, f^* \neq 1\}. \tag{4.43}$$

Proof. Let $\log S$ be an analytic logarithm of S and let M be the Möbius transformation

$$z \mapsto (z+1)/(z-1), \quad z \in \hat{\mathrm{C}}.$$

Define $f = M \circ \log S$. Since $\log S$ has a negative real part and M maps $\{\operatorname{Re} z < 0\}$ onto Δ, we have f is an analytic mapping of Δ into itself.

If $\eta \in C$ such that $|S^*(\eta)| = 1$, then $(\log S)^*(\eta)$ is pure imaginary. Now M maps the imaginary axis onto $C \setminus \{1\}$. Thus $f^*(\eta) \in C \setminus \{1\}$ and (4.43) is verified. Since $|S^*| = 1$ a.e., we have $|f^*| = 1$ a.e. so it follows that $f \in \mathscr{I}$.

If $\eta \in C$ such that $S^*(\eta) = 0$, then $(\log S)^*(\eta) = \infty$. Since $M(\infty) = 1$, we conclude that $f^*(\eta) = 1$. The inclusion (4.41) follows.

Finally, if $\eta \in C$ such that $0 < |S^*(\eta)| < 1$, then $(\log S)^*(\eta)$ has negative real part. Since M maps $\{\operatorname{Re} z < 0\}$ onto Δ, we conclude that $0 \le |f^*(\eta)| < 1$ and (4.42) is verified.

The proof of Lemma 4.4 is complete.

The second assertion of the following theorem was pointed out to the author by Professor G. Piranian.

THEOREM 4.9. *Let $\alpha \in C$.*

(i) *If $E \in \mathscr{F}_\sigma{}^0 \cap \mathscr{G}_\delta \setminus \{\emptyset\}$, then there exists $B \in \mathscr{B}_\rho^0$ such that $\{B^* = \alpha\} = E$. In fact, B may be chosen so that $|B^*(\eta)| = 1$ for all $\eta \in C$.*

(ii) *If $E \in \mathscr{F}_\sigma{}^0 \setminus \{\emptyset\}$, then there exists $f \in \mathscr{I}$ such that $f^*(\eta) = \alpha$, $\eta \in E$.*

Proof. (i) By Theorem 4.1, there exists $S \in \mathscr{S}_\rho^0$ such that $\{S^* = 0\} = E$. Let f be as in Lemma 4.4 relative to S and $B = \alpha f$. It follows from Lemma 4.4 that $B \in \mathscr{I}_\rho^0$, $\{B^* = \alpha\} = E$, and $|B^*(\eta)| = 1$ for all η in C. By Theorem 4.2, we have B is a Blaschke product so that $B \in \mathscr{B}_\rho^0$.

(ii) By Theorem 4.6, there exists $S \in \mathscr{S}$ such that $\{S^* = 0\} = E$. The existence of f having the required properties now follows from Lemma 4.4.

5. Applications for singular monotone functions

In this section, μ will always denote a singular generating function (cf. the paragraph containing (4.1)). The object of this section is to apply some of the results of the preceding sections to the study of the set $\{\mu' = +\infty\}$. All of the results stated in the following theorem are valid when $\mu'(t)$ is interpreted as the full derivative

$$\lim_{h \to 0} [\mu(t+h) - \mu(t)]/h$$

or the symmetric derivative

$$\lim_{h \to 0} [\mu(t+h) - \mu(t-h)]/(2h)$$

at each point t in $\mathbf{R}$ where the limit exists (finite of infinite).

THEOREM 5.1. (1) *The set* $\{e^{it}: 0 < \mu'(t) \leq +\infty\}$ *is contained in some* $\mathscr{F}_\sigma^0$ *set. On the other hand, if E is an* $\mathscr{F}_\sigma^0$ *set, then there exists* μ *for which* $E = \{e^{it}: \mu'(t) = \infty\}$.

(2) *If* μ' *exists at each point, then* μ *is locally constant at each point of an open dense subset of* $\mathbf{R}$ *and in particular,*

$$\{e^{it}: 0 < \mu'(t) \leq +\infty\}$$

is nowhere dense. In this case

$$\{e^{it}: T \leq \mu'(t) \leq +\infty\}$$

is an $(\mathscr{F}_\sigma^0)_\delta \cap \mathscr{G}_\delta$ *set for each* $T \in (0, +\infty]$.

(3) *If* μ' *exists and is* 0 *or* $+\infty$ *at each point, then*

$$\{e^{it}: \mu'(t) = +\infty\}$$

is an $\mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta$ *set. Conversely, if E is an* $\mathscr{F}_\sigma^0 \cap \mathscr{G}_\delta$ *set, then there exists* μ *such that*

$$E = \{e^{it}: \mu'(t) = +\infty\} \quad \text{and} \quad C \setminus E = \{e^{it}: \mu'(t) = 0\}$$

(*Lohwater and Piranian*).

Proof. Observe first that the results of §3 concerning the classes $\mathcal{MI}$, $\mathcal{MI}_\rho$, and $\mathcal{MI}_\rho^0$ do not depend on the analytic structure of the functions involved. Thus if f: $\Delta \to \hat{\mathbf{C}}$ is a continuous function with

$$\lim_{r\to 1} f(r\eta) \in C$$

for almost all η in C, then we can apply the results of that section for $\mathcal{MI}$ to f. If we add the assumption that $\lim_{r\to 1} f(r\eta)$ exists (resp. is of modulus 1 or 0) at each point of C, then the results for $\mathcal{MI}_\rho$ (resp. $\mathcal{MI}_\rho^0$) apply.

Let $f = |S_\mu|$. Then by the Fatou radial limit theorem we have

$$\lim_{r\to 1} f(re^{it}) = \exp[-\mu'(t)]$$

for each $t \in \mathbf{R}$ where $\mu'(t)$ exists. On applying the third assertion of Proposition 3.1, Corollary 3.3, and inclusion (3.8) for the present contexts, we conclude that the first assertion of (1), the second assertion of (2), and the first assertion in (3) hold.

By [3; Cor. 2.3], a function $g \in \mathcal{M}$ is analytic and of modulus 1 in an open dense subset of C if $\lim_{r\to 1}|g(r\eta)| = 1$ a.e. and $\lim_{r\to 1}|g(r\eta)|$ exists at each point of C. On recalling that S_μ is analytic at e^{it} if and only if μ is locally constant at t, for each $t \in \mathbf{R}$, the first assertion of (2) follows.

The second assertions of (1) and (3) follow from Theorems 4.6 and 4.1 respectively. This completes the proof.

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