ILLINOIS JOURNAL OF MATHEMATICS
Volume 29, Number 3, Fall 1985

INVERSE IMAGES OF THETA DIVISORS

BY

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In this paper we will explain a method for computing the linear equivalence class of the inverse image of the theta divisor via a morphism. Then we will give some illustrations of the method. First we will recall the basic definitions (for instance [2]).

Let C be a smooth complete algebraic curve of genus g over an algebraically closed field. A family $\mathscr{L}$ of invertible sheaves on C parameterized by a scheme S is an invertible sheaf $\mathscr{L}$ on the product $C \times S$. Such a family $\mathscr{L}$ is said to have degree d if, for all points of S, the invertible sheaf $\mathscr{L}|_{C\times s}$ on C has degree d.

For any integer d we have the Picard variety P_d of degree d together with a family $\mathscr{Q}_d$ of invertible sheaves on C of degree d parameterized by P_d. The family $\mathscr{Q}_d$ is universal in the following sense. For any family $\mathscr{L}$ of degree d parameterized by S, we have a classifying morphism $\psi_{\mathscr{L}}\colon S \to P_d$ such that

$$\mathscr{L} \approx (1_C \times \psi_{\mathscr{L}})^* \mathscr{Q}_d \otimes \pi_S^* \mathscr{N}$$

where $\mathscr{N}$ is an invertible sheaf on S. There is a one-to-one correspondence between such families $\mathscr{L}$ modulo $\otimes\, \pi_S^*\mathscr{N}$ and morphisms $S \to P_d$. The Picard variety P_d is a smooth projective variety.

We will be studying the variation of the cohomology in a family of degree $g-1$ on C parameterized by S. For this purpose let D be an effective divisor of degree g on C. We have a short exact sequence,

$$0 \to \mathscr{L} \to \mathscr{L}(D \times S) \to \mathscr{L}(D \times S)|_{D\times S} \to 0$$

of sheaves on $C \times S$. Taking direct images under the projection π_S onto S, we have the long exact sequence

(+)
$$0 \to \pi_{S*}\mathscr{L} \to \pi_{S*}(\mathscr{L}(D \times S)) \xrightarrow{\alpha_{\mathscr{L}}} \pi_{S*}(\mathscr{L}(D \times S)|_{D\times S}) \to R^1\pi_{S*}\mathscr{L} \to 0$$

as $R^1\pi_{S*}(\mathscr{L}(D \times S))$ is zero because $\mathscr{L}(D \times S)$ is a family of degree $2g-1$.

Received April 22, 1983.

Furthermore $\alpha_{\mathscr{L}}$ is a homomorphism between locally free $\mathcal{O}_S$-modules of rank g whose formation commutes with base extension; i.e., for any morphism f: $T \to S$, $\alpha_{f^*\mathscr{L}}$ is naturally isomorphic to $f^*(\alpha_{\mathscr{L}})$.

Recall that the theta divisor is the closed subset

$$\theta \equiv \left\{ p \in P_{g-1} | H^0\left(C, \mathscr{Q}_{g-1}|_{C\times p}\right) \neq 0\right\}$$

of the Picard variety P_{g-1}. We will abbreviate $\mathscr{Q}_{g-1}$ by $\mathscr{Q}$, P_{g-1} by P and $\alpha_{\mathscr{Q}}$ by α. The subset θ is a divisor and satisfies the following:

THEOREM 1. (a) *θ is the scheme of zeroes of the homomorphism $\Lambda^g\alpha$ between two invertible $\mathcal{O}_P$-modules.*

(b) *We have an isomorphism*

$$\mathcal{O}_P(\theta) \approx \Lambda^g\pi_{P*}\left(\mathscr{Q}(D\times P)|_{D\times P}\right) \otimes \left(\Lambda^g\pi_{P*}\mathscr{Q}(D\times P)\right)^{\otimes -1}.$$

(c) *The sheaf $\pi_{P*}\mathscr{Q} = 0$ and θ is the first Chern class of the coherent sheaf $R^1\pi_{P*}\mathscr{Q}$, which is a torsion sheaf with support θ.*

Proof. The statement (a) is a local statement on P and appears implicitly in [1]. It is explained in detail in the notes [2]. The isomorphism in (b) follows directly from (a) and the definitions. By the sequence $(+)$ the sheaf $\pi_{P*}\mathscr{Q}$ is a torsion-free coherent sheaf and by (a) it is supported in the proper subset θ of P. Hence $\pi_{P*}\mathscr{Q}$ is zero. Furthermore by $(+)$ and (a), θ is the support of the coherent sheaf $R^1\pi_{P*}\mathscr{Q}$. Also by $(+)$ and the vanishing of $\pi_{P*}\mathscr{Q}$ the first Chern class of $R^1\pi_{P*}\mathscr{Q}$ is

$$\begin{aligned} &c_1\left(\pi_{P*}\left(\mathscr{Q}(D\times P)|_{D\times P}\right) - c_1\left(\pi_{P*}\mathscr{Q}(D\times P)\right)\right.\\ &\qquad = c_1\left(\Lambda^g\pi_{P*}\left(\mathscr{Q}(D\times P)|_{D\times P}\right) \otimes \left(\Lambda^g\pi_{P*}\mathscr{Q}(D\times P)\right)^{\otimes -1}\right)\end{aligned}$$

which by (b) equals $c_1(\mathcal{O}_P(\theta)) = \theta$. This proves (c), Q.E.D.

The previous result in the universal case may be specialized to more general families of invertible sheaves. Let $\mathscr{L}$ be a family of invertible sheaves of degree $g-1$ on C parameterized by a variety X. In this case we have:

THEOREM 2. *Assume that the divisor $\psi_{\mathscr{L}}^{-1}(\theta)$ of X is defined. Then:*

(a) *$\psi_{\mathscr{L}}^{-1}(\theta)$ is the scheme of zeroes of the homomorphism $\Lambda^g\alpha_{\mathscr{L}}$ between two invertible $\mathcal{O}_X$-modules.*

(b) *We have an isomorphism*

$$\mathcal{O}_X\left(\psi_{\mathscr{L}}^{-1}(\theta)\right) \approx \Lambda^g\pi_{X*}\left(\mathscr{L}(D\times X)|_{D\times X}\right) \otimes \left(\Lambda^g\pi_{X*}\mathscr{L}(D\times X)\right)^{\otimes -1}.$$

(c) *The sheaf $\pi_{X*}\mathscr{L}$ is zero and $R^1\pi_{X*}\mathscr{L}$ has support $\psi_{\mathscr{L}}^{-1}(\theta)$.*

Proof. The statement (b) follows from (a) as before. Similarly (c) follows from (a) and the sequence $(+)$ as $\psi_{\mathscr{L}}^{-1}(\theta)$ is a proper subset of X. For (a), if we apply the statement (a) of Theorem 1, the naturality of $\alpha_{\mathscr{Q}}$ and the projection formula for the complication due to the factor $\pi_X^*\mathscr{N}$ in the isomorphism

$$(1, \psi_{\mathscr{L}})^*\mathscr{Q} \approx \mathscr{L} \otimes \pi_X^*\mathscr{N},$$

we immediately deduce the truth of (a), Q.E.D.

In a similar vein we may even say something when $\psi_{\mathscr{L}}^{-1}(\theta)$ is not defined.

THEOREM 3. *Assume that X is a smooth variety. Then*

(a) *we have an isomorphism*

$$\psi_{\mathscr{L}}^*(\mathscr{O}_P(\theta)) \approx \Lambda^g \pi_{X*}(\mathscr{L}(D \times X)|_{D \times X}) \otimes (\Lambda^g \pi_{X*}\mathscr{L}(D \times X))^{\otimes -1}$$

or

(b) *the divisor class of $\psi_{\mathscr{L}}^*(\theta)$ is $c_1(R^1\pi_{X*}\mathscr{L}) - c_1(\pi_{X*}\mathscr{L})$.*

Proof. The two statements are equivalent by the exact sequence $(+)$. The proof of (b) is analogous to the last proof. The only difference is that one pulls back the isomorphism from statement (b) of Theorem 1, Q.E.D.

The rest of this paper will consist of examples.

Application 1. Consider the family $\mathscr{L} \equiv \pi_1^*\mathscr{M}(-\Delta)$ parameterized by C where $\mathscr{M}$ is an invertible sheaf on C of degree g and Δ is the diagonal of $C \times C$. The classifying morphism $\psi_{\mathscr{L}}$ sends a point d of C to the point $c_1(\mathscr{M}) - d$ of P. The inverse image $\psi_{\mathscr{L}}^{-1}(\theta)$ is defined if and only if $H^0(C, \mathscr{M}) = 1$. The long exact sequence of direct images of $0 \to \pi_1^*\mathscr{M}(-\Delta) \to \pi_1^*\mathscr{M} \to \pi_1^*\mathscr{M}|_\Delta \to 0$ is just

$$0 \to \pi_{2*}\mathscr{L} \to H^0(C, \mathscr{M}) \otimes \mathscr{O}_C \to \mathscr{M} \to R^1\pi_{2*}\mathscr{L} \to H^1(C, \mathscr{M}) \otimes \mathscr{O}_C \to 0.$$

Thus $c_1(R^1\pi_{2*}\mathscr{L}) - c_2(\pi_{2*}\mathscr{L}) = c_1(\mathscr{M})$. Hence by Theorem 3(b), $\psi_{\mathscr{L}}^*(\theta)$ is $c_1(\mathscr{M})$. If $H^0(C, \mathscr{M}) = 1$, $\psi_{\mathscr{L}}^{-1}(\theta)$ is the sole point of the linear system of $\mathscr{M}$.

The next application is similar.

Application 2. Consider the family $\mathscr{L} \equiv \pi_1^*\mathscr{M}(\Delta)$ parameterized by C where $\mathscr{M}$ is an invertible sheaf on C of degree $g-2$. The classifying morphism $\psi_{\mathscr{L}}$ sends a point d of C to the point $c_1(\mathscr{M}) + d$ of P. The inverse image $\psi_{\mathscr{L}}^{-1}(\theta)$ is defined if and only if $H^0(C, \mathscr{M}) = 0$. The long exact sequence of direct images of $0 \to \pi_1^*\mathscr{M} \to \pi_1^*\mathscr{M}(\Delta) \to \pi_1^*\mathscr{M}(\Delta)|_\Delta \to 0$ is just

$$\begin{aligned} 0 \to H^0(C, \mathscr{M}) \times \mathscr{O}_C \to \pi_{2*}\mathscr{L} \to \mathscr{M} \otimes \Omega_C^{\otimes -1} \\ \to H^1(C, \mathscr{M}) \otimes \mathscr{O}_C \to R^1\pi_{2*}\mathscr{L} \to 0. \end{aligned}$$

In this case we conclude that $\psi^*_{\mathscr{L}}(\theta) = -c_1(\mathscr{M}) + c_1(\Omega_C)$. If $\psi^{-1}_{\mathscr{L}}(\theta)$ is defined, it is the sole point of the linear system of $\Omega_C \otimes \mathscr{M}^{\otimes -1}$.

The following example is a combination of these two results.

Application 3. Consider the family $\mathscr{L} \equiv \pi_1^*\mathscr{H}(\Delta_2 - \Delta_3)$ parameterized by the product $C \times C$ where $\mathscr{H}$ is an invertible sheaf on C of degree $g - 1$ and Δ_i denotes the inverse image $\pi_{1i}^{-1}(\Delta)$ of the diagonal under projection onto the first and i-th factors. The classifying morphism $\psi_{\mathscr{L}}$ sends a point (d_1, d_2) in the product $C \times C$ to the point $c_1(\mathscr{H}) + d_1 - d_2$ of P. The inverse image $\psi^{-1}_{\mathscr{L}}(\theta)$ is defined if and only if $H^0(C, \mathscr{H}) \leq 1$. The long exact sequence of direct images of

$$0 \to \pi_1^*\mathscr{H}(\Delta_2 - \Delta_3) \to \pi_1^*\mathscr{H}(\Delta_2) \to \pi_1^*\mathscr{H}(\Delta_2)|_{\Delta_3} \to 0$$

is just

$$0 \to \pi_{23*}\mathscr{L} \to \pi_{23*}\mathscr{I} \to \pi_2^*\mathscr{H}(\Delta) \to R^1\pi_{23*}\mathscr{L} \to R^1\pi_{23*}\mathscr{I} \to 0$$

where $\mathscr{I} = \pi_1^*\mathscr{H}(\Delta_2)$. Thus we have the relation

$$c_1\big(R^1\pi_{23*}\mathscr{L}\big) \to c_1(\pi_{23*}\mathscr{L}) = c_1\big(R^1\pi_{23*}\mathscr{I}\big) - c_1(\pi_{23*}\mathscr{I}) + C \times c_1(\mathscr{H}) + \Delta.$$

The long exact sequence of direct images of

$$0 \to \pi_1^*\mathscr{H} \to \pi_1^*\mathscr{H}(\Delta_2) \to \pi_1^*\mathscr{H}(\Delta_2)|_{\Delta_2} \to 0$$

is just

$$\begin{aligned} 0 &\to H^0(C_1\mathscr{H}) \otimes \mathscr{O}_{C\times C} \to \pi_{23*}\mathscr{I} \to \pi_1^*\big(\mathscr{H} \otimes \Omega_C^{\otimes -1}\big) \\ &\to H^1(C, \mathscr{H}) \otimes \mathscr{O}_{C\times C} \to R^1\pi_{23*}\mathscr{I} \to 0. \end{aligned}$$

Hence $c_1(R^1\pi_{23*}\mathscr{I}) - c_1(\pi_{23*}\mathscr{I}) = -c_1(\mathscr{H} \otimes \Omega_C^{\otimes -1}) \times C$. All in all we have

$$\begin{aligned} \psi^{-1}_{\mathscr{L}}(\theta) &= c_1\big(R^1\pi_{23*}\mathscr{L}\big) - c_1(\pi_{23*}\mathscr{L}) \\ &= C \times c_1(\mathscr{H}) + c_1\big(\Omega_C \otimes \mathscr{H}^{\otimes -1}\big) \times C + \Delta. \end{aligned}$$

If $H^0(C, \mathscr{H}) = 1$, then $\psi^{-1}_{\mathscr{L}}(\theta)$ contains the diagonal Δ and, hence

$$\psi^{-1}_{\mathscr{L}}(\theta) = C \times D + D' \times C + \Delta$$

where D and D' are the sole points of the linear systems of $\mathscr{H}$ and $\Omega_C \otimes \mathscr{H}^{\otimes -1}$ respectively.

We can do another example where the inverse image $\psi_{\mathscr{L}}^{-1}(\theta)$ is not defined because it is related to the self-intersection of θ.

Application 4. Let D be the universal divisor of degree $g-1$ on C. So

$$D = \{(c, E) \in C \times C^{(g-1)} | c \text{ is contained in } E\}.$$

Consider the family $\mathscr{L} \equiv \mathscr{O}_{C \times C^{(g-1)}}(D)$ parameterized by the symmetric product $C^{(g-1)}$. The classifying morphism $\psi_{\mathscr{L}}$ of this family is the abelian integral $\int: C^{(g-1)} \to P = P_{g-1}$. The image of this morphism is exactly the theta divisor θ. In this case we can use Theorem 3 to compute the linear equivalence class $(\int)^*(\theta)$. The long exact sequence of direct images of

$$0 \to \mathscr{O}_{C \times C^{(g-1)}} \to \mathscr{O}_{C \times C^{(g-1)}}(D) \to \mathscr{O}_{C \times C^{(g-1)}}(D)|_D \to 0$$

is

$$0 \to \mathscr{O}_{C^{(g-1)}} \to \pi_{C^{(g-1)}*}\mathscr{L} \to \Theta_{C^{(g-1)}} \to H^1(C, \mathscr{O}_C) \otimes \mathscr{O}_{C^{(g-1)}} \to R^1\pi_{C^{(g-1)}*}\mathscr{L} \to 0$$

because $\pi_{C \times C^{(g-1)}*}\mathscr{O}(D)|_D$ is isomorphic to the sheaf $\Theta_{C^{(g-1)}}$ of regular vector fields on $C^{(g-1)}$ by [4]. Therefore we may conclude that

$$\left(\int\right)^*(\theta) = -c_1(\Theta_{C^{(g-1)}}) = \text{the canonical class of } C^{(g-1)}.$$

The last example will involve correspondences [3]. Recall that an invertible sheaf $\mathscr{L}$ on the product $C \times D$ with another smooth complete curve is called a correspondence from D to C. If $\deg_C \mathscr{L}$ denotes the degree of $\mathscr{L}$ along the fibers over D, $\mathscr{L}$ is a family of invertible sheaves on C of degree $\deg_C \mathscr{L}$ parameterized by D. The classifying morphism $\psi_{\mathscr{L}}: D \to P_{\deg_C \mathscr{L}}$ is constant if and only if $\mathscr{L}$ is a trivial correspondence by the universal mapping property of the Picard variety.

Application 5. Let $\mathscr{L}$ be a correspondence from D to C where $\deg_C \mathscr{L} = g-1$. Then $\psi_{\mathscr{L}}^{-1}(\theta)$ is defined if and only if there is a point d of D such that $H^0(C, \mathscr{L}|_{C \times d}) = 0$.

LEMMA 6. $\deg(\psi_{\mathscr{L}}^*(\theta)) = N(\mathscr{L})$ *where* $N(\mathscr{L})$ *is the numerical function of* $\mathscr{L}$ [3].

Proof. By the definition of N we need to see that $\deg(\psi_{\mathscr{L}}^*(\theta)) = -\chi(\mathscr{L})$ where χ is the Euler characteristic as $\deg_C \mathscr{L} = g-1$. By the Leray spectral sequence for the projection π_D, $\chi(\mathscr{L}) = \chi(\pi_{D*}\mathscr{L}) - \chi(R^1\pi_{D*}\mathscr{L})$. On the other hand by the Riemann-Roch formula for D,

$$\chi(\pi_{D*}\mathscr{L}) - \chi(R^1\pi_{D*}\mathscr{L}) = \deg c_1(\pi_{D*}\mathscr{L}) + \operatorname{rank}(\pi_{D*}\mathscr{L}) \cdot\chi(\mathscr{O}_D) - \deg c_1(R^1\pi_{D*}\mathscr{L}) - \operatorname{rank}(R^1\pi_{D*}\mathscr{L})\cdot\chi(\mathscr{O}_D).$$

As the two ranks are equal using the Theorem 3 this equals $-\deg(\psi_{\mathscr{L}}^{-1}(\theta))$, Q.E.D.

Thus the fundamental theorem, "(a) $N(\mathscr{L}) \geq 0$ and (b) $N(\mathscr{L}) = 0 \Leftrightarrow \mathscr{L}$ is a trivial correspondence", is equivalent to the numerical positivity of the theta divisor θ on P. Also by the theory of abelian varieties this is equivalent to the amplitude of θ.

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