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A MODEL OF ADAMS-HILTON TYPE FOR FIBER SQUARES

BY

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Introduction

Let

$$\begin{array}{ccc} A & \longrightarrow & E \\ {\scriptstyle \hat{g}}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{\;f\;} & B \end{array} \qquad (1)$$

be a fiber square; i.e., g: $E \to B$ is a fibration and $\hat{g}$ the pull-back of g via f. Eilenberg and Moore [8] have given a very beautiful theory for the cohomology of the pull-back total space A, which has been expanded upon by many researchers [4], [9], [11], [14]. In this paper a new chain complex is presented whose homology equals that of A when all spaces are simply connected. It relies heavily upon the Adams-Hilton construction and is therefore described as having "Adams-Hilton type".

Certain properties and consequences are especially significant. The new chain complex may be taken to be locally finite whenever X, B, and E have locally finite homology. Over a field, we obtain another proof of the existence of the Eilenberg-Moore spectral sequence, and an explicit perturbation theory is constructed for its E_1 term. We also demonstrate that this spectral sequence is a homotopy invariant. As applications, the Eilenberg-Moore spectral sequence is shown to degenerate when f and g are "p-homogeneous" maps and we compute anew the cohomology of the free loop space on any formal space.

1. Chain ringoids

In Sections 1 and 2 we shall develop the algebraic tools and techniques needed in the remainder of the paper. In particular, we study objects called "chain ringoids", which are a generalization of the objects obtained from the two-sided cobar construction. As motivation, chain ringoids will later provide a framework into which both cobar constructions and constructions of Adams-

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Hilton type conveniently fit, thus allowing us to make the connection between them. The main result of this section, Theorem 1.5, asserts that quasi-isomorphisms of certain chain complexes always have inverses with certain desirable properties.

We begin with a series of algebraic definitions. The geometric motivation, as we shall see later in greater detail, is the desire to understand the singular chains on the space of those paths on a given space which start in one subspace and end in another. Allowing only paths which start and end at the base point yields an H-group, whose cubical singular chains form a ring. Allowing more paths yields an H-groupoid, whose cubical singular chains form something we call a "ringoid".

A *ringoid* is an abelian group $(P, +)$, together with two subgroups K and L and a pairing μ: $K \otimes L \to P$ satisfying the following axioms:

(a) If $R = K \cap L$, then $\mu(R \otimes R) \subseteq R$, and $\mu|_{R \otimes R}$ makes R into a (generally non-commutative) ring with unity.

(b) $\mu(K \otimes R) \subseteq K$ and $\mu|_{K \otimes R}$ makes K into a right R-module.

(c) $\mu(R \otimes L) \subseteq L$ and $\mu|_{R \otimes L}$ makes L into a left R-module.

(d) This diagram commutes:

$$\begin{array}{ccc} K \otimes R \otimes L & \xrightarrow{\mu \otimes 1} & K \otimes L \\ {\scriptstyle 1 \otimes \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ K \otimes L & \xrightarrow{\mu} & P. \end{array}$$

The action of the pairing μ on a ringoid will be called multiplication and will be denoted by a raised dot or by juxtaposition. A *morphism* between two ringoids f: $P \to P'$ must have $f(K) \subseteq K'$, $f(L) \subseteq L'$, and be a morphism of rings, modules or groups when suitably restricted.

Ideally one might expect a "ringoid" to be for rings what a "groupoid" is for groups. That is, one of the operations, in this case multiplication, would not always be defined, but it would satisfy the usual axioms whenever they made sense. The object we have defined would be a very special kind of ringoid with this more general definition. The geometric motivation for this restriction is that we shall consider composing two paths in our path space only when the first ends at the base point and the second starts there. Roughly speaking, the K, R, and L of the ringoid correspond respectively to the subspaces of paths which end at the base point, which are loops at the base point, and which start at the base point.

By a *graded group* we mean a $\mathbf{Z}$-graded abelian group which is zero in all negative dimensions. A *graded ringoid* is a graded group P together with two graded subgroups K and L and a pairing μ: $K \otimes L \to P$ satisfying (a) through (d) above. Here we assume the pairing is of graded groups, i.e.,

$\mu(K_k \otimes L_l) \subseteq P_{k+l}$. A *morphism* of graded ringoids must have degree zero. Let $f, g\colon P \to P'$ be two morphisms of graded ringoids. A *derivation of degree r from f to g* is a degree r homomorphism $F\colon P \to P'$ of graded groups such that

$$F(x \cdot y) = F(x) \cdot g(y) + (-1)^{r \cdot |x|} f(x) \cdot F(y) \tag{2}$$

whenever $x \cdot y$ is defined in P. Note that this implies that we must have $F(K) \subseteq K'$, $F(L) \subseteq L'$, and $F(R) \subseteq R'$. Note also that setting $x = y = 1$ yields $F(1) = 0$.

Fix a commutative ring with unity S. A graded S-module $M = \{M_n\}$ is *augmented* if and only if it comes with a surjection $\varepsilon\colon M \to M_0 \to S$ and a right inverse $\varepsilon'\colon S \to M_0 \to M$, in which case we let $\overline{M}$ denote $\ker(\varepsilon)$. For any graded S-module V, TV is the tensor algebra (over S) on V, which is always augmented via

$$S \xrightarrow{=} T_0V \to TV.$$

It is always assumed that TV inherits its grading from V; the gradation on TV induced by the number of factors will be called the *lower gradation* and will only occasionally be used.

Assume henceforth that all tensor products are over S. A *tensor ringoid* is any object of the form $P = M \otimes TV \otimes N$, for V a free graded and M and N augmented free graded S-modules, with $\overline{M}$ and $\overline{N}$ also free. A tensor ringoid becomes a graded ringoid by setting $K = M \otimes TV$, $L = TV \otimes N$, choosing the obvious pairing, and letting the augmentations dictate how $R = TV$ embeds in K and in L. Note that a tensor ringoid is itself naturally augmented. A *morphism* of tensor ringoids is any map of graded ringoids which also preserves the augmentation. The lower gradation on TV is extended over $M \otimes TV \otimes N$ by assigning all of M and N to lower degree zero.

As $\overline{M}$, V, and $\overline{N}$ are required to be S-free, we may define a set of *generators* for the tensor ringoid $P = M \otimes TV \otimes N$ to be a disjoint union of S-bases for $\overline{M}$, V, and $\overline{N}$. By a slight abuse of terminology we shall sometimes refer to "the" generators of P even though the set of generators is by no means canonical. None of our results will depend upon which set of generators is used.

Our first four lemmas will be used to show that it suffices to do certain constructions and to check certain formulas on the generators of a tensor ringoid. We state them here for convenient reference, but the proofs, which are straightforward formal verifications, are postponed to the appendix.

LEMMA 1.1. *Let $P = M \otimes TV \otimes N$ and $P' = M' \otimes TV' \otimes N'$ be two tensor ringoids. Suppose for $i = 0,1,2$, f_i': $(\overline{M} \oplus V \oplus \overline{N}) \to P'$ has*

$$f_i'(\overline{M}) \subseteq \overline{M' \otimes TV'}, \quad f_i'(V) \subseteq \overline{TV'} \quad \text{and} \quad f_i'(\overline{N}) \subseteq \overline{TV' \otimes N'},$$

where f_0', f_1', f_2' are maps of graded S-modules of degree $0,0,r$ respectively. Viewing $\overline{M}$, V, and $\overline{N}$ as submodules of P, f_0', f_1' extend uniquely to morphisms of tensor ringoids f_0, f_1: $P \to P'$ and f_2' extends uniquely to a derivation f_2 from f_0 to f_1.

LEMMA 1.2. *Let $P = M \otimes TV \otimes N$ and $P' = M' \otimes TV' \otimes N'$ be tensor ringoids and let f, g: $P \to P'$ be two morphisms of tensor ringoids. Suppose h_0, h_1: $P \to P$ are morphisms of tensor ringoids and h_2 is a derivation of degree r from h_0 to h_1, and likewise for h_0', h_1', h_2': $P' \to P'$. Suppose further that $gh_i = h_i'f$ on the generators of P for $i = 0,1,2$. Then $gh_i = h_i'f$ on all of P, $i = 0,1,2$.*

Keeping in mind that the motivating examples for studying ringoids will be certain singular chain complexes and certain approximations to them, we now introduce a differential on tensor ringoids. A *chain ringoid* is a tensor ringoid P, together with a derivation d from the identity to the identity of degree -1, such that $d^2 = 0$ and $d(\overline{P}) \subseteq \overline{P}$. A morphism of chain ringoids is a morphism of tensor ringoids commuting with the respective differentials. By Lemma 1.2 this happens if and only if it commutes for a set of generators. It is easily seen that a derivation d of odd degree from the identity to itself has $d^2 = 0$ if and only if $d^2(x_j) = 0$ for each generator x_j of P.

The reader familiar with two-sided cobar constructions will no doubt recognize chain ringoids as a generalization of them. We shall make this connection precise in Section 4.

Chain ringoids will be a central concept throughout this paper. They are obviously a special kind of chain complex over S, and a morphism of chain ringoids is a special kind of chain map. We also restrict our attention to a special kind of chain homotopy. Let f, g: $(P, \delta) \to (Q, d)$ be two morphisms between chain ringoids and let F be a derivation of degree $+1$ from f to g. F is a *derivation homotopy* from f to g if and only if

$$(F\delta + dF)(x) = f(x) - g(x) \tag{3}$$

for all $x \in P$. This is a generalization of "algebra homotopy" as defined in [5].

LEMMA 1.3. *Let f, g: $(P, \delta) \to (Q, d)$ be two morphisms of chain ringoids and let F be a derivation of degree $+1$ from f to g. Then F is a derivation homotopy from f to g if and only if formula* (3) *holds when x is a generator of P.*

We shall also make use of a kind of "derivation homotopy between derivations".

LEMMA 1.4. *Let* $f, g\colon (P, \delta) \to (Q, d)$ *be two morphisms of chain ringoids and suppose* F, G, H *are derivations from* f *to* g *of degree* $+1, +1, +2$ *respectively. Then the formula* $(H\delta - dH)(x) = (F - G)(x)$ *holds for all* $x \in P$ *if and only if it holds for all* x *belonging to a set of generators for* P.

In Section 2 we shall construct a way of combining certain simple chain ringoids into more complicated ones. We shall want to deduce that a map between chain ringoids induces an isomorphism of homology when this is known to be true for the smaller pieces. The first step is to show that any suitable quasi-isomorphism has a suitable inverse. Although it is known [7] that a chain map inducing an isomorphism of homology has a chain homotopy inverse when the complexes in question are free, we shall need a stronger result concerning inverses up to derivation homotopy.

A *chain tensor algebra* is a chain ringoid $(P = M \otimes TV \otimes N, d)$ for which $M = N = S$. We denote it by (TV, d) or simply TV.

THEOREM 1.5. *Let* $f\colon (TM, \delta) \to (TV, d)$ *be a morphism of chain tensor algebras inducing an isomorphism of homology. Then there is a morphism*

$$g\colon (TV, d) \to (TM, \delta)$$

of chain tensor algebras together with derivation homotopies F *from* 1 *to* gf *and* G *from* 1 *to* fg.

Note. This is an immediate consequence of [5] when S is a field.

Proof. See the appendix.

One application of Theorem 1.5 concerns the "homogeneous homology" of a chain ringoid $P = (M \otimes TV \otimes N, \delta)$. Since $\delta(\overline{P}) \subseteq \overline{P}$, δ is non-decreasing on lower degree; i.e., δ applied to a product involving t or more generators from V must equal a linear combination of products involving t or more generators from V. For x a product of generators of lower degree t, write $\delta(x) = x_0 + x_1 + x_2 + \cdots$, where x_i has lower degree $t + i$, and set $\delta_0(x) = x_0$. Extend δ_0 to an S-module map $\delta_0\colon P \to P$. (P, δ_0) is also a chain ringoid, called the *homogeneous approximation* to (P, δ_0), and the homology of (P, δ_0) is called the *homogeneous homology* of (P, δ).

LEMMA 1.6. *There is a functor* $\mathscr{H}$ *from chain ringoids to chain ringoids which sends each chain ringoid to its homogeneous approximation. Furthermore, whenever* F *is a derivation homotopy from* f *to*

$$g\colon (P, \delta) \to (Q, d),$$

then there is a derivation homotopy F_0 from $\mathscr{H}(f)$ to

$$\mathscr{H}(g)\colon (P, \delta_0) \to (Q, d_0).$$

Proof. For $f\colon (P, \delta) \to (Q, d)$ a morphism of chain ringoids, and for x a product of generators in P of lower degree t, let $f_0(x)$ be the component of $f(x)$ in Q with lower degree t. Extending f_0 to a map of S-modules $f_0\colon P \to Q$ gives a map satisfying $f_0\delta_0 = d_0 f_0$, so we may define $\mathscr{H}(f)$ to be f_0. Likewise let $F_0(x)$ be the component of $F(x)$ with the same lower degree as x when x is a product of generators, and extend linearly. The desired formulas which assert that F_0 is a derivation homotopy from f_0 to g_0 are just the lower degree t components of the formulas (2) and (3).

An immediate corollary which we state without further proof is:

LEMMA 1.7. *Suppose the morphism $f\colon (TM, \delta) \to (TV, d)$ of chain tensor algebras induces an isomorphism on homology. Then f also induces an isomorphism of homogeneous homology.*

Note. A converse to this lemma when S is a field is supplied in [5].

2. A homotopy invariance theorem

The main result of this section asserts that the homology of a certain construction, which associates a chain ringoid to a diagram of spaces and maps, is a homotopy invariant. We shall later use this result frequently to simplify computations and comparisons. We begin with several definitions and observations which will remain relevant throughout the paper. Then we state the theorem; its proof, which utilizes eight technical lemmas, is contained in the appendix.

Our first definition is very general. Let $\mathscr{D}$ be any category. The category of *corners* in $\mathscr{D}$ has as objects all diagrams in $\mathscr{D}$ of the form

$$X \xrightarrow{f} B \xleftarrow{g} E \tag{4}$$

and as morphisms all commuting diagrams in $\mathscr{D}$ of the form

$$\begin{array}{ccccc} X' & \xrightarrow{f'} & B' & \xleftarrow{g'} & E' \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ X & \xrightarrow{f} & B & \xleftarrow{g} & E. \end{array} \tag{5}$$

The terminology "corner" comes from the fact that we wish to think of (4) as the maps involved in the lower right corner of the pull-back diagram (1). For $\mathscr{F}$: $\mathscr{D} \to \mathscr{D}'$ a functor, $\mathscr{F}$ induces a functor from corners of $\mathscr{D}$ to corners of $\mathscr{D}'$. By a slight abuse of notation, the latter functor will also be denoted $\mathscr{F}$.

For an arbitrary graded S-module M, let sM denote its suspension, i.e., $(sM)_n = M_{n-1}$, and let M^+ denote the augmented graded module $M^+ = S \oplus sM$, the first summand occurring in degree zero. For (TM, d) a chain tensor algebra, consider $M^+ \otimes TM$. We can extend

$$s\colon M \overset{\approx}{\to} sM \hookrightarrow M^+ \to M^+ \otimes TM$$

to an S-module map $s\colon TM \to M^+ \otimes TM$ of degree $+1$ by setting $s(1) = 0$ and

$$s(x_{j_1} \cdot \cdots \cdot x_{j_t}) = s(x_{j_1}) x_{j_2} \cdot \cdots \cdot x_{j_t}.$$

This done, we may extend d over all of $M^+ \otimes TM$ by the formula

$$ds(x) = x - sd(x) \tag{6}$$

for $x \in \overline{TM}$. Define $\mathscr{K}(TM, d)$ to be $(M^+ \otimes TM, d)$. $\mathscr{K}$ is a functor from chain tensor algebras to chain ringoids. It is known (e.g., the proof in [2] works for the general case) that $d^2 = 0$ and that $\mathscr{K}(TM, d)$ is acyclic, i.e., the augmentation $(S, 0) \leftrightarrows \mathscr{K}(TM, d)$ is a chain equivalence.

Similarly we may define s' as the composition

$$M \overset{\approx}{\to} sM \to M^+ \hookrightarrow TM \otimes M^+$$

and extend s' over TM by $s'(1) = 0$ and

$$s'(x_{j_1} \cdot \cdots \cdot x_{j_t}) = (-1)^{|x_{j_1} \cdots x_{j_{t-1}}|} x_{j_1} \cdot \cdots \cdot x_{j_{t-1}} s'(x_{j_t}).$$

Formula (6) with s' replacing s extends the differential d over $TM \otimes M^+$, and we let $\mathscr{L}(TM, d)$ denote $(TM \otimes M^+, d)$. $\mathscr{L}$ is also a functor from chain tensor algebras to acyclic chain ringoids. The need for a sign in the formula for s' may come as a surprise, but it is there because the formula

$$d(xy) = d(x)y + (-1)^{|x|} x d(y), \tag{7}$$

which specializes (2), is not right-left symmetric, so (TM, d) is not a right-left symmetric object.

A construction of central significance in this paper is called the "associated chain ringoid".

DEFINITION 2.1. Let

$$(TM, d_M) \xrightarrow{f} (TV, d_V) \xleftarrow{g} (TN, d_N) \tag{8}$$

be a corner of chain tensor algebras. f and g induce maps of tensor ringoids

$$f^+ \colon M^+ \otimes TM \to M^+ \otimes TV, \qquad g^+ \colon TN \otimes N^+ \to TV \otimes N^+.$$

Extend d_M, d_N as above over $M^+ \otimes TM, TN \otimes N^+$ to obtain $\mathscr{K}(TM, d_M)$, $\mathscr{L}(TN, d_N)$. Define d on generators of $M^+ \otimes TV \otimes N^+$ by

$$d|_{M^+} = f^+ \circ d_M, \qquad d|_V = d_V, \qquad d|_{N^+} = g^+ \circ d_N,$$

and extend d over all of $M^+ \otimes TV \otimes N^+$ by formula (7). $d^2 = 0$ on generators of $M^+ \otimes TV \otimes N^+$ hence $d^2 = 0$. The *associated chain ringoid* to the corner (8) is

$$\mathscr{T}\left(TM \xrightarrow{f} TV \xleftarrow{g} TN\right) = (M^+ \otimes TV \otimes N^+, d).$$

We will abbreviate

$$\mathscr{T}\left(TM \xrightarrow{f} TV \xleftarrow{g} TN\right)$$

to $\mathscr{T}(TM, TV, TN)$ or to $\mathscr{T}(f, g)$ when no confusion can result. It is easily checked that $\mathscr{T}$ is a functor from corners of chain tensor algebras to chain ringoids. Note that

$$\mathscr{T}\left(TM \xrightarrow{=} TM \xleftarrow{\varepsilon'} S\right) = \mathscr{K}(TM, d_M)$$

and

$$\mathscr{T}\left(S \xrightarrow{\varepsilon'} TM \xleftarrow{=} TM\right) = \mathscr{L}(TM, d_M).$$

We are now almost ready to state the homotopy invariance theorem. Let CTA denote the category of chain tensor algebras over S. Let TOP denote the category of pointed topological spaces and continuous maps, or a subcategory. A functor $\mathscr{F}$: TOP $\to$ CTA satisfies the *derivation homotopy axiom* if and only if whenever $f, g \colon X \to Y$ are homotopic, there is a derivation homotopy from $\mathscr{F}(f)$ to $\mathscr{F}(g) \colon \mathscr{F}(X) \to \mathscr{F}(Y)$. We shall observe in Sections 3 and 4 that both the Adams-Hilton construction and the cobar functor satisfy this axiom.

THEOREM 2.2. *Let* $\mathscr{F}$: TOP $\to$ CTA *be any functor from a subcategory of pointed topological spaces to chain tensor algebras which satisfies the derivation homotopy axiom. Let*

$$\begin{array}{ccccc} X' & \xrightarrow{f'} & B' & \xleftarrow{g'} & E' \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ X & \xrightarrow{f} & B & \xleftarrow{g} & E \end{array} \tag{9}$$

be any diagram in TOP *which commutes up to homotopy. Suppose* α, β, *and* γ *are homotopy equivalences or, more generally, that the induced maps* $H_*\mathscr{F}(\alpha)$, $H_*\mathscr{F}(\beta)$, *and* $H_*\mathscr{F}(\gamma)$ *on homology are isomorphisms. Then there is a morphism*

$$\psi\colon \mathscr{T}\mathscr{F}\left(X' \xrightarrow{f'} B' \xleftarrow{g'} E'\right) \to \mathscr{T}\mathscr{F}\left(X \xrightarrow{f} B \xleftarrow{g} E\right)$$

between the associated chain ringoids of these corners such that ψ *induces an isomorphism on homology and on homogeneous homology.*

Proof. See the appendix.

3. The model of Adams-Hilton type

In this section we develop the main theorem, proving the existence of a "small" chain complex whose homology equals that of the fiber homotopy pull-back A for a corner (4) of simply-connected spaces. When the coefficient ring S is a field, the dual of this chain complex is filtered and gives rise to a spectral sequence converging to $H^*(A; S)$. We observe that this spectral sequence is a homotopy invariant and explore some of its properties.

Our overall goal in this paper is to understand (1) better, in particular to evaluate $H_*(A; S)$ when g is a fibration. When g is not required to be a fibration, the pull-back

$$X \times_B E = \{(x, e) \in X \times E \mid f(x) = g(e)\}$$

still exists, but the homotopy type of $X \times_B E$ is no longer a homotopy invariant of f and g. The proper generalization for our purposes of (1) to arbitrary corners is called the "fiber homotopy pull-back". Starting with any corner

$$X \xrightarrow{f} B \xleftarrow{g} E$$

of spaces, make g into a fibration $\hat{g}$: $\hat{E} \to B$ in the usual way, i.e.,

$$\hat{E} = \{(\omega, e) \in B^I \times E | \omega(1) = g(e)\} \quad \text{and} \quad \hat{g}(\omega, e) = \omega(0).$$

The *fiber homotopy pull-back* of the corner

$$X \xrightarrow{f} B \xleftarrow{g} E$$

is the pull-back $X \times_B \hat{E}$. This is well known to be a homotopy invariant, and to have the same homotopy type as the pull-back if g was a fibration to start with. A morphism of corners (5) in which α, β, γ are weak homotopy equivalences gives rise to a weak homotopy equivalence of the fiber homotopy pull-backs.

A very important special kind of fiber homotopy pull-back occurs when f and g are inclusions. In that case, $\hat{E} = \{\omega \in B^I | \omega(1) \in E\}$ and the fiber homotopy pull-back is $A = \{\omega \in B^I | \omega(1) \in E$ and $\omega(0) \in X\}$, the space of paths in B starting in X and ending in E. The space A has the same homotopy type as the space of Moore paths in B from X to E, i.e.,

$$\{\omega: [0, r] \to B \text{ for some } r \geq 0 | \omega(0) \in X, \omega(r) \in E\}.$$

The latter space will henceforth be denoted P^B_{XE}. For simplicity we define a *bipair* (B, X, E) to be a pointed space B together with two (pointed) subspaces X and E such that $X \cap E = *$. A bipair (B, X, E) is *m-connected* if and only if B, X, and E are all m-connected. Clearly, a bipair is a special kind of corner in which both morphisms are inclusions. For a bipair (B, X, E), the map

$$p: P^B_{XE} \to X \times E, \qquad p(\omega) = (\omega(0), \omega(r))$$

is a fibration.

We recall next some basic facts and notation about the Adams-Hilton construction [2]. Let X be any 1-connected CW complex and let $CU_*(\Omega X)$ denote the chain complex of non-degenerate cubical singular chains on $\Omega X = P^X_{**}$ with coefficients in S. The Adams-Hilton construction over S, which we denote by $\mathscr{A}$, associates to each such X an associative differential graded algebra $(\mathscr{A}(X), d_X)$ and a chain map θ_X: $\mathscr{A}(X) \to CU_*(\Omega X)$ such that

$$(\theta_X)_*: H_*(\mathscr{A}(X), d_X) \xrightarrow{\approx} H_*(\Omega X; S)$$

is an isomorphism. The choice of $\mathscr{A}(X)$ depends upon the CW decomposition of X and the choice of d_x is not canonical in general even for a given CW structure. However, $\mathscr{A}$ is functorial in the sense that for any f: $X \to Y$ and for any suitable models $\mathscr{A}(X)$ and $\mathscr{A}(Y)$ for X and Y, there is a chain map

$\mathscr{A}(f)$: $(\mathscr{A}(X), d_X) \to (\mathscr{A}(Y), d_Y)$ making

$$\begin{array}{ccc} H_*(\mathscr{A}(X), d_X) & \xrightarrow{H_*(\mathscr{A}(f))} & H_*(\mathscr{A}(Y), d_Y) \\ \downarrow{\scriptstyle (\theta_X)_*} & & \downarrow{\scriptstyle (\theta_Y)_*} \\ H_*(\Omega X; S) & \xrightarrow{(\Omega f)_*} & H_*(\Omega Y; S) \end{array} \tag{10}$$

commute. If in addition $g: Y \to Z$, $\mathscr{A}(gf)$ can be taken to be $\mathscr{A}(g) \circ \mathscr{A}(f)$.

Algebraically, $\mathscr{A}(X)$ is a free tensor algebra over S. In particular, if $X = * \cup \bigcup_{i \in \Gamma} e_i$ is a CW decomposition for X with each $|e_i| \geq 2$, we may take $\mathscr{A}(X) = S\langle a_i | i \in \Gamma\rangle$ with $|a_i| = |e_i| - 1$. $(\mathscr{A}(X), d_X)$ is in fact a chain tensor algebra and the induced maps $\mathscr{A}(f)$ are morphisms in CTA. This makes it possible to define $\mathscr{T}\mathscr{A}$, a (multi-valued) functor from corners of 1-connected CW complexes to chain ringoids. As we shall see shortly, the "model of Adams-Hilton type" referred to in the title is precisely $\mathscr{T}\mathscr{A}$ applied to a corner of spaces.

Another property of $\mathscr{A}$ we shall use is that the differential d_X on $\mathscr{A}(X)$ can be extended over the larger S-module $\mathscr{B}(X) \approx C_*^c(X) \otimes \mathscr{A}(X)$, $C_*^c(X)$ denoting the cellular chains of X with coefficients in S. When this is done, $(\mathscr{B}(X), d_X)$ is acyclic and θ_X extends to a chain map

$$\theta_X: \mathscr{B}(X) \to CU_*(P_{X*}^X).$$

As is easy to see from the formulas in [2], $(\mathscr{B}(X), d_X)$ is precisely $\mathscr{K}(\mathscr{A}(X), d_X)$. Using parallel reasoning, θ_X could as well be extended to a chain map

$$\theta_X: \mathscr{L}(\mathscr{A}(X), d_X) \to CU_*(P_{*X}^X).$$

A final property worth noting is that when $Y \subseteq X$ is a subcomplex, d_X may be chosen to be an extension of any suitable d_Y, under the natural embedding $\mathscr{A}(Y) \hookrightarrow \mathscr{A}(X)$.

For bipairs, the main theorem may be deduced from these observations alone. A *CW bipair* (B, X, E) is assumed to exhibit X and E as subcomplexes of B, and a *1-connected CW bipair* is taken to have trivial 1-skeleton. When we consider the Adams-Hilton models of the spaces in such a bipair (B, X, E), we always assume that d_B, θ_B are chosen so as to extend d_X and d_E, θ_X and θ_E, respectively.

A 1-connected CW bipair (B, X, E) determines two chain complexes, namely $CU_*(P_{XE}^B)$ and, when it is viewed as a corner, $\mathscr{T}\mathscr{A}(X \hookrightarrow B \hookleftarrow E)$.

THEOREM 3.1. *There is a natural isomorphism* $\tilde{\theta}_*: H_*\mathscr{T}\mathscr{A} \to H_*(P_{\cdot\cdot}^{\cdot}; S)$ *of functors from* 1*-connected CW bipairs to graded S-modules.*

Proof. For any such bipair (B, X, E), note that

$$\theta_X \colon \mathscr{K}(\mathscr{A}(X), d_X) \to CU_*(P^X_{X*}) \hookrightarrow CU_*(P^B_{X*}),$$
$$\theta_B \colon (\mathscr{A}(B), d_B) \to CU_*(P^B_{**}),$$

and

$$\theta_E \colon \mathscr{L}(\mathscr{A}(E), d_E) \to CU_*(P^E_{*E}) \hookrightarrow CU_*(P^B_{*E})$$

are multiplicative and consistent where their domains overlap, so together they define $\tilde{\theta}\colon \mathscr{T}\mathscr{A}(X \hookrightarrow B \hookleftarrow E) \to CU_*(P^B_{XE})$. In the sense of diagram (10), $\tilde{\theta}$ induces a natural transformation $\tilde{\theta}_*$ between the homologies of these two functors.

To prove $\tilde{\theta}_*$ an isomorphism, we first reduce to the case when E is locally finite, since we will then want to use that $X \times E$ is also a CW complex. The reduction is made in the usual way: $E = \varinjlim_{E_\alpha \in \mathscr{E}} \{E_\alpha\}$, where $\mathscr{E}$ is the partially ordered collection of all locally finite subcomplexes of E. That

$$\mathscr{T}\mathscr{A}\left(X \overset{f}{\hookrightarrow} B \overset{g}{\hookleftarrow} E\right) = \varinjlim_{\mathscr{E}} \mathscr{T}\mathscr{A}\left(X \overset{f}{\hookrightarrow} B \overset{g_\alpha}{\hookleftarrow} E_\alpha\right)$$

is obvious, so

$$H_*\mathscr{T}\mathscr{A}\left(X \overset{f}{\hookrightarrow} B \overset{g}{\hookleftarrow} E\right) = \varinjlim_{\mathscr{E}} H_*\mathscr{T}\mathscr{A}\left(X \overset{f}{\hookrightarrow} B \overset{g_\alpha}{\hookleftarrow} E_\alpha\right).$$

Fortunately we also have $H_*(P^B_{XE}; S) = \varinjlim_{\mathscr{E}} H_*(P^B_{XE_\alpha}; S)$, because any compact subset of P^B_{XE} actually lies in some $P^B_{XE_\alpha}$.

To show that each $\tilde{\theta}_*$ is an isomorphism when E is locally finite, we use a spectral sequence comparison. Filter

$$\mathscr{T}\mathscr{A}(X \hookrightarrow B \hookleftarrow E) \approx C^c_*(X) \otimes \mathscr{A}(B) \otimes C^c_*(E)$$

by setting

$$D_n = \bigoplus_{r+t \le n} C^c_r(X) \otimes \mathscr{A}(B) \otimes C^c_t(E).$$

Recalling that $p\colon P^B_{XE} \to X \times E$ is a fibration, filter $CU_*(P^B_{XE})$ so as to obtain the Serre spectral sequence, i.e., by inverse images under p of the skeleta of $X \times E$. By construction and [2], $\tilde{\theta}$ preserves these filtrations and consequently induces a map of spectral sequences. The "E^1" term for D_* is

$$\begin{aligned} C^c_*(X) \otimes H_*(\mathscr{A}(X), d_B) \otimes C^c_*(E) &\approx C^c_*(X) \otimes H_*(\Omega B; S) \otimes C^c_*(E) \\ &\approx C^c_*(X \times E; H_*(\Omega B; S)) \end{aligned}$$

and its "E^2" term is isomorphic with $H_*(X \times E; H_*(\Omega B; S))$. This agrees with the second term in the Serre spectral sequence and $\tilde{\theta}$ is easily seen to induce this isomorphism. By standard arguments (e.g., [6]),

$$\tilde{\theta}_*\colon H_*\mathscr{T}\mathscr{A}(X \hookrightarrow B \hookleftarrow E) \to H_*(P^B_{XE}; S)$$

is an isomorphism.

Theorem 3.1 asserts that for 1-connected CW bipairs, $H_*\mathscr{T}\mathscr{A}$ agrees with the homology of the fiber homotopy pull-back. To extend this to corners of 1-connected spaces which are not bipairs, we reason as follows. First, $\mathscr{A}$ operates only on CW complexes, so if necessary we replace each space by a CW approximation. This does not affect the weak homotopy type of the fiber homotopy pull-back. Second, we replace a corner of CW complexes with a CW bipair by using a double (reduced) mapping cylinder. Theorem 2.2 is used to deduce that $H_*\mathscr{T}\mathscr{A}(\ \ )$ is unchanged during this step. To apply Theorem 2.2, however, we need to know that $\mathscr{A}$ satisfies the derivation homotopy axiom, and this requires an understanding of how the Adams-Hilton model looks on a (reduced) cylinder.

Let $X = * \cup \bigcup_{i \in \Gamma} e_i$ be a 1-connected CW complex with trivial 1-skeleton and let $\tilde{X} = (X \times I)/(* \times I)$. We have

$$\mathscr{A}(X) = S\langle a_i | i \in \Gamma\rangle \quad \text{and} \quad A(\tilde{X}) = S\langle a_i', a_i'', b_i | i \in \Gamma\rangle,$$

where $|a_i| = |a_i'| = |a_i''| = |e_i| - 1$ and $|b_i| = |e_i|$. Assuming we have chosen d_X, we may define $d_{\tilde{X}}$ as follows. First define morphisms $f, g\colon A(X) \to A(\tilde{X})$ of tensor ringoids and a derivation F of degree $+1$ from f to g by $f(a_i) = a_i'$, $g(a_i) = a_i''$, and $F(a_i) = b_i$. $d_{\tilde{X}}$ may be taken to be an extension of d_X on the subcomplexes $X \times 0$ and $X \times 1$, and this suggests that a candidate d for $d_{\tilde{X}}$ is obtained by setting $d(a_i') = fd_X(a_i)$ and $d(a_i'') = gd_X(a_i)$. Setting

$$d(b_i) = (f - g - Fd_X)(a_i),$$

f and g become morphisms of chain tensor algebras $(\mathscr{A}(X), d_X) \to (\mathscr{A}(\tilde{X}), d)$ with F a derivation homotopy from f to g. To verify that d is a valid choice for $d_{\tilde{X}}$, it suffices to show this when the cells of $\tilde{X}$ are attached one at a time, and the only possible uncertainty is for the cells corresponding to the b_i's. For these it suffices to check that $H_*(\mathscr{A}(\tilde{X}^n), d|_{\tilde{X}^n}) \approx H_*(\Omega\tilde{X}^n; S)$ for n-skeleta X^n of X. But this is obvious because f is a chain equivalence (its inverse h has $h(a_i') = h(a_i'') = a_i$, $h(b_i) = 0$) and consequently

$$H_*(\mathscr{A}(\tilde{X}^n), d|_{\tilde{X}^n}) \approx H_*(\mathscr{A}(X^n), d_X|_{X^n}) \approx H_*(\Omega X^n; S) \approx H_*(\Omega\tilde{X}^n; S).$$

LEMMA 3.2. *As a functor from pointed* 1-*connected CW complexes and continuous maps between them to CTA,* $\mathscr{A}$ *satisfies the derivation homotopy axiom.*

Proof. The essence of the model for $\tilde{X}$ described above is that if f_0 and f_1 are the inclusions of X into $X \times 0$ and $X \times 1$, then there is a derivation homotopy F from $\mathscr{A}(f_0)$ to $\mathscr{A}(f_1)$. Suppose g_0, g_1: $X \to Y$ and h is a homotopy between them. h: $\tilde{X} \to Y$ extends

$$g_0 \vee g_1: (X \times 0) \vee (X \times 1) \to Y,$$

and it is implicit in [2] that $\mathscr{A}(h)$ may be chosen so as to extend any suitable $\mathscr{A}(g_0 \vee g_1)$. $\mathscr{A}(h)$ is a morphism of chain tensor algebras, so $\mathscr{A}(h) \circ F$ is a derivation homotopy from

$$\mathscr{A}(h) \circ \mathscr{A}(f_0) = \mathscr{A}(h \circ f_0) = \mathscr{A}(g_0)$$

to

$$\mathscr{A}(h) \circ \mathscr{A}(f_1) = \mathscr{A}(h \circ f_1) = \mathscr{A}(g_1).$$

THEOREM 3.3. *Let*

$$X \xrightarrow{f} B \xleftarrow{g} E$$

be any corner of pointed 1-*connected CW complexes and let A be its fiber homotopy pull-back. Then*

$$H_* \mathscr{T}\mathscr{A}\left(X \xrightarrow{f} B \xleftarrow{g} E \right)$$

is isomorphic with $H_*(A; S)$ as graded S-modules.

Proof. Replacing B by its double reduced mapping cylinder B', we obtain a diagram (9) commuting up to homotopy in which α and γ are the identity, β is a homotopy equivalence, and the corner

$$X' \xrightarrow{f'} B' \xleftarrow{g'} E'$$

is a bipair. Apply Theorems 2.2 and 3.1 and the homotopy invariance of the fiber homotopy pull-back.

Theorem 3.3 already fulfills one of the promises of the introduction, namely, a free chain complex with homology equal to $H_*(A; S)$, whose basis is locally finite whenever X, B, and E are locally finite CW complexes. However, when S is a field, we would like something as small as possible for two reasons. First, in practice the size of

$$D_* = \mathscr{T}\mathscr{A}\left(X \xrightarrow{f} B \xleftarrow{g} E \right)$$

grows rapidly as the number of cells in B increases, so actual computations are simplified by keeping this number small. Second, we shall see in the following sections that a lot of information can be deduced from D_* if the cellular models we choose are "minimal". We recall next the relevant definitions.

Suppose now that S is a field and let p be its characteristic ($p = 0$ is allowed). A 1-connected space X with a specified CW decomposition is *p-minimal* if and only if the cellular chain complex for X with coefficients in S has all differentials vanish. By the universal coefficient theorem, this depends only on p. It is equivalent [5] to the vanishing of the differential in $\mathscr{H}(\mathscr{A}(X), d_X)$. When X is locally finite, p-minimality is equivalent [3] to the existence of a bijection in each dimension between the cells of X and an S-basis for $H_*(X; S)$.

For any 1-connected space X with locally finite p-homology, there is a p-minimal approximation to X, i.e., a p-minimal CW complex X' and a p-homotopy equivalence $f\colon X' \to X$ [3]. We extend this to corners in the next lemma. All spaces and maps are henceforth taken to be pointed.

LEMMA 3.4. *Let*

$$X \xrightarrow{f} B \xleftarrow{g} E$$

be any corner of 1*-connected spaces with locally finite p-homology. Then there are p-minimal spaces X', B', E' and a diagram commuting up to homotopy*

$$\begin{array}{ccccc} X' & \xrightarrow{f'} & B' & \xleftarrow{g'} & E' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ X & \xrightarrow{f} & B & \xleftarrow{g} & E \end{array} \tag{11}$$

in which α, β, γ are p-homotopy equivalences.

Proof. First pick a p-minimal approximation to B, $\beta\colon B' \to B$. Replacing β by a fibration $\hat\beta$, taking pull-backs of $\hat\beta$ via f and g, and choosing p-minimal approximations to the pull-backs results in a diagram

$$\begin{array}{ccccccccc} X' & \xrightarrow{\alpha'} & \hat X & \xrightarrow{\hat f} & \hat B & \xleftarrow{\hat g} & \hat E & \xleftarrow{\gamma'} & E' \\ & & {\scriptstyle\hat\alpha}\downarrow & & {\scriptstyle\hat\beta}\downarrow\ \searrow{\scriptstyle h} & & \downarrow{\scriptstyle\hat\gamma} & & \\ & & & & \qquad B' & & & & \\ & & & & \swarrow{\scriptstyle\beta} & & & & \\ & & X & \xrightarrow{f} & B & \xleftarrow{g} & E & & \end{array}$$

in which the squares commute, the triangle commutes up to homotopy, and h is a homotopy equivalence. Since the fiber of $\hat{\beta}\colon \hat{B} \to B$ has trivial p-homotopy, $\hat{\alpha}$ and $\hat{\gamma}$ are also p-homotopy equivalences. It follows that setting $\alpha = \hat{\alpha} \circ \alpha'$, $\gamma = \hat{\gamma} \circ \gamma'$, $f' = h \circ \hat{f} \circ \alpha'$, and $g' = h \circ \hat{g} \circ \gamma'$ yields the desired diagram (11).

THEOREM 3.5. *Let*

$$X \xrightarrow{f} B \xleftarrow{g} E$$

be any corner of 1*-connected spaces with locally finite p-homology, let A be its fiber homotopy pull-back, and suppose S is a field of characteristic p. Then there is a corner*

$$X' \xrightarrow{f'} B' \xleftarrow{g'} E'$$

in which X', B', E' are p-minimal such that

$$H_*(A;S) \approx H_*\mathscr{T}\mathscr{A}(X' \to B' \leftarrow E')$$

as graded S-modules. In particular, $H_*(A;S)$ equals the homology of

$$H_*(X;S) \otimes Ts^{-1}\overline{H}_*(B;S) \otimes H_*(E;S) \tag{12}$$

with respect to a certain explicitly computable differential.

Proof. Apply Lemma 3.4 to obtain a diagram (11) subject to the conditions of that lemma. If we localized all spaces at (p), α, β, γ would become homotopy equivalences, so the fiber homotopy pull-backs A' and A of the corners (f', g') and (f, g) have the same p-homotopy type. Thus $H_*(A;S) \approx H_*(A';S) \approx H_*\mathscr{T}\mathscr{A}(f', g')$. $\mathscr{T}\mathscr{A}(f', g')$ is always equal as a tensor ringoid to

$$C_*^c(X') \otimes Ts^{-1}\overline{C}_*^c(B') \otimes C_*^c(E'),$$

C_*^c denoting cellular chains with coefficients in S and s^{-1} denoting desuspension. This agrees with

$$H_*(X';S) \otimes Ts^{-1}\overline{H}_*(B';S) \otimes H_*(E';S) \tag{13}$$

precisely when all three spaces are p-minimal, and the p-homotopy equivalences allow us to equate (12) and (13).

When S is a field, yet another very important construction is possible. Let

$$X \xrightarrow{f} B \xleftarrow{g} E$$

be a corner of 1-connected CW complexes. The dual of $\mathscr{T}\mathscr{A}(f, g)$ which equals

$$D^* = C_c^*(X) \otimes Ts^{-1}\overline{C}_c^*(B) \otimes C_c^*(E),$$

where $C_c^*(X) = \operatorname{Hom}(C_*^c(X), S)$, may be viewed as a filtered cochain complex by grading it with

$$D^n = C_c^*(X) \otimes \underbrace{s^{-1}\overline{C}_c^*(B) \otimes \cdots \otimes s^{-1}\overline{C}_c^*(B)}_{n \text{ factors}} \otimes C_c^*(E).$$

As D^n is also graded by dimension, we may write $D^{n,q}$ for the component of D^n in dimension q. For the sake of comparison with the Eilenberg-Moore spectral sequence [8] later, it is most convenient to replace $D^{*,*}$ by the equivalent complex $\hat{D}^{*,*}$, where

$$\hat{D}^{n,t} = \Big[C_c^*(X) \otimes \underbrace{\overline{C}_c^*(B) \otimes \cdots \otimes \overline{C}_c^*(B)}_{n \text{ factors}} \otimes C_c^*(E)\Big]^t,$$

so that $\hat{D}^{n,t} \approx D^{n,t-n}$ as S-modules. The differential $\hat{d}$ on $\hat{D}^{*,*}$ satisfies

$$\hat{d}(\hat{D}^{n,t}) \subseteq \hat{D}^{n,t+1} \oplus \hat{D}^{n-1,t} \oplus \hat{D}^{n-2,t-1} \oplus \cdots \oplus \hat{D}^{0,t-n+1},$$

so the first grading gives rise to a spectral sequence. We collect information about this spectral sequence, called the *dual spectral sequence*, in the following lemma.

LEMMA 3.6. *Suppose S is a field. Let*

$$X \xrightarrow{f} B \xleftarrow{g} E$$

be any corner of 1-connected CW complexes and let A be its fiber homotopy pull-back. The dual spectral sequence for this corner has

$$E_1^{n,t} = \Big[H^*(X;S) \otimes \underbrace{\overline{H}^*(B;S) \otimes \cdots \otimes \overline{H}^*(B;S)}_{n \text{ factors}} \otimes H^*(E;S)\Big]^t$$

and has

$$\bigoplus_{t-n=q} E_\infty^{n,t} \approx H^q(A;S)$$

as graded S-modules. The differential $\hat{d}_r$ has bidegree $(-r, -r+1)$.

The next lemma asserts that the dual spectral sequence is a p-homotopy invariant.

LEMMA 3.7. *Let S be a field of characteristic p and let*

$$\begin{array}{ccccc} X' & \xrightarrow{f'} & B' & \xleftarrow{g'} & E' \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ X & \xrightarrow{f} & B & \xleftarrow{g} & E \end{array}$$

be a diagram of 1*-connected CW complexes which commutes up to homotopy and in which α, β, γ are p-homotopy equivalences. Then the dual spectral sequences for the corners (f', g') and (f, g) coincide from their E_1 terms onward.*

Proof. By [6] it suffices to construct a filtration preserving chain map from $\hat{D}$ for (f, g) to $\hat{D}$ for (f', g') which induces an isomorphism on the E_1 terms. This is equivalent to constructing a morphism of tensor ringoids ψ: $\mathcal{T}\mathcal{A}(f', g') \to \mathcal{T}\mathcal{A}(f, g)$ whose dual is isomorphic on the E_1 terms. The latter requirement is equivalent to the demand that ψ induce an isomorphism of homogeneous homology. This is guaranteed by theorem 2.2 if we know that $\mathcal{A}(\alpha), \mathcal{A}(\beta), \mathcal{A}(\gamma)$ induce isomorphisms of homology. This in turn follows from (10), since a p-homotopy equivalence induces an isomorphism of loop space homology with coefficients in S.

We may combine Lemmas 3.7 and 3.4 to obtain the E_2 term of the dual spectral sequence.

LEMMA 3.8. *Let*

$$X \xrightarrow{f} B \xleftarrow{g} E$$

be a corner of 1*-connected CW complexes with locally finite p-homology and suppose S is a field of characteristic p. The E_2 term of the dual spectral sequence for this corner is*

$$E_2^{n,t} = \mathrm{Tor}_{n,t}^{H^*(B;S)}\big(H^*(X; S), H^*(E; S)\big).$$

Here $H^*(X; S)$ and $H^*(E; S)$ are viewed as $H^*(B; S)$-modules via f^* and g^*.

Proof. By Lemmas 3.4 and 3.7, we need only compute the E_2 term when X, B, and E are p-minimal, where $p = \mathrm{char}(S)$. Writing $d = d_0 + d_1 + d_2 + \cdots$ for the differential d on $\mathcal{T}\mathcal{A}(f, g)$, where d_i is the component of d which raises the lower degree by i, we have $d_0 = 0$ and d_1 being the dual of the cup product [5]. From this we may deduce that $(E_1, \hat{d}_1)$ coincides with the bigraded two-sided bar resolution for $H^*(X; S)$ and $H^*(E; S)$ and $H^*(B; S)$-modules. Then

$$E_2^{n,t} = H_*\left(E_1^{n,t}, \hat{d}_1\right) \approx \operatorname{Tor}_{n,t}^{H^*(B;S)}\left(H^*(X;S), H^*(E;S)\right).$$

4. The Eilenberg-Moore spectral sequence

In this section we show that the dual spectral sequence coincides with the spectral sequence of Eilenberg and Moore [8]. In the process we offer a variation on the proof of the existence of the Eilenberg-Moore spectral sequence. Lastly, we observe that an explicit perturbation theory may be given for its E_1 term and state a very general homotopy invariance theorem.

We begin with a review of the basic notation and facts about the two-sided cobar construction. Given any free S chain complex (C_*, δ) with $C_0 \approx S$ which is a chain coalgebra over S via $\lambda\colon C_* \to C_* \otimes C_*$, we may form the cobar construction for C_*, denoted $\underline{\Omega} C_*$:

$$\underline{\Omega} C_* = \left(Ts^{-1}\overline{C}_*, \delta_T\right)$$

where $\delta_T = \delta_1 + \delta_2$, $\delta_1 = -s^{-1}\delta$, and if $\lambda(a) = \Sigma a_i' \otimes a_i''$, then

$$\delta_2(s^{-1}a) = \sum (-1)^{|a_i'|}\left(s^{-1}a_i'\right)\cdot\left(s^{-1}a_i''\right).$$

The axioms for an associative chain coalgebra require that

$$\delta_1^2 = \delta_2^2 = \delta_1\delta_2 + \delta_2\delta_1 = 0$$

and consequently $\delta_T^2 = 0$. δ_T is extended over $Ts^{-1}\overline{C}_*$ so as to make it into a chain tensor algebra, i.e., by formula (7).

The two-sided cobar construction begins with a chain coalgebra C_* and a right and a left chain comodule, C_*' and C_*''. The cases that we shall be concerned with all have the additional property that C_*' and C_*'' are themselves free associative coalgebras, the comodule structures coming from chain coalgebra maps $f'\colon C_*' \to C_*$ and $f''\colon C_*'' \to C_*$. The formulas for the two-sided cobar construction, sometimes denoted $\underline{\Omega}(C_*', C_*, C_*'')$, may easily be recognized as coinciding with the formulas of Section 2 for the associated chain ringoid to the corner

$$\underline{\Omega} C_*' \xrightarrow{\underline{\Omega} f'} \underline{\Omega} C_* \xleftarrow{\underline{\Omega} f''} \underline{\Omega} C_*''$$

of chain tensor algebras. An excellent reference, containing many explicit formulas, for the cobar construction is [4].

Eilenberg and Moore [8] were the first to observe the connection between the two-sided cobar construction and the cohomology of the pull-back of a fibration. Let $\mathscr{C}(X)$ denote the singular chain complex on X with coefficients in S, viewed as a free chain coalgebra via the Alexander-Whitney diagonal. Let $\mathscr{C}^m(X)$ denote the subcoalgebra spanned by those singular simplices which send the m-skeleton of the standard simplex to the base point of X. With this notation, a key step in the Eilenberg-Moore paper may be described as the proof that for 0-connected spaces,

$$H_*(A;S) \approx H_*\mathscr{T}\underline{\Omega}\mathscr{C}^0\left(X \xrightarrow{f} B \xleftarrow{g} E\right) \tag{14}$$

when g is a fibration and A is the pull-back as in (1). Interestingly, (14) can also be obtained using only Adams' result [1] and the techniques introduced here, if we require all spaces to be simply connected. This proof is given next.

LEMMA 4.1. *The functors* $\underline{\Omega}\mathscr{C}^m$, $m \geq 0$, *satisfy the derivation homotopy axiom.*

Proof. As in the proof of Lemma 3.2, it suffices to show that the two inclusions $g_i\colon X \to \tilde{X} = (X \times I)/(* \times I)$ by $g_i(x) = (x,i)$ for $i = 0,1$ induce derivation homotopic chain maps. For a singular p-simplex $\phi\colon \sigma^p \to X$, the "standard" homotopy between $\phi^{(0)} = g_{0\#}(\phi)$ and $\phi^{(1)} = g_{1\#}(\phi)$ is given by the formula

$$G(\phi) = \sum_{n=0}^{p} (-1)^n \phi_{\langle n \rangle},$$

where $\phi_{\langle n \rangle}\colon \sigma^{p+1} \to \tilde{X}$ is defined by

$$\begin{aligned} &\phi_{\langle n \rangle}(x_0, x_1, \dots, x_{p+1}) \\ &\quad = \left(\phi(x_0, \dots, x_n + x_{n+1}, \dots x_{p+1}), x_{n+1} + \cdots + x_{p+1}\right). \end{aligned}$$

Extend G over $\underline{\Omega}\mathscr{C}^m(X)$ so that it is a derivation from $g_{0\#}$ to $g_{1\#}$. We want G to be a derivation homotopy from $g_{0\#}$ to $g_{1\#}$, and by Lemma 1.3 we need only verify that

$$\delta_T G(\phi) + G\delta_T(\phi) = \phi^{(0)} - \phi^{(1)},$$

where we are abusing notation slightly by writing ϕ instead of $s^{-1}\phi$ for this typical generator of $\underline{\Omega}\mathscr{C}^m(X)$. Since $\delta_T = \delta_1 + \delta_2$ and the fact that

$$(\delta_1 G + G\delta_1)(\phi) = \phi^{(0)} - \phi^{(1)}$$

is just the standard proof that $\mathscr{C}^m$ satisfies the (usual) homotopy axiom, it remains to check that

$$(\delta_2 G + G\delta_2)(\phi) = 0$$

for each singular simplex ϕ.

Again we rely on explicit formulas. Following accepted notation, write

$$\delta_2(\phi) = \sum_{i=0}^{p} (-1)^i {}_i\phi \cdot \phi_{p-i},$$

where ${}_i\phi$ is the front i-face and ϕ_j the back j-face of ϕ.

$$\delta_2 G(\phi) = \sum_{n=0}^{p} \sum_{i=0}^{p+1} (-1)^n (-1)^i {}_i(\phi_{\langle n\rangle}) \cdot (\phi_{\langle n\rangle})_{p+1-i},$$

$$\begin{aligned}
G\delta_2(\phi) &= \sum_{j=0}^{p} (-1)^j G({}_j\phi \cdot \phi_{p-j}) \\
&= \sum_{j=0}^{p} (-1)^j G({}_j\phi) \cdot \phi_{p-j}^{(1)} - \sum_{j=0}^{p} {}_j\phi^{(0)} \cdot G(\phi_{p-j}) \\
&= \sum_{j=0}^{p} \sum_{t=0}^{j} (-1)^{j+t} ({}_j\phi)_{\langle t\rangle} \cdot \phi_{p-j}^{(1)} \\
&\quad - \sum_{j=0}^{p} \sum_{t=0}^{p-j} (-1)^t {}_j\phi^{(0)} \cdot (\phi_{p-j})_{\langle t\rangle} \\
&= \sum_{j=0}^{p} \sum_{t=0}^{j} (-1)^{j+t} {}_{j+1}(\phi_{\langle t\rangle}) \cdot (\phi_{\langle t\rangle})_{p-j} \\
&\quad - \sum_{j=0}^{p} \sum_{t=j}^{p} (-1)^{t-j} {}_j(\phi_{\langle t\rangle}) \cdot (\phi_{\langle t\rangle})_{p-j+1} \\
&= \sum_{t=0}^{p} \sum_{j=t}^{p} (-1)^{j+t} {}_{j+1}(\phi_{\langle t\rangle}) \cdot (\phi_{\langle t\rangle})_{p-j} \\
&\quad - \sum_{t=0}^{p} \sum_{j=0}^{t} (-1)^{t-j} {}_j(\phi_{\langle t\rangle}) \cdot (\phi_{\langle t\rangle})_{p-j+1} \\
&= - \sum_{t=0}^{p} \sum_{j=0}^{p+1} (-1)^j (-1)^t {}_j(\phi_{\langle t\rangle}) \cdot (\phi_{\langle t\rangle})_{p+1-j} \\
&= -\delta_2 G(\phi).
\end{aligned}$$

Lemma 4.1 and Theorem 2.2 allow us always to replace any corner by a bipair and the fiber homotopy pull-back by a path space when working with the two-sided cobar construction. A more subtle consequence of Lemma 4.1 is that the inclusion $\underline{\Omega}\mathscr{C}^m \hookrightarrow \underline{\Omega}\mathscr{C}^0$ is a natural derivation homotopy equivalence when restricted to m-connected spaces. This follows because the inverse to $\mathscr{C}^m(X) \hookrightarrow \mathscr{C}^0(X)$ may be given, on each singular simplex, in terms of a homotopy between two maps of the standard simplex to X (for example, see the proof of the relative Hurewicz theorem in [16]).

THEOREM 4.2. *There is a natural isomorphism between the two functors $H_*\mathscr{T}\underline{\Omega}\mathscr{C}^0_*$ and $H_*(P\,.\!\dot{.}\,.; S)$ from 1-connected bipairs to graded S-modules.*

Proof. The proof follows the same lines as that of Theorem 3.1. In place of the Adams-Hilton construction and the natural transformation θ we have from [1] the cobar construction and a natural transformation

$$\eta_X\colon \underline{\Omega}\mathscr{C}^1(X) \to CU_*(P^X_{**})$$

which extends to a map $\eta_X\colon \mathscr{L}(\underline{\Omega}\mathscr{C}^1(X)) \to CU_*(P^X_{*X})$. Similar reasoning to that in [1] allows η_X also to be extended to

$$\eta_X\colon \mathscr{K}(\underline{\Omega}\mathscr{C}^1(X)) \to CU_*(P^X_{X*}).$$

For a 1-connected bipair (B, X, E) we may piece together η_B, η_X, and η_E to obtain a natural transformation

$$\eta\colon \mathscr{T}\underline{\Omega}\mathscr{C}^1(X \to B \leftarrow E) \to CU_*(P^B_{XE}).$$

That η induces an isomorphism of homology follows from the same spectral sequence comparison as was used in proving theorem 3.1. Finally, the fact that $\underline{\Omega}\mathscr{C}^1 \to \underline{\Omega}\mathscr{C}^0$ is a natural derivation homotopy equivalence means by the methods of Section 2 that $\mathscr{T}\underline{\Omega}\mathscr{C}^1 \to \mathscr{T}\underline{\Omega}\mathscr{C}^0$ is a natural equivalence on 1-connected bipairs. The desired natural isomorphism is a composition

$$H_*\mathscr{T}\underline{\Omega}\mathscr{C}^0 \overset{\approx}{\leftarrow} H_*T\underline{\Omega}\mathscr{C}^1 \overset{\approx}{\to} H_*(P\,.\!\dot{.}\,.\,S).$$

Results 4.1 and 4.2 combine to yield (cf. (14)):

THEOREM 4.3. *Let*

$$X \xrightarrow{f} B \xleftarrow{g} E$$

be any corner of 1-connected spaces and let A be its fiber homotopy pull-back.

Then

$$H_*(A;S) \approx H_* \mathscr{T}\underline{\Omega}\mathscr{C}^0\left(X \xrightarrow{f} B \xleftarrow{g} E\right).$$

When S is a field, the dual of $\mathscr{T}\underline{\Omega}\mathscr{C}^0(X \xrightarrow{f} B \xleftarrow{g} E)$ may be filtered, just as in Lemma 3.6, to obtain a spectral sequence converging to $H^*(A;S)$. Clearly this spectral sequence is precisely the Eilenberg-Moore spectral sequence. Lemmas 3.6 and 3.7, without the restriction that the spaces be CW complexes, are also true of the Eilenberg-Moore spectral sequence, since their proofs remain valid when $\mathscr{A}$ and θ are replaced throughout by $\underline{\Omega}\mathscr{C}^1$ and η.

With so many similarities between the dual spectral sequence and the Eilenberg-Moore spectral sequence, one suspects that they must coincide. The proof of this is given next.

THEOREM 4.4. *Suppose S is a field and let*

$$X \xrightarrow{f} B \xleftarrow{g} E$$

be any corner of 1*-connected CW complexes. The dual spectral sequence and the Eilenberg-Moore spectral sequence for this corner coincide from their E_1 terms onward.*

Proof. By the previous remarks it suffices to demonstrate this equivalence for bipairs, so assume (B, X, E) is a 1-connected CW bipair. As in the proof of lemma 3.7, we need only construct a morphism of chain ringoids

$$\xi: \mathscr{T}\mathscr{A}(X \to B \leftarrow E) \to \mathscr{T}\underline{\Omega}\mathscr{C}^1(X \to B \leftarrow E)$$

which induces an isomorphism on homogeneous homology. We have observed that there is an explicit map

$$\eta: \mathscr{T}\underline{\Omega}\mathscr{C}^1(X \to B \leftarrow E) \to CU_*\left(P^B_{XE}\right)$$

as well as a map

$$\theta: \mathscr{T}\mathscr{A}(X \to B \leftarrow E) \to CU_*\left(P^B_{XE}\right),$$

both of which induce isomorphisms on homology. The map θ is not canonical. It can be defined [2] on one generator at a time, and it has indeterminacy which allows the addition of certain boundaries when choosing the θ-images of generators. We see readily that in fact θ can be chosen so that the θ-image of each generator lies in the image of η. Then θ factors through η and we may write $\theta = \eta\xi$ for some morphism ξ of chain ringoids. Since θ and η induce isomorphisms on homology, ξ does also.

Performing the construction of ξ separately for X, E, and B gives a commuting diagram in CTA,

$$\begin{array}{ccccc} \mathscr{A}(X) & \hookrightarrow & \mathscr{A}(B) & \hookleftarrow & \mathscr{A}(E) \\ \downarrow{\scriptstyle \xi_X} & & \downarrow{\scriptstyle \xi_B} & & \downarrow{\scriptstyle \xi_E} \\ \underline{\Omega}\mathscr{C}^1(X) & \hookrightarrow & \underline{\Omega}\mathscr{C}^1(B) & \hookleftarrow & \underline{\Omega}\mathscr{C}^1(E), \end{array} \tag{15}$$

where ξ_B is chosen to extend both ξ_X and ξ_E. In (15), all three vertical maps induce isomorphisms of homology. ξ is just $\mathscr{T}(\xi_E, \xi_B, \xi_E)$, the induced map on associated chain ringoids, and the essence of Theorem 2.2 is that under such conditions ξ induces an isomorphism on homology and on homogeneous homology.

THEOREM 4.5. *Let*

$$X \xrightarrow{f} B \xleftarrow{g} E$$

be any corner of 1*-connected spaces with locally finite p-homology, let A be its fiber homotopy pull-back, and suppose S is a field of characteristic p. Then there is an explicitly computable "perturbation theory" for the E_1 term of the Eilenberg-Moore spectral sequence for this corner. That is, there is a map $\hat{d}$: $E_1 \to E_1$ with $\hat{d}^2 = 0$, with respect to which E_1 is a filtered complex, such that $H_*(E_1, \hat{d}) \approx H^*(A; S)$ as graded S-modules, and by filtering $(E_1, \hat{d})$ one recovers the Eilenberg-Moore spectral sequence starting with the E_1 term.*

Proof. By Theorem 4.4 we may consider the dual spectral sequence instead and by Lemmas 3.4 and 3.7 we may replace X, B, and E by p-minimal spaces. The result then follows from Theorem 3.5.

Note. The claim that $\hat{d}$ gives an "explicitly computable" perturbation theory deserves to be qualified somewhat. $\hat{d}$ is computable directly from the Adams-Hilton models for the spaces in the corner

$$X \xrightarrow{f} B \xrightarrow{g} E.$$

These models may be deduced from some fairly basic information about the spaces, and various examples appear in [2] and [4]. However, what is not so explicit in this treatment is the actual chain homotopy equivalence between $(E_1, \hat{d})$ and (E_0, d_T).

For completeness, we state the most general homotopy invariance result possible (cf. [11]).

LEMMA 4.6. *Let S be a field, and suppose*

$$\begin{array}{ccccc} X' & \xrightarrow{f'} & B' & \xleftarrow{g'} & E' \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ X & \xrightarrow{f} & B & \xleftarrow{g} & E \end{array}$$

is a diagram of 0-connected spaces which commutes up to homotopy. Suppose further that α, β, and γ induce isomorphisms on homology with coefficients in S. Then the Eilenberg-Moore spectral sequences for the corners (f', g') and (f, g) coincide from their E_1 terms onward.

Proof. As in the proof of Lemma 3.7, we may apply Lemma 4.1 and Theorem 2.2 if we show that $\underline{\Omega}\mathscr{C}^0(\alpha)$, $\underline{\Omega}\mathscr{C}^0(\beta)$, and $\underline{\Omega}\mathscr{C}^0(\gamma)$ induce isomorphisms on homology. Clearly this amounts to proving the claim that whenever Y and Y' are 0-connected and h: $Y' \to Y$ induces an isomorphism on homology with coefficients in S, then $\underline{\Omega}\mathscr{C}^0(h)$ is a quasi-isomorphism.

To prove this claim, dualize the map $\underline{\Omega}\mathscr{C}^0(h)$ and consider the induced map on the resulting spectral sequences. The map induced on the E_1 terms,

$$\hat{h}_1^*: T\overline{H}^*(Y; S) \to T\overline{H}^*(Y'; S),$$

is an isomorphism because h^*: $H^*(Y; S) \to H^*(Y'; S)$ is an isomorphism. We conclude that the dual of $\underline{\Omega}\mathscr{C}^0(h)$, and hence $\underline{\Omega}\mathscr{C}^0(h)$ itself, is a quasi-isomorphism.

5. Application to formal spaces

In this section we apply our results to show that the Eilenberg-Moore spectral sequence for (1) degenerates over characteristic zero whenever f and g are formal maps. This result has recently been proved via minimal models [19], but the approach adopted here is valuable in that it generalizes to positive characteristics.

Our main result will in fact be valid over arbitrary characteristics, so we begin by describing a generalization of the concept of formal spaces [10], [12] to non-zero characteristics. We restrict ourselves in what follows to the case where S is a field, and let $p = \text{char}(S)$. By [5] and [12], a 0-minimal 1-connected CW complex X is *formal* if and only if the Adams-Hilton model $(\mathscr{A}(X), d_X)$ when $p = 0$ can be chosen such that, for each generator a_i of $\mathscr{A}(X)$, we may write

$$d_X(a_i) = \sum_{j,l} c_{ijl} a_j a_l \tag{16}$$

for some scalars $c_{ijl} \in S$. As noted in [3], condition (16) provides an excellent generalization of formal spaces to arbitrary characteristics. A p-minimal (p arbitrary) 1-connected CW complex X is called *p-quadratic* if and only if d_X may be chosen on $\mathscr{A}(X)$ such that (16) holds. This depends only on $p = \operatorname{char}(S)$ and, up to sign, the scalars c_{ijl} reflect the cup coproduct structure of X.

A map $f\colon X \to Y$ between formal spaces is said to be a *formal* map when its minimal model is a formal consequence of the map it induces on cohomology. Assuming X and Y are 0-minimal 1-connected CW complexes, this is equivalent to the condition that $\mathscr{A}(f)\colon \mathscr{A}(X) \to \mathscr{A}(Y)$ may be chosen so as to send generators of $\mathscr{A}(X)$ to linear combinations of generators in $\mathscr{A}(Y)$. Ignoring the differentials, $\mathscr{A}(f)$ would then agree with its homogeneous approximation $\mathscr{H}\mathscr{A}(f)$. This concept may be generalized to maps $f\colon X \to Y$ between p-minimal 1-connected CW complexes for any p. Call such a map *p-homogeneous* whenever $\mathscr{A}(f)$ may be chosen so as to preserve lower degrees.

Formal spaces have many beautiful properties, and one particular property deserves mention here since we shall generalize it. When $p = \operatorname{char}(S) = 0$, the Pontrjagin homology ring $H_*(\Omega X; S)$ of the loop space on a formal space X may be given the structure of a bigraded S-algebra, and this additional structure is natural in the category of formal spaces and formal maps. For X locally finite there is also a natural transformation from the Ext algebra of $H^*(X; S)$ to the Pontrjagin ring $H_*(\Omega X; S)$, which is an isomorphism of bigraded algebras if the proper multiplication is adopted on the Ext algebra. By [3], these properties still hold when $p = \operatorname{char}(S)$ is arbitrary and we restrict ourselves to the category of p-quadratic spaces and p-homogeneous maps, which we call the *p-formal category*.

THEOREM 5.1. *Let*

$$X \xrightarrow{f} B \xleftarrow{g} E$$

be a corner in the p-formal category and let A be its fiber homotopy pull-back. Then:

(a) $H^*(A; S)$ *has a natural bigrading, and we may write*

$$H^q(A; S) = \bigoplus_{t=0}^{\infty} H^{t,q}(A; S).$$

By "natural" here we mean that morphisms of corners in this category induce maps which respect the bigrading on their fiber homotopy pull-back cohomology.

(b) *There is a natural isomorphism of bigraded S-modules*

$$H^{t,q}(A; S) \approx \operatorname{Tor}_{t,q+t}^{H^*(B;S)}\big(H^*(X; S), H^*(E; S)\big). \tag{17}$$

(c) *When X and E are contractible, A has the homotopy type of ΩB, and the bigrading given here on A agrees with that of* [3] *or (for $p = 0$)* [10] *on $H^*(\Omega B; S)$.*

(d) *The Eilenberg-Moore spectral sequence for this corner degenerates at the E_2 term.*

Proof. This is a fairly direct consequence of Theorem 3.5 and Lemma 3.8, where the automatic p-minimality of p-quadratic spaces renders unnecessary the finiteness condition. In the p-formal category, the formulas for $\mathscr{T}\mathscr{A}$ together with (16) show that the differential $\hat{d}_1$ on the E_1 term is actually the whole differential $\hat{d}$.

This proves part (d) and the formula

$$H^q(A; S) \approx \bigoplus_{t=0}^{\infty} \operatorname{Tor}_{t,q+t}^{H^*(B;S)}\big(H^*(X; S), H^*(E; S)\big). \tag{18}$$

The right-hand side of (18) is bigraded and functorial as a functor on corners, so, from (18), $H^*(A; S)$ inherits a natural bigrading. This proves parts (a) and (b). Part (c) follows because one of the definitions [3] for the bigrading already known on ΩB for B p-quadratic agrees with the one obtained by specializing the natural isomorphism of part (b).

Examples of formal spaces are suspensions, spaces whose rational cohomology is a polynomial algebra, and r-connected manifolds of dimension $\leq 4r + 2$. Wedges, fat wedges, and products of formal spaces are formal. Examples of formal maps are the inclusion of a subcomplex into a 0-minimal formal complex, the projection of a product of formal spaces onto one factor, and the suspension of any map. The diagonal map $\Delta\colon X \to X \times X$ is formal when X is a formal space, and the fact leads directly to:

THEOREM 5.2 (cf. [20]). *Let S be a field of characteristic zero and let X be a formal space. Let ΛX denote X^{S^1}, the space of free (i.e., non-based) loops on X. Then $H^*(\Lambda X; S)$ is naturally bigraded and*

$$H^{t,q}(\Lambda X; S) \approx \operatorname{Tor}_{t,q+t}^{R\otimes R}(R, R), \tag{19}$$

where $R = H^*(X; S)$. In particular

$$H^q(\Lambda X; S) \approx \bigoplus_{t=0}^{\infty} \operatorname{Tor}_{t,q+t}^{R\otimes R}(R, R). \tag{20}$$

Proof. This is immediate from Theorem 5.1 once we note by [15] that ΛX is the fiber homotopy pull-back of the corner

$$X \xrightarrow{\Delta} X \times X \xleftarrow{\Delta} X$$

for any X.

Unfortunately, the diagonal need not be p-homogeneous for X p-quadratic when $p \neq 0$. As a simple example take Y to be the suspension of $\mathbf{C}P^2$ and $S = \mathbf{Z}/2\mathbf{Z}$. The decomposition of Y as $* \cup e^3 \cup e^5$ is 2-minimal, so we may write $\mathscr{A}(Y) = S\langle a_2, a_4 \rangle$, subscripts corresponding to degree. By dimensional constraints, $d_Y = 0$. Let $b_{ij} \in \mathscr{A}(Y \times Y)$ be the generator corresponding to the product of the i-cell and the j-cell when they both exist. Using the fact that the transgressive generator of $H^4(\Omega Y; S)$ is the square of the generator of $H^2(\Omega Y; S)$ (they are connected by an Sq^2) we may deduce

$$\mathscr{A}(\Delta)(a_4) = b_{05} + b_{50} + b_{03}b_{30},$$

with indeterminacy $d_{Y \times Y}(b_{33}) = b_{30}b_{03} - b_{03}b_{30}$. So $\mathscr{A}(\Delta)$ cannot be chosen so as to preserve the lower gradation. It remains an open question whether or not (20) holds for X p-quadratic when $p > 0$.

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Appendix

We assemble here proofs which were omitted from Sections 1 and 2 to improve their readability. The proofs of Lemma 1.1 through 1.4 are straightforward checks, and the proof of Theorem 1.5 is in essence a large "diagram-chase". The poof of Theorem 2.2 is spread out over eight technical lemmas, but the underlying idea is very simple: we make use of the high degree of naturalness intrinsic in Definition 2.1 for the associated chain ringoid.

Proof of Lemma 1.1. Let $\{x_j\}$ be a set of generators for P, hence also an S-basis for $\overline{M} \oplus V \oplus \overline{N}$. Let $f_0(1) = f_1(1) = 1$ and $f_2(1) = 0$. For $i = 0, 1$, set

$$f_i(x_{j_1} \cdot \cdots \cdot x_{j_t}) = f_i'(x_{j_1}) \cdot \cdots \cdot f_i'(x_{j_t})$$

whenever $x_{j_1} \cdot \cdots \cdot x_{j_t}$ is defined in P, and recursively set

$$f_2(x_{j_1} \cdot \cdots \cdot x_{j_t}) = f_2'(x_{j_1}) f_1(x_{j_2} \cdot \cdots \cdot x_{j_t}) + (-1)^{r \cdot |x_{j_1}|} f_0(x_{j_1}) \cdot f_2(x_{j_2} \cdot \cdots \cdot x_{j_t}).$$

The key point here is that the tensor ringoid P is free with an S-basis consisting precisely of $\{1\}$ and products of the form $x_{j_1} \cdot \cdots \cdot x_{j_t}$, $t \geq 1$, where $x_{j_m} \notin \overline{M}$ for $m > 1$ and $x_{j_m} \notin \overline{N}$ for $m < t$. This means that these formulas define three unique maps of S-modules of degrees $0, 0, r$ respectively. f_0 and f_1 are clearly morphisms of tensor ringoids, and it suffices to check formula (2) when x and y are products of generators.

Assume inductively that formula (2) holds when x and y are products of generators and $x \cdot y$ is a product of fewer than t generators. If now $x \cdot y$ is a product of length t, write $x = x' \cdot x''$, where x' is a generator and $x'' \cdot y$ has product length $t - 1$. We obtain

$$\begin{aligned} f_2(x \cdot y) = f_2(x' \cdot x''y) &= f_2(x')f_1(x''y) + (-1)^{r \cdot |x'|} f_0(x')f_2(x''y) \\ &= f_2(x')f_1(x'')f_1(y) + (-1)^{r \cdot |x'|} f_0(x')\big[f_2(x'')f_1(y) \\ &\quad + (-1)^{r \cdot |x''|} f_0(x'')f_2(y)\big] \\ &= f_2(x')f_1(x'')f_1(y) + (-1)^{r \cdot |x'|} f_0(x')f_2(x'')f_1(y) \\ &\quad + (-1)^{r \cdot |x|} f_0(x')f_0(x'')f_2(y) \\ &= \big[f_2(x')f_1(x'') + (-1)^{r \cdot |x'|} f_0(x')f_2(x'')\big] f_1(y) \\ &\quad + (-1)^{r \cdot |x|} f_0(x'x'')f_2(y) \\ &= f_2(x)f_1(y) + (-1)^{r \cdot |x|} f_0(x)f_2(y), \end{aligned}$$

as desired.

Proof of Lemma 1.2. Again we may restrict our attention to products of generators and by induction on product length it suffices to check for $i = 0, 1, 2$ that

$$gh_i(xy) = h_i'f(xy)$$

when these equalities are known separately for x and for y. For $i = 0$ or 1,

$$\begin{aligned} gh_i(xy) &= g(h_i(x) \cdot h_i(y)) = gh_i(x) \cdot gh_i(y) = h_i'f(x) \cdot h_i'f(y) \\ &= h_i'(f(x) \cdot f(y)) = h_i'f(xy), \\ gh_2(xy) &= g\big(h_2(x) \cdot h_1(y) + (-1)^{r \cdot |x|} h_0(x) \cdot h_2(y)\big) \\ &= gh_2(x) \cdot gh_1(y) + (-1)^{r \cdot |x|} gh_0(x) \cdot gh_2(y) \\ &= h_2'f(x) \cdot h_1'f(y) + (-1)^{r \cdot |x|} h_0'f(x) \cdot h_2'f(y) \\ &= h_2'(f(x) \cdot f(y)) \\ &= h_2'f(x \cdot y). \end{aligned}$$

Proof of Lemma 1.3. As usual everything is additive so it suffices inductively to verify formula (3) for a product xy when it is known to hold separately for x and for y:

$$\begin{aligned}(F\delta + dF)(xy) &= F\big(\delta(x)\cdot y + (-1)^{|x|}x\cdot\delta(y)\big)\\ &\quad + d\big(F(x)\cdot g(y) + (-1)^{|x|}f(x)\cdot F(y)\big)\\ &= F\delta(x)\cdot g(y) - (-1)^{|x|}f\delta(x)\cdot F(y)\\ &\quad + (-1)^{|x|}F(x)\cdot g\delta(y) + f(x)\cdot F\delta(y)\\ &\quad + dF(x)\cdot g(y) - (-1)^{|x|}F(x)\cdot dg(y)\\ &\quad + (-1)^{|x|}df(x)\cdot F(y) + f(x)\cdot dF(y)\\ &= (F\delta + dF)(x)\cdot g(y) + f(x)\cdot(F\delta + dF)(y)\\ &= (f(x) - g(x))\cdot g(y) + f(x)\cdot(f(y) - g(y))\\ &= f(x)\cdot g(y) - g(x)\cdot g(y) + f(x)\cdot f(y) - f(x)\cdot g(y)\\ &= f(xy) - g(xy),\end{aligned}$$

as desired.

Proof of Lemma 1.4. The formula we are to check is

$$(H\delta - dH)(x) = (F - G)(x). \tag{A-1}$$

It suffices to verify (A-1) on a product xy when it is known to hold on each factor. Assuming (A-1) is valid for x and for y,

$$\begin{aligned}(H\delta - dH)(xy) &= H\big(\delta(x)\cdot y + (-1)^{|x|}x\cdot\delta(y)\big)\\ &\quad - d(H(x)\cdot g(y) + f(x)\cdot H(y))\\ &= H\delta(x)\cdot g(y) + f\delta(x)\cdot H(y) + (-1)^{|x|}H(x)\cdot g\delta(y)\\ &\quad + (-1)^{|x|}f(x)\cdot H\delta(y) - dH(x)\cdot g(y) - (-1)^{|x|}H(x)\\ &\quad \cdot dg(y) - df(x)\cdot H(y) - (-1)^{|x|}f(x)\cdot dH(y)\\ &= (H\delta - dH)(x)\cdot g(y) + (-1)^{|x|}f(x)\cdot(H\delta - dH)(y)\\ &= (F(x) - G(x))\cdot g(y) + (-1)^{|x|}f(x)\cdot(F(y) - G(y))\\ &= F(x)\cdot g(y) + (-1)^{|x|}f(x)\cdot F(y) - G(x)\cdot g(y)\\ &\quad - (-1)^{|x|}f(x)\cdot G(y)\\ &= F(xy) - G(xy).\end{aligned}$$

Proof of Theorem 1.5. We shall proceed by induction on the dimension to construct g, G, F, and a derivation H of degree $+2$ from f to fgf which satisfies

$$H\delta - dH = Gf - fF. \tag{A-2}$$

We assume that g, G, F, and H exist and have all the stated properties in dimension $< n$, clearly true for $n = 0$. Lemmas 1.1 through 1.4 assure us that it suffices to define and check the formulas for these on a set of generators for TM and TV.

First we define g and G on a typical generator y in $(TV)_n$. $x_1 = gd(y)$ has already been chosen and $\delta(x_1) = \delta gd(y) = gd^2(y) = 0$, so x_1 is a cycle in TM. $f(x_1) = fgd(y)$ is a cycle in TV, and the formula

$$(Gd + dG)(dy) = (1 - fg)(dy)$$

shows that

$$fg(dy) = d(y - Gd(y)),$$

i.e., $f(x_1)$ is a boundary. As f induces a monomorphism on homology, the cycle x_1 must also be a boundary. Choose x_2 such that $\delta(x_2) = x_1$ and consider $y_1 = y - f(x_2) - Gd(y)$. We have

$$\begin{aligned} d(y_1) &= d(y) - df(x_2) - dGd(y) \\ &= d(y) - f(x_1) + (fg - 1)(dy) = -f(x_1) + fgd(y) = 0, \end{aligned}$$

so y_1 is a cycle in TV. As f induces an epimorphism of homology, there is some cycle $x_3 \in (TM)_n$ such that $f(x_3)$ differs from y_1 by a boundary, i.e., we may write $f(x_3) = y_1 - d(y_2)$. Define $g(y)$ by

$$g(y) = x_2 + x_3.$$

Then $\delta g(y) = \delta(x_2) = x_1 = gd(y)$, so g continues to be a chain map. Define $G(y)$ by

$$G(y) = y_2.$$

Then

$$\begin{aligned} dG(y) &= d(y_2) = y_1 - f(x_3) = (y - f(x_2) - Gd(y)) - f(g(y) - x_2) \\ &= (1 - fg - Gd)(y), \end{aligned}$$

so G continues to be a derivation homotopy from 1 to fg.

Now we define F and H on a typical generator x in $(TM)_n$. Consider

$$w_1 = (1 - gf - F\delta)(x).$$

We have

$$\delta(w_1) = \delta(x) - gf\delta(x) - \delta F\delta(x) = (1 - gf - \delta F)(\delta x) = F\delta^2(x) = 0,$$

so w_1 is a cycle in TM. However,

$$\begin{aligned} f(w_1) &= (f - fgf - fF\delta)(x) \\ &= (1 - fg)(f(x)) - fF(\delta(x)) \\ &= (Gd + dG)f(x) + (H\delta - dH - Gf)\delta(x) \\ &= d(Gf - H\delta)(x) \end{aligned}$$

is a boundary, and since f is one-to-one in homology, w_1 is a boundary, say $w_1 = \delta(w_2)$. Consider $y_3 = Gf(x) - H\delta(x) - f(w_2)$. $d(y_3) = f(w_1) - df(w_2) = 0$, so y_3 is a cycle. As f is onto in homology, there is a cycle w_3 and an element y_4 such that $f(w_3) = y_3 + d(y_4)$. Define $F(x)$ by

$$F(x) = w_2 + w_3;$$

then $\delta F(x) = \delta(w_2) = w_1 = (1 - gf - F\delta)(x)$, so F continues to be a derivation homotopy from 1 to gf. Finally, set

$$H(x) = y_4.$$

Then

$$\begin{aligned} dH(x) &= d(y_4) \\ &= f(w_3) - y_3 \\ &= fF(x) - f(w_2) - (Gf(x) - H\delta(x) - f(w_2)) \\ &= H\delta(x) + fF(x) - Gf(x) \\ &= (H\delta + fF - Gf)(x), \end{aligned}$$

so H continues to satisfy (A-2). This completes the inductive step, and the proof.

Proof of Theorem 2.2. The remainder of the paper is devoted to a proof of Theorem 2.2. The desired result will be rendered almost trivial after we complete a series of lemmas. The first two lemmas assert that the associated chain ringoid functor $\mathscr{T}$ behaves well with respect to derivation homotopies. For brevity we henceforth write TM for a chain tensor algebra (TM, d_M) when the differential is unambiguous or does not need to be specified.

LEMMA A.1. *Let h: $TN \to TV$ be a morphism in CTA and suppose*

$$f, g\colon TM \to TV$$

are two morphisms in CTA with F a derivation homotopy from g to f. Define a map of tensor ringoids ϕ_F: $\mathscr{T}(f, h) \to \mathscr{T}(g, h)$ by setting

$$\phi_F|_{TV\otimes N^+} = 1 \quad \text{and} \quad \phi_F(s(a)) = s(a) - F^+ d_{\mathscr{X}(TM)} s(a)$$

for a a generator of M, where F^+ is the derivation of degree $+1$ from g^+ to f^+: $M^+ \otimes TM \to M^+ \otimes TV$ having $F^+(M^+) = 0$ and $F^+|_{TM} = F$. Then ϕ_F is an isomorphism of chain ringoids.

Proof. We first show that ϕ_F is a morphism of chain ringoids, i.e., that $d_g\phi_F = \phi_F d_f$, where d_g, d_f are the respective differentials on $\mathscr{T}(g, h)$, $\mathscr{T}(f, h)$. Because the differential d_M on the chain tensor algebra TM must preserve the kernel of the augmentation, we may for each generator a of TM write $d_M(a) = \Sigma a_i' b_i$, where $a_i' \in M$ are generators, $b_i \in TM$, and the sum is finite. By (6), $d_{\mathscr{X}(TM)} s(a) = a - \Sigma s(a_i') b_i$ and the formula for $\phi_F(s(a))$ may be written as

$$\phi_F(sa) = sa - F(a) - \sum (-1)^{|a_i'|} sa_i' F(b_i). \tag{A-3}$$

We must verify that

$$(d_g\phi_F - \phi_F d_f)(x) = 0 \tag{A-4}$$

for each generator x of $\mathscr{T}(f, h) = (M^+ \otimes TV \otimes N^+, d_f)$. Formula (A-4) clearly holds for x a generator of $TV \otimes N^+$. So let a be a generator of M, with $d_M(a) = \Sigma a_i' b_i$ as above, and for $d_M(a_i')$ write $\Sigma a_{ij}'' b_{ij}'$ with a_{ij}'' generators and $b_{ij}' \in TM$. The fact that $d_M^2 = 0$ means that

$$\sum (-1)^{|a_i'|} a_i' d_M(b_i) + \sum \sum a_{ij}'' b_{ij}' b_i = 0$$

and consequently that

$$\sum \sum (-1)^{|a_{ij}''|} a_{ij}'' b_{ij}' b_i = -\sum a_i' d_M(b_i).$$

Using this we obtain

$$
\begin{aligned}
&(d_g\phi_F - \phi_F d_f)(sa)\\
&\quad = d_g\Big(sa - F(a) - \textstyle\sum(-1)^{|a_i'|} sa_i' \cdot F(b_i)\Big) - \phi_F\big(f(a) - \sum sa_i' \cdot f(b_i)\big)\\
&\quad = g(a) - \textstyle\sum sa_i' \cdot g(b_i) - d_V F(a) - \sum(-1)^{|a_i'|} g(a_i') \cdot F(b_i)\\
&\qquad + \textstyle\sum(-1)^{|a_i'|}\sum sa_{ij}'' \cdot g(b_{ij}') \cdot F(b_i) + \sum sa_i' d_V F(b_i) - f(a)\\
&\qquad + \textstyle\sum sa_i' \cdot f(b_i) - \sum F(a_i') \cdot f(b_i)\\
&\qquad - \textstyle\sum\sum(-1)^{|a_{ij}''|} sa_{ij}'' \cdot F(b_{ij}') \cdot f(b_i)\\
&\quad = g(a) - f(a) - d_V F(a) - \textstyle\sum\Big(F(a_i') \cdot f(b_i) + (-1)^{|a_i'|} g(a_i') \cdot F(b_i)\Big)\\
&\qquad - \textstyle\sum sa_i' \cdot (g(b_i) - f(b_i) - d_V F(b_i))\\
&\qquad - \textstyle\sum\sum(-1)^{|a_{ij}''|} sa_{ij}'' \cdot \Big(F(b_{ij}') f(b_i) + (-1)^{|b_{ij}'|} g(b_{ij}') F(b_i)\Big)\\
&\quad = (g - f - d_V F - F d_M)(a) - \textstyle\sum sa_i' \cdot (g - f - d_V F - F d_M)(b_i)\\
&\quad = 0.
\end{aligned}
$$

It remains to verify that ϕ_F is an isomorphism. Let $P = M^+ \otimes TV \otimes N^+$, which equals both $\mathscr{T}(f, h)$ and $\mathscr{T}(g, h)$ as a tensor ringoid, and filter P by setting $P^{(-1)} = 0$ and

$$P^{(r)} = \bigoplus_{j+k \le r} (M^+)_j \otimes TV \otimes (N^+)_k \quad \text{for } r \ge 0.$$

Note that ϕ_F preserves this filtration, i.e., $\phi_F(P^{(r)}) \subseteq P^{(r)}$ for all $r \ge -1$. ϕ_F is isomorphic if each of the induced maps

$$\phi_F^{(r)}\colon P^{(r)}/P^{(r-1)} \to P^{(r)}/P^{(r-1)}$$

on the quotients is an isomorphism. Formula (A-3) shows that $\phi_F^{(r)}$ is in fact the identity on each such quotient.

LEMMA A.2. *Let $f\colon TM \to TV$ be a morphism in CTA and suppose*

$$g, h\colon TN \to TV$$

are morphisms in CTA and G is a derivation homotopy from g to h. Then there is an isomorphism of chain ringoids $\phi_G\colon \mathscr{T}(f, g) \to \mathscr{T}(f, h)$.

Proof. Define ϕ_G by letting it be the identity on $M^+ \otimes TV$ and let

$$\phi_G(s'(b)) = s'(b) + G^+ d_{\mathscr{L}(TN)} s'(b).$$

The proof is parallel step for step to that of the previous lemma.

Essentially, Lemmas A.1 and A.2 say that the maps f and g in the corner (8) may be replaced by derivation homotopic maps, without altering the homology of the associated chain ringoid. In order to be able to handle a diagram like (9) in full generality, however, we must also be able to replace the chain tensor algebras themselves with suitably equivalent counterparts. An important special case of this occurs when the diagram commutes precisely and only one of the vertical maps, say the left-most, is not the identity:

$$\begin{array}{ccccc} TM & \xrightarrow{f\alpha} & TV & \xleftarrow{g} & TN \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\ TM' & \xrightarrow{f} & TV & \xleftarrow{g} & TN. \end{array} \qquad \text{(A-5)}$$

One might expect that when α: $TM \to TM'$ induces an isomorphism of homology in (A-5), then the induced map on corners,

$$\mathscr{T}(\alpha, 1, 1): \mathscr{T}(TM, TV, TN) \to \mathscr{T}(TM', TV, TN),$$

would also be isomorphic on homology. One might expect similar results when only the middle or right-most map is not the identity. This intuition turns out to be correct but the proofs are rather technical. The proofs must in fact wait until after the next three lemmas.

The next three lemmas prove that the isomorphisms ϕ_F and ϕ_G introduced in Lemmas A.1 and A.2 are in some sense "natural". We first make the observation that if

$$TM \xrightarrow{f} TN \underset{g_2}{\overset{g_1}{\rightrightarrows}} TV \xrightarrow{h} TW$$

are morphisms in CTA and G is a derivation homotopy from g_1 to g_2, then Gf is a derivation homotopy from $g_1 f$ to $g_2 f$ and hG is a derivation homotopy from hg_1 to hg_2.

LEMMA A.3. *Let h_1, h_2: $TM \to TM'$ be two morphisms in CTA with H a derivation homotopy from h_2 to h_1. Suppose*

$$TM' \xrightarrow{f} TV \xleftarrow{g} TN$$

is any corner in CTA. Let

$$\phi_{fH}\colon \mathscr{T}\left(TM \overset{fh_1}{\to} TV \overset{g}{\leftarrow} TN\right) \to \mathscr{T}\left(TM \overset{fh_2}{\to} TV \overset{g}{\leftarrow} TN\right)$$

be the isomorphism of Lemma A.1. Then the diagram

$$\begin{array}{ccc} \mathscr{T}(fh_1, g) & \xrightarrow{\ \phi_{fH}\ } & \mathscr{T}(fh_2, g) \\ & {\scriptstyle \mathscr{T}(h_1,1,1)} \searrow \quad \swarrow {\scriptstyle \mathscr{T}(h_2,1,1)} & \\ & \mathscr{T}(f, g) & \end{array}$$

commutes up to derivation homotopy. In particular, it induces a commuting diagram on homology and on homogeneous homology.

Proof. Let h_{i*} denote $\mathscr{T}(h_i, 1, 1)$ for $i = 1, 2$. Let $G = \mathscr{T}(-H, 0, 0)$, i.e., G is the unique derivation from $h_{2*} \circ \phi_{fH}$ to h_{1*} such that $G(TV \otimes N^+) = 0$ and for a a generator of M, $G(sa)$ is the image of a under the composition

$$M \overset{-H}{\to} TM' \overset{s}{\to} (M')^+ \otimes TM' \overset{f^+}{\to} (M')^+ \otimes TV \to (M')^+ \otimes TV \otimes N^+ = \mathscr{T}(f, g).$$

We assert that G is a derivation homotopy from $h_{2*} \circ \phi_{fH}$ to h_{1*}. By Lemma 1.3 we need only check this on the generators, and this check is trivial except for sa a generator of M^+.

Writing $d_M(a) = \Sigma a_i' b_i$ with $a_i' \in M$ generators and $b_i \in TM$, we have

$$\begin{aligned} &(Gd_{fh_1} + d_f G)(sa) \\ &\quad = G\left(fh_1(a) - \textstyle\sum sa_i' \cdot fh_1(b_i)\right) - d_f f^+ sH(a) \\ &\quad = \textstyle\sum f^+ sH(a_i') \cdot fh_1(b_i) - f^+ H(a) + f^+ s d_{M'} H(a), \\ &(h_{2*} \circ \phi_{fH} - h_{1*})(sa) \\ &\quad = h_{2*}(sa - fH(a) + f^+ H^+ s d_M(a)) - h_{1*}(sa) \\ &\quad = f^+ sh_2(a) - f^+ sh_1(a) - fH(a) + h_{2*} f^+ H^+ s d_M(a) \\ &\quad = f^+ s(d_{M'} H + H d_M)(a) - fH(a) - \textstyle\sum (-1)^{|a_i'|} f^+ sh_2(a_i') \cdot fH(b_i). \end{aligned}$$

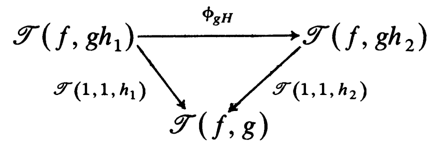
These are equal because $f^+ H(a) = fH(a)$ and

$$\begin{aligned} f^+ sHd_M(a) &= \textstyle\sum f^+ sH(a_i' b_i) \\ &= \textstyle\sum f^+ s\left(H(a_i') \cdot h_1(b_i) + (-1)^{|a_i'|} h_2(a_i') \cdot H(b_i)\right) \\ &= \textstyle\sum f^+ sH(a_i') \cdot fh_1(b_i) + \textstyle\sum (-1)^{|a_i'|} f^+ sh_2(a_i') \cdot fH(b_i). \end{aligned}$$

LEMMA A.4. *Let H be a derivation homotopy from h_1 to h_2: $TN \to TN'$ in CTA and let*

$$TM \xrightarrow{f} TV \xleftarrow{g} TN'$$

be any corner in CTA. Then the diagram



commutes up to derivation homotopy and induces a commuting diagram on homology and on homogeneous homology.

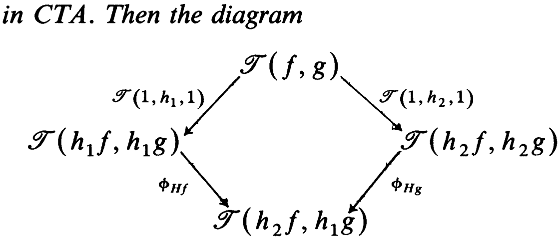
Proof. Follow the same argument as for the previous lemma. The derivation homotopy from h_{1*} to $h_{2*} \circ \phi_{gH}$ is given by $F = \mathscr{T}(0, 0, -H)$, where $F(M^+ \otimes TV) = 0$ and for b a generator of N, $F(s'b)$ is the image of b under the composition

$$N \xrightarrow{-H} TN' \xrightarrow{s'} TN' \otimes (N')^+ \xrightarrow{g^+} TV \otimes (N')^+ \to M^+ \otimes TV \otimes (N')^+ = \mathscr{T}(f, g).$$

LEMMA A.5. *Let H be a derivation homotopy from h_2 to h_1: $TV \to TV'$ in CTA and let*

$$TM \xrightarrow{f} TV \xleftarrow{g} TN$$

be any corner in CTA. Then the diagram



commutes up to derivation homotopy and induces a commuting diagram on homology and on homogeneous homology.

Proof. Let $F = \mathscr{T}(0, H, 0)$ be the derivation from $\phi_{Hg} \circ h_{2*}$ to $\phi_{Hf} \circ h_{1*}$ with $F(M^+) = F(N^+) = 0$ and $F(x) = H(x)$ for x a generator of TV. Let d_1 and d_0 denote the differentials on $\mathscr{T}(f, g)$ and $\mathscr{T}(h_2 f, h_1 g)$ respectively. By Lemma 1.3 we need only check that

$$(d_0 F + F d_1)(x) = (\phi_{Hg} \circ h_{2*} - \phi_{Hf} \circ h_{1*})(x)$$

for x a generator of $\mathscr{T}(f, g) = (M^+ \otimes TV \otimes N^+, d_1)$.

For x a generator of TV, we have

$$(d_0F + Fd_1)(x) = (d_VH + Hd_V)(x) = (h_2 - h_1)(x)$$
$$= \phi_{Hg}(h_{2*}(x)) - \phi_{Hf}(h_{1*}(x)).$$

For sa a generator of M^+,

$$\begin{aligned}(d_0F + Fd_1)(sa) &= Fd_1s(a) = F(f(a) - f^+sd_M(a))\\ &= Hf(a) - H^+f^+sd_M(a)\\ &= s(a) - (s(a) - Hf(a) + H^+f^+sd_M(a))\\ &= s(a) - \phi_{Hf}(s(a))\\ &= \phi_{Hg}(h_{2*}(sa)) - \phi_{Hf}(h_{1*}(sa)).\end{aligned}$$

The check for a generator $s'b$ of N^+ is similar.

The next lemma uses the results assembled so far to show that diagram (A-5) induces a quasi-isomorphism of associated chain ringoids when α is a quasi-isomorphism. The remaining two lemmas prove similar results when one of the other two vertical maps is the quasi-isomorphism.

LEMMA A.6. *Suppose*

$$TM' \xrightarrow{f} TV \xleftarrow{g} TN$$

is a corner in CTA and the morphism α: $TM \to TM'$ *in CTA induces an isomorphism on homology. Then* $\alpha_* = \mathscr{T}(\alpha, 1, 1)$: $\mathscr{T}(f\alpha, g) \to \mathscr{T}(f, g)$ *induces an isomorphism on homology and on homogeneous homology.*

Proof. By Theorem 1.5 choose α': $TM' \to TM$ and derivation homotopies F from 1 to $\alpha'\alpha$ and F' from 1 to $\alpha\alpha'$. Consider the diagram

$$\begin{array}{rl} & \mathscr{T}\left(TM \xrightarrow{f\alpha\alpha'\alpha} TV \xleftarrow{g} TN\right)\\ \alpha''_* = \mathscr{T}(\alpha,1,1) \downarrow & \\ & \mathscr{T}\left(TM' \xrightarrow{f\alpha\alpha'} TV \xleftarrow{g} TN\right)\\ \alpha'_* = \mathscr{T}(\alpha',1,1) \downarrow & \\ & \mathscr{T}\left(TM \xrightarrow{f\alpha} TV \xleftarrow{g} TN\right)\\ \alpha_* = \mathscr{T}(\alpha,1,1) \downarrow & \\ & \mathscr{T}\left(TM' \xrightarrow{f} TV \xleftarrow{g} TN\right).\end{array}$$

By Lemma A.3, $\alpha_* \alpha'_*$ and $\phi_{fF'}$ induce the same map on homology and the latter is an isomorphism by Lemma A.1, so α'_* induces an injection on homology. Likewise $\alpha'_* \alpha''_*$ agrees on homology with $\phi_{f\alpha F}$, so α'_* is surjective on homology. Thus α'_* and hence α_* are isomorphic on homology. The same proof works if "homology" is replaced throughout by "homogeneous homology".

The proofs of the last two lemmas are similar to the proof of Lemma A.6 and will be omitted.

LEMMA A.7. *Suppose in CTA we have a corner*

$$TM \xrightarrow{f} TV \xleftarrow{g} TN'$$

and a quasi-isomorphism γ: $TN \to TN'$. *Then*

$$\gamma_* = \mathscr{T}(1,1,\gamma) \colon \mathscr{T}(f, g\gamma) \to \mathscr{T}(f, g)$$

induces an isomorphism on homology and on homogeneous homology

LEMMA A.8. *Suppose in CTA we have a corner*

$$TM \xrightarrow{f} TV \xleftarrow{g} TN$$

and a quasi-isomorphism β: $TV \to TV'$. *Then*

$$\beta_* = \mathscr{T}(1,\beta,1) \colon \mathscr{T}(f, g) \to \mathscr{T}(\beta f, \beta g)$$

induces an isomorphism on homology and on homogeneous homology.

We are at last in a position to prove Theorem 2.2.

Proof of Theorem 2.2. Referring to (9), use the derivation homotopy axiom to choose derivation homotopies F from $\mathscr{F}(f\alpha)$ to $\mathscr{F}(\beta f')$ and G from $\mathscr{F}(\beta g')$ to $\mathscr{F}(g\gamma)$. The map ψ: $\mathscr{T}\mathscr{F}(f', g') \to \mathscr{T}\mathscr{F}(f, g)$ given by

$$\psi = \mathscr{F}(\gamma)_* \circ \phi_G \circ \mathscr{F}(\alpha)_* \circ \phi_F \circ \mathscr{F}(\beta)_*$$

is a morphism of chain ringoids, and each factor in this composition has already been shown to induce an isomorphism on homology and on homogeneous homology.

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