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HIGHER ORDER SWEEPING OUT

BY

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1. Introduction

Let T act in a measure space X with measure m and $m(X) = 1$. We say T *sweeps out* [3] if $m(A) > 0$ implies $m(\bigcup_{n=1}^{\infty} T^{k_n}A) = 1$ for all increasing sequences (k_n). We will say T is *lightly mixing* if $m(A) > 0$ and $m(B) > 0$ imply

$$\liminf_{n \to \infty} m(T^n A \cap B) > 0.$$

Lightly mixing implies mildly mixing [5] and weakly mixing [2]. In particular, if T is lightly mixing, then T is mixing on a sequence of density one [3], [6]. It is shown in [1] that the conditions for sweeping out and lightly mixing are equivalent. The term sequence mixing is used in [1] but might be confused with mixing on a sequence.

The definitions of higher order sweeping out and higher order lightly mixing are given in §2. In §3 it is shown that k-sweeping out is equivalent to lightly k-mixing, $k \geq 1$. The examples of transformations that are partially k-mixing but not partially $(k + 1)$-mixing [4] are also examples of transformations that are lightly k-mixing but not lightly $(k + 1)$-mixing, $k \geq 1$. The construction in [1] for $k = 1$ is generalized in §3 to obtain transformations that are lightly k-mixing but not partially k-mixing, $k \geq 1$.

A transformation T *uniformly sweeps out* if for each set A of positive measure and $\varepsilon > 0$ there exists $N = N(A, \varepsilon)$ such that the measure of the union of any N iterates of A is greater than $1 - \varepsilon$. If T is mixing, then T uniformly sweeps out [3]. It is not known if the converse is true. Higher order uniform sweeping out is introduced and in §4 it is shown that $(2k - 1)$-mixing implies uniform k-sweeping out, $k \geq 1$.

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2. Definitions

Let $(X, \mathscr{B}, m)$ be a measure space isomorphic to the unit interval with Lebesgue measure. Let T be an invertible ergodic measure preserving transformation mapping X into X. A transformation T is *partially k-mixing* [4] if there exists $\beta > 0$ such that for all A_i, $0 \leq i \leq k$ $(t_0 = 0)$,

$$\liminf_{t_{i+1} - t_i \to \infty} m\left(\bigcap_{i=0}^{k} T^{t_i} A_i \right) \geq \beta \prod_{i=0}^{k} m(A_i). \tag{2.1}$$

Given α, $0 < \alpha \leq 1$, T is $\alpha - k$*-mixing* [4] if (2.1) holds for $\beta = \alpha$ but not for $\beta > \alpha$. If $\alpha = 1$, then the limit in (2.1) exists and T is k-mixing.

We shall say that T is *lightly k-mixing* if for all A_i of positive measure, $0 \leq i \leq k$ $(t_0 = 0)$,

$$\liminf_{t_{i+1} - t_i \to \infty} m\left(\bigcap_{i=0}^{k} T^{t_i} A_i \right) > 0. \tag{2.2}$$

If T is lightly mixing $(k = 1)$, then T is weakly mixing and also mildly mixing [5]. In particular, there exists an increasing sequence s of density one such that T is mixing on s [3], [6]. A transformation that is lightly mixing but not partially mixing was constructed in [1]. This construction will be extended in §3 to obtain a transformation that is lightly k-mixing but not partially k-mixing.

Let d_i be positive integers and A_i sets of positive measure, $1 \leq i \leq k$. Let

$$d^{(k)} = (d_1, d_2, \ldots, d_k), \qquad A^{(k)} = (A_1, A_2, \ldots, A_k).$$

A k-fold intersection is denoted by

$$I(T, d^{(k)}, A^{(k)}) = T^{d_1}\left(A_1 \cap T^{d_2}\left(A_2 \cap \cdots \cap T^{d_k} A_k\right) \ldots\right). \tag{2.3}$$

If $d_i = t_i - t_{i-1}$, $1 \leq i \leq k$ $(t_0 = 0)$, then

$$\bigcap_{i=1}^{k} T^{t_i} A_i = I(T, d^{(k)}, A^{(k)}). \tag{2.4}$$

3. Sweeping out

Let $d_n^{(k)} = (d_{n,i}\colon 1 \leq i \leq k)$, $n \geq 1$, and assume that the entries of $d_n^{(k)}$ are positive and that $d_n^{(k)}$ is increasing. That is, $0 < d_{n,i} < d_{n+1,i}$, $1 \leq i \leq k$,

$n \geq 1$. We will say that T *k-sweeps out* if for all increasing $d_n^{(k)}$ and $A^{(k)}$, we have

$$(3.1) \qquad m\left(\bigcup_{n=1}^{\infty} I\left(T, d_n^{(k)}, A^{(k)}\right)\right) = 1.$$

The following result is proved in [1] for $k = 1$, where lightly mixing is called sequence mixing.

(3.2) THEOREM. *A transformation T k-sweeps out if and only if T is lightly k-mixing.*

Proof. Suppose (3.1) is not satisfied for some $d_n^{(k)}$ and $A^{(k)}$. Define B by

$$(1) \qquad B = \left(\bigcup_{n=1}^{\infty} I\left(T, d_n^{(k)}, A^{(k)}\right)\right)^c.$$

Hence $m(B) > 0$ and $m(I(T, d_n^{(k)}, A^{(k)}) \cap B) = 0$, $n \geq 1$, so T is not lightly k-mixing. Conversely, if T is not lightly k-mixing, then there exist increasing $d_n^{(k)}$, $A^{(k)}$, and B of positive measure such that

$$(2) \qquad \liminf_{n \to \infty} m\left(I\left(T, d_n^{(k)}, A^{(k)}\right) \cap B\right) = 0.$$

Choose n_i such that

$$(3) \qquad m\left(I\left(T, d_{n_i}^{(k)}, A^{(k)}\right) \cap B\right) < m(B)/5^i, \quad i \geq 1.$$

From (3) we obtain

$$(4) \qquad m\left(\bigcup_{i=1}^{\infty} I\left(T, d_{n_i}^{(k)}, A^{(k)}\right)\right) < 1 - 3m(B)/4.$$

Thus T does not k-sweep out.

(3.3) *Example.* In [4], transformations T_k, $k \geq 1$, were constructed such that T_k is $(1 - j/(k+1)) - j$-mixing, $1 \leq j \leq k+1$. In particular, T_k is $(1/(k+1) - k)$-mixing but not partially $(k+1)$-mixing. Moreover, there exist $p_{n,i} \to \infty$, $1 \leq i \leq k+1$, such that for every set A,

$$(3.4) \qquad \lim_{n \to \infty} m\left(T_k^{p_{n,k+1}}\left(\ldots\left(T_k^{p_{n,1}} A \cap A^c\right) \cap A\right)\ldots\right) \cap B\right) = 0,$$

where $B = A^c$ if k is even and $B = A$ if k is odd. Thus T_k is partially

k-mixing and hence lightly k-mixing but (3.4) implies T_k is not lightly $k+1$-mixing, $k \geq 1$.

(3.5) *Example.* Given a positive integer k, we will now construct T that is lightly k-mixing but not partially k-mixing by extending the case for $k = 1$ in [1]. The idea is to start with a partially k-mixing transformation S and form the product T of S with itself countably many times. It is straightforward to check that T is not partially k-mixing by considering cylinder sets. However, one also sees that T is lightly k-mixing for cylinder sets. To prove that T is lightly k-mixing, we will approximate measurable sets by cylinder sets as in [1].

Let S be $\alpha - k$-mixing and define the product transformation

$$T = \prod_{i=1}^{\infty} S_i, \quad S_i = S, i \geq 1.$$

Thus T is defined on the direct product space $(Y, \mathscr{F}, \mu)$, where $Y = \prod_{i=1}^{\infty} X_i$, $\mathscr{F} = \prod_{i=1}^{\infty} \mathscr{B}_i$, and $\mu = \prod_{i=1}^{\infty} m_i$, $X_i = X$, $\mathscr{B}_i = \mathscr{B}$, $m_i = m$, $i \geq 1$.

Since S is $\alpha - k$-mixing, there exist increasing $d_n^{(k)}$ and A_i of positive measure, $0 \leq i \leq k$, such that

$$(1) \qquad \lim_{n \to \infty} m\left(I\left(S, d_n^{(k)}, A^{(k)}\right) \cap A_0\right) = \alpha \prod_{i=0}^{k} m(A_i).$$

Let $F_{i,l} = A_i \times A_i \times \cdots \times A_i \times X \times X \times \cdots$, $0 \leq i \leq k$, where A_i appears l times. Now (1) implies

$$(2) \qquad \lim_{n \to \infty} \mu\left(I\left(T, d_n^{(k)}, F_l^{(k)}\right) \cap F_{0,l}\right) = \alpha^l \prod_{i=0}^{k} \mu(F_{i,l}).$$

Since $\alpha^l \to 0$, (2) implies T cannot be partially k-mixing. It remains to verify that T is lightly k-mixing.

Fix F_r with $\mu(F_r) > 0$, $0 \leq r \leq k$. We need to show

$$(3) \qquad \liminf_{t_{r+1} - t_r \to \infty} \mu\left(\bigcap_{r=0}^{k} T^{t_r} F_r\right) > 0.$$

The proof is a generalization of the proof for $k = 1$ in [1]. Since there are several modifications, as well as some omissions in [1], the proof will be given in detail.

Let β_i, $0 < \beta_i < 1$, satisfy

$$(4) \qquad \prod_{i=1}^{\infty} \beta_i = \gamma > (2k+1)/(2k+2).$$

Let $\mathscr{C}$ denote the class of finite dimensional cylinder sets in $\mathscr{F}$; hence $\mathscr{C}$ is dense in $\mathscr{F}$. Fix $F \in \mathscr{F}$. We now show there exists $C(1) \in \mathscr{C}$ such that

$$\mu(C(1)) < \mu(F) \quad \text{and} \quad \mu(F\Delta C(1)) < (1-\beta_1)\mu(F).$$

Let G be a subset of F satisfying $\delta < \mu(F-G) < \varepsilon$ where

$$\varepsilon = \tfrac{9}{10}(1-\beta_1)\mu(F) \quad \text{and} \quad \delta = \tfrac{1}{2}(1-\beta_1)\mu(F).$$

Let $C(1) \in \mathscr{C}$ be chosen so that

$$\mu(G\Delta C(1)) < \eta \quad \text{where } \eta = \tfrac{1}{10}(1-\beta_1)\mu(F).$$

In the next few computations, let us suppress the index, writing C instead of $C(1)$.

Then, first of all,

$$\begin{aligned}\mu(C) &\le \mu(C-G) + \mu(G) \le \eta + \mu(F) - \mu(F-G)\\ &\le \mu(F) + \eta - \delta < \mu(F),\end{aligned}$$

as desired. Secondly,

$$\mu(F\Delta C) = \mu(F-C) + \mu(C-F) = \mu(G-C) + \mu(F-G-C) + \mu(C-F).$$

Since the last term is no greater than $\mu(C-G)$, the inequality continues:

$$\begin{aligned}&\le \mu(G-C) + \mu(C-G) + \mu(F-G-C) \le \mu(G\Delta C) + \mu(F-G)\\ &\le \eta + \varepsilon = (1-\beta_1)\mu(F),\end{aligned}$$

which is the second requirement on $C = C(1)$. This latter inequality implies $\mu(C(1) \cap F) \ge \beta_1 \mu(F)$.

Let $F(1) = C(1) \cap F$. Choose $C(2) \in \mathscr{C}$ such that

(5) $$\mu(C(2)\Delta F(1)) \le (1-\beta_2)\mu(F(1)).$$

Replace $C(2)$ by $C(2) \cap C(1)$ if $C(2) \not\subset C(1)$. Hence (5) still holds and

(6) $$\mu(C(2) \cap F(1)) \ge \beta_2 \mu(F(1)) \ge \beta_2\beta_1\mu(F).$$

Let $F(2) = C(2) \cap F(1)$. Proceeding inductively, assume we have $F(i-1)$ and choose $C(i) \in \mathscr{C}$ such that

(7) $$\mu(C(i)\Delta F(i-1)) \le (1-\beta_i)\mu(F(i-1)).$$

Replace $C(i)$ by $C(i) \cap C(i-1)$ if $C(i) \not\subset C(i-1)$. Define

$$F(i) = C(i) \cap F(i-1).$$

We assume

$$\mu(F(i-1)) \geq \prod_{j=0}^{i-1} \beta_j \mu(F). \tag{8}$$

Now (7) implies (8) holds with $i-1$ replaced by i. Let

$$C_\infty = \bigcap_{i=1}^{\infty} C(i) \quad \text{and} \quad F_\infty = \bigcap_{i=1}^{\infty} F(i). \tag{9}$$

We conclude that $\mu(C_\infty) = \mu(F_\infty) \geq \gamma\mu(F)$ and $C_\infty \subset F$ except possibly for a null set.

The sets $C(i)$ will now be considered in more detail. Let $C(i)$ have dimension N_i as a cylinder set, $i \geq 1$. We have $C(i) = \bigcup_j C(i,j)$, where j is in a finite index set and $C(i,j)$ is a product of N_i sets in $\mathscr{B}$ for each j; hence

$$C(i,j) = \prod_{l=1}^{N_i} C(i,j,l) \times X \times X \times \cdots. \tag{10}$$

Therefore $C(i,j) = G(i,j) \cap H(i,j)$, where

$$G(i,j) = \prod_{l=1}^{N_1} C(i,j,l) \times X \times X \times \cdots, \tag{11}$$

$$H(i,j) = X \times X \times \cdots \times X \times \prod_{l=N_1+1}^{N_j} C(i,j,l) \times X \times \cdots. \tag{12}$$

By subdividing the $C(i,j)$ if necessary, we can assume the $G(i,j)$ satisfy $G(i,a) \cap G(i,b) = \emptyset$ if $G(i,a) \neq G(i,b)$. Since $C(i) \subset C(1)$, we have $\bigcup_j G(i,j) \subset C(1)$. Define $U(i,a) = \bigcup_j H(i,j)$, where the union is over those $H(i,j)$ with $G(i,j) = G(i,a)$. Let

$$J(i) = \{a \colon \mu(U(i,a)) > 2\gamma - 1\}. \tag{13}$$

Now we have

$$(14)\qquad \mu(C(i)) = \mu\Big(\bigcup_j G(i,j)\cap H(i,j)\Big)$$
$$\le \mu\Big(\bigcup_{a\in J(i)} G(i,a)\Big) + \mu\Big(\bigcup_{a\notin J(i)} G(i,a)\cap H(i,a)\Big)$$
$$\le \mu\Big(\bigcup_{a\in J(i)} G(i,a)\Big) + (2\gamma-1)\mu\Big(\bigcup_{a\notin J(i)} G(i,a)\Big).$$

Now since $\mu(C(1)) < \mu(F)$, it is easy to check that $\gamma\mu(C(1) \le \mu(C(i))$. This, together with (14), implies

(15)

$$\gamma\mu(C(1)) \le \mu\Big(\bigcup_{a\in J(i)} G(i,a)\Big) + (2\gamma-1)\Big(\mu(C(1)) - \mu\Big(\bigcup_{a\in J(i)} G(i,a)\Big)\Big).$$

From (15) we obtain

$$(16)\qquad \mu(C(1))/2 \le \mu\Big(\bigcup_{a\in J(i)} G(i,a)\Big).$$

Now consider $F = F_r$, $0\le r\le k$, and the corresponding sets $G_r(i)$, $C_r(i,j)$, $G_r(i,j)$, $H_r(i,j)$, $J_r(i)$, etc. For each i we can assume $C_r(i)$ have the same dimension N_i, $0\le r\le k$. Now

$$(17)\qquad \bigcap_{r=0}^{k} T^{t_r}\Big(\bigcup_{a\in J_r(i)} C_r(i,a)\Big) \subset \bigcap_{r=0}^{k} T^{t_r}\Big(\bigcup_j C_r(i,j)\Big).$$

Note that for each i and r, $J_r(i)\ne\varnothing$, by (16). Choose $a_r\in J_r(i)$ for each i and r, and define $S(i,r) = \{s\,|\,G_r(i,s) = G_r(i,a_r)\}$. Then

$$(18)\qquad \mu\Big(\bigcap_{r=0}^{k} T^{t_r}\Big(\bigcup_{s\in S(i,r)} C_r(i,s)\Big)\Big)$$
$$= \mu\Big(\bigcap_{r=0}^{k} T^{t_r}\Big(\bigcup_{s\in S(i,r)} G_r(i,s)\cap H_r(i,s)\Big)\Big)$$
$$= \mu\Big(\bigcap_{r=0}^{k} T^{t_r}\Big(G_r(i,a_r)\cap \bigcup_{s\in S(i,r)} H_r(i,s)\Big)\Big)$$
$$= \mu\Big(\bigcap_{r=0}^{k} T^{t_r}(G_r(i,a_r))\Big)\mu\Big(\bigcap_{r=0}^{k} T^{t_r}\Big(\bigcup_{s\in S(i,r)} H_r(i,s)\Big)\Big)$$
$$= \mu\Big(\bigcap_{r=0}^{k} T^{t_r}(G_r(i,a_r))\Big)\mu\Big(\bigcap_{r=0}^{k} T^{t_r}(U_r(i,a_r))\Big).$$

By (13),

$$\mu\left(\bigcap_{r=0}^{k} T^{t_r}(U_r(i,a_r))\right) \geq \sum_{r=0}^{k} \mu(U_r(i,a_r)) - k \tag{19}$$
$$\geq (k+1)(2\gamma-1)-k$$
$$= \gamma(2k+2)-(2k+1)$$
$$= p > 0.$$

Thus (18), (19), and S being $\alpha - k$-mixing imply

$$\liminf_{t_{r+1}-t_r\to 0} \mu\left(\bigcap_{r=0}^{k} T^{t_r}\left(\bigcup_{s\in S(i,r)} C_r(i,s)\right)\right) \geq \liminf p\mu\left(\bigcap_{r=0}^{k} T^{t_r}G_r(i,a_r)\right) \tag{20}$$
$$\geq p\alpha^{N_1}\prod_{r=0}^{k}\mu(G_r(i,a_r)).$$

Summing, for one r at a time, over the different disjoint $G_r(i,a)$'s, $a \in J_r(i)$ gives, by (16),

$$\liminf \mu\left(\bigcap_{r=0}^{k} T^{t_r}\left(\bigcup_{a\in J_r(i)} C_r(i,a)\right)\right) \geq p\alpha^{N_1}\prod_{r=0}^{k}\frac{\mu(C_r(1))}{2} = P > 0. \tag{21}$$

Therefore

$$\liminf \mu\left(\bigcap_{r=0}^{k} T^{t_r}C_r(i)\right) = \liminf \mu\left(\bigcap_{r=0}^{k} T^{t_r}\left(\bigcup_S C_r(i,s)\right)\right) \geq P. \tag{22}$$

Since $C_r(i) \to C_r(\infty)$ as $i \to \infty$, (22) holds with $C_r(i)$ replaced by $C_r(\infty)$, $0 \leq r \leq k$. Now $C_r(\infty) \subset F_r$ except for possibly a null set, $0 \leq r \leq k$, so we obtain (3) as required. Thus T is lightly k-mixing.

(3.6) *Example.* Consider $S = T_k$ in Example (3.3). Thus S is $\alpha_j - j$-mixing, $\alpha_j = (1 - j/(k+1))$, $1 \leq j \leq k$. Hence Example (3.5) yields T that is not partially 1-mixing but is lightly j-mixing, $1 \leq j \leq k$.

Given $1 \leq l < k$, we do not have an example of a transformation that is partially j-mixing only for $1 \leq j \leq l$ and lightly j-mixing only for $j \leq k$. In particular, for $l = 1$ and $k = 2$, we do not have a transformation that is partially 1-mixing, not partially 2-mixing, but is lightly 2-mixing.

A long-open problem is whether mixing implies 2-mixing. A more basic problem is whether mixing even implies lightly 2-mixing. The transformation T_2 in Example (3.3) is $\frac{1}{2}$-mixing but not lightly 2-mixing. We do not know if there exists T that is α-mixing and not lightly 2-mixing for $\alpha > \frac{1}{2}$.

The argument in Example (3.5) can be extended to yield the following result. The only change is that in (20), α^{N_1} will be replaced by $\prod_{i=1}^{N_1}\alpha_i$.

(3.7) THEOREM. *If T_i is $\alpha_i - k$-mixing, $\alpha_i > 0$, $i \geq 1$, then $T = \prod_{i=1}^{\infty} T_i$ is lightly k-mixing.*

Note that Theorem (3.7) does not depend on the size of α_i. However, if $\lim_i \alpha_i = 0$, then T will not be partially k-mixing. If $\alpha_i = \alpha$, $i \geq 1$, then it is possible that T will also be $\alpha - k$-mixing. This depends on the timing of the α-mixing of the T_i's. To illustrate this behavior, we will construct T and R that are both $\frac{1}{2}$-mixing and $T \times R$ is also $\frac{1}{2}$-mixing. This example can be extended to obtain T_i that are all $\frac{1}{2}$-mixing, $i \geq 1$, and $\prod_{i=1}^{\infty} T_i$ is also $\frac{1}{2}$-mixing.

(3.8) *Example.* The transformations will be constructed on the unit interval X with Lebesgue measurable sets $\mathscr{B}$ and Lebesgue measure m. T and R will both be $\frac{1}{2}$-mixing. However, while T is $\frac{1}{2}$-mixing for certain intervals, R will be mixing very well. Then while R is $\frac{1}{2}$-mixing for certain intervals, T will be mixing very well. Thus while one transformation is $\frac{1}{2}$-mixing, the other transformation will be essentially mixing with respect to certain intervals. The result of the construction is that for any intervals I_i, $1 \leq i \leq 4$, we will have

$$\liminf_{n\to\infty} m(T^n I_1 \cap I_2) m(R^n I_3 \cap I_4) \geq \tfrac{1}{2} \prod_{i=1}^{4} m(I_i). \tag{1}$$

Let $\mu = m \times m$, $A = I_1 \times I_3$ and $B = I_2 \times I_4$; hence (1) implies

$$\liminf_{n\to\infty} \mu\left((T \times R)^n A \cap B\right) \geq \tfrac{1}{2}\mu(A)\mu(B). \tag{2}$$

Measurable sets in $\mathscr{B} \times \mathscr{B}$ can be approximated arbitrarily well by disjoint rectangles $I \times J$ for intervals I and J. Thus (2) will also hold for A, $B \in \mathscr{B} \times \mathscr{B}$.

Both transformations are constructed using the same induction step. It consists in first mixing only half the space and then mixing the whole space. The induction step is diagrammed in Fig. 1.

The construction is described in terms of towers and cutting and stacking [3], [4]. A column C is an ordered set of disjoint left-closed right-open intervals of the same length. A tower G is an ordered set of disjoint columns. An interval in a column in G is referred to as a *level in G*. We picture levels in a column arranged vertically as rungs on a ladder. If x is not in the top level in a column in G, then $T_G(x)$ is the point above x.

We let SG denote the tower formed by independent cutting and stacking of G [3], [4]. The set consisting of the union of levels in columns in G is also denoted by G. The transformation $T(G) = \lim_{n\to\infty} T_{S^n G}$ is defined on G. An *M-tower G* is a tower with two column heights that are relatively prime. In this

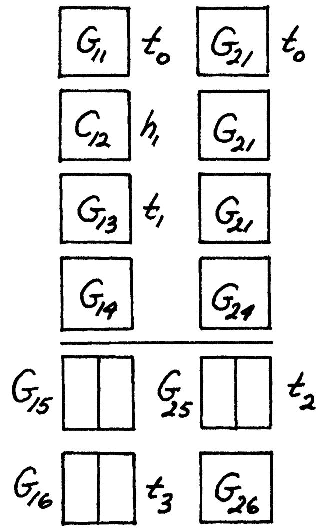

FIG. 1.

case $T(G)$ is mixing. Thus if G is an M-tower, then for $\varepsilon > 0$ there exists $t^* = t(G, \varepsilon)$ such that for all levels I and J in G,

$$(1) \qquad |m(T(G)^t I \cap J)m(G)/m(I)m(J) - 1| < \varepsilon, \quad t^* \leq t.$$

Let $t^{**} \geq t^*$. The definition of $T(G)$ implies there exists n so large that if T extends $T_{S^n G}$, then for all levels I and J in G,

$$(2) \qquad |m(T^t I \cap J)m(G)/m(I)m(J) - 1| < \varepsilon, \quad t^* \leq t \leq t^{**}.$$

The stacking construction for introducing periodicity in a tower will now be briefly described. Only towers with rational widths will be considered. In this case the columns in a tower G can be cut into subcolumns of equal width w. These columns are then stacked to form a single column C_1 of width w. Each level in G is a union of levels in C_1. Let p be a positive integer. We cut C_1 into p equal subcolumns and stack them to form a column C of width w/p. Let h be the height of C_1. If T extends T_C and I is a level in C_1, then the construction implies

$$(3) \qquad m(T^h I \cap I) > (1 - 1/p)m(I).$$

Since each level J in G is a union of levels I in C_1, (3) also holds for J in G. We denote $C_p(G) = C$. Thus $C_p(G)$ is a column obtained by converting G to a column, cutting this column into p equal subcolumns, and stacking these subcolumns.

The induction step in Figure 1 will now be described. We begin with M-towers G_{i1}, $i = 1, 2$, of equal measure and $\varepsilon > 0$. There exists t_0 such that if $G = G_{i1}$ and I and J are levels in G, then (1) holds with $t^* = t_0$.

Choose p so that $1/p < \varepsilon$. Form $C_{12} = C_p(G_{11})$. Thus (3) implies there is a positive integer h_1 so that if T extends $T_{C_{12}}$ and J is a level in G_{11}, then

$$(4) \qquad m(T^{h_1}J \cap J) > (1 - \varepsilon)m(J).$$

We now convert C_{12} into an M-tower as follows. Cut C_{12} into q equal subcolumns and add one extra level to the last subcolumn to obtain an M-tower G_{13} with q columns. The measure of the extra level is certainly less than $1/q$ and can be made arbitrarily small by choosing q large.

Since G_{13} is an M-tower, we can choose $t_1 \geq t(G_{13}, \varepsilon)$ so that $t_1 > t_0$ and if I and J are levels in G_{13}, then (1) holds with $G = G_{13}$ and $t^* = t_1$. We can now choose n so large that if $G_{14} = S^n G_{13}$ and T extends $T_{G_{14}}$, then (3) implies

$$(5) \qquad |m(T^tI \cap J)m(G_{13})/m(I)m(J) - 1| < \varepsilon, \quad t = t_1,$$

where I and J are levels in G_{13}.

Now choose n so large that if $G_{24} = S^n G_{21}$ and T extends $T_{G_{24}}$, then

$$(6) \qquad |m(T^tI \cap J)m(G_{21})/m(I)m(J) - 1| < \varepsilon, \quad t_0 \leq t \leq t_1,$$

where I and J are levels in G_{21}.

We now want to mix G_{14} with G_{24} while preserving the previous mixing. Cut each column in G_{i4} in half and form two copies $g_{ji} = G_{i4}/2$, $j = 1, 2$, $i = 1, 2$. Let $G_{15} = (g_{11}, g_{12})$ and $G_{25} = (g_{21}, g_{22})$. Thus G_{i5} is a half-size copy of G_{14} next to a half-size copy of G_{24}, $i = 1, 2$. Choose $t_2 \geq t(G_{25}, \varepsilon)$ so that $t_2 > t_2$. In particular, levels in g_{21} mix with levels in g_{22} at t_2. To preserve $\frac{1}{2}$-mixing during $[t_1, t_2]$, we will continue to mix g_{1j} separately, $j = 1, 2$. Thus we can choose n so large that if

$$G_{16} = (S^n g_{11}, S^n g_{12})$$

and T extends T_{16}, then (5) implies

$$(7) \qquad |m(T^tI \cap J)m(g_{11})/m(I)m(J) - 1| < \varepsilon, \quad t_1 \leq t \leq t_2,$$

where I and J are levels in g_{11}. Also, (6) implies

$$(8) \qquad |m(T^tI \cap J)m(g_{12})/m(I)m(J) - 1| < \varepsilon, \quad t_1 \leq t \leq t_2,$$

where I and J are levels in g_{12}. Here we used the fact that g_{11} is a copy of G_{14} and g_{12} is a copy of G_{24}.

Choose $t_3 \geq t(G_{16}, \varepsilon)$ so that $t_3 > t_2$. Now choose n so large that if $G_{26} = S^n G_{25}$ and T extends $T_{G_{26}}$, then

$$(9) \qquad |m(T^t I \cap J)m(G_{25})/m(I)m(J) - 1| < \varepsilon, \quad t_2 \leq t \leq t_3,$$

where I and J are levels in G_{25}.

The induction step will now be used to construct T and R. Let G_0 be an M-tower and let $G_{11} = G_{11}^1 = G_0/2$ and $G_{21} = G_{21}^1 = G_0/2$. Let $\varepsilon = \varepsilon_1$. Apply the induction step to obtain $G_{1j}^1 = G_{1j}$, $2 \leq j \leq 6$, $G_{2j}^1 = G_{2j}$, $4 \leq j \leq 6$, and $t_{1i} = t_i$, $0 \leq i \leq 3$.

For convenience we set $T_{kij} = T_{G_{ij}^k}$. The towers defining R will be denoted by H_{ij}^k and we let $R_{kij} = T_{H_{ij}^k}$.

We have T_{1i6}, $i = 1, 2$. If T extends T_{1i6}, $i = 1, 2$, and I and J are levels in G_0, then the induction step guarantees

$$(10) \qquad m(T^t I \cap J) \geq (1 - \varepsilon_1)m(I)m(J)/2m(G_0), \quad t_{10} \leq t \leq t_{13}.$$

If I is a level in G_0, then $I/2$ is a level in G_{11}^1. Hence (4) of the induction step implies there is a positive integer $h = h_{11}$ such that

$$(11) \qquad m(T^h I \cap I) > (1 - \varepsilon_1)m(I)/2.$$

We will now begin defining R so that R is mixing well for $[t_{10}, t_{13}]$. Let $H_0 = G_0$. Choose n so large that if $H_1 = S^n H_0$ and R extends T_{H_1}, then for levels I and J in H_0,

$$(12) \qquad |m(R^t I \cap J)m(H_0)/m(I)m(J) - 1| < \varepsilon_1, \quad t_{10} \leq t \leq t_{13}.$$

During $[t_{10}, t_{13}]$, (10) and (12) imply T is essentially $\frac{1}{2}$-mixing and R is essentially mixing for levels in $G_0 = H_0$. Condition (11) is used to verify T is not α-mixing for $\alpha > \frac{1}{2}$.

Now let $H_{11}^1 = H_1/2$ and $H_{21}^1 = H_1/2$. Apply the induction step with $\varepsilon = \varepsilon_1$ and $G_{i1} = H_1/2$, $i = 1, 2$. We obtain $H_{1j}^1 = G_{1j}$, $2 \leq j \leq 6$, $H_{2j}^1 = G_{2j}$, $4 \leq j \leq 6$, and $r_{1i} = t_i$, $0 \leq i \leq 3$. We can choose $r_{10} > t_{13}$.

We have R_{1i6}, $i = 1, 2$. If I is a level in H_0, then $I/2$ is in H_{i1}^1, $i = 1, 2$. Therefore the induction step guarantees that if R extends R_{1i6}, $i = 1, 2$, then for levels I and J in H_0,

$$(13) \qquad m(R^t I \cap J) \geq (1 - \varepsilon_1)m(I)m(J)/2m(H_0), r_{10} \leq t \leq r_{13}.$$

Since $H_{21}^1 = H_1/2$, the continuation of mixing in H_{21}^1 to form H_{24}^1 also guarantees $\frac{1}{2}$-mixing for levels I in H_0 during $[t_{13}, r_{10}]$. Thus we also have

$$(14) \qquad m(R^t I \cap J) \geq (1 - \varepsilon_1) m(I) m(J)/2m(H_0), \quad t_{13} \leq t \leq r_{10},$$

for levels I and J in H_0.

If I is a level in H_0, then $I/2$ is a union of levels in H^1_{11}. Hence (4) implies there is a positive integer $h = h_{12}$ such that

$$(15) \qquad m(R^h I \cap I) > (1 - \varepsilon_1) m(I)/2.$$

Now we have $r_{13} > r_{10} > t_{13} > t_{12}$.

For the kth stage in the construction, assume we have $r_{k3} > r_{k0} > t_{k3} > t_{k2}$. We also have $\varepsilon_k > 0$, towers G^k_{i6}, $i = 1, 2$, of equal measure, and $t_{k3} \geq t(G^k_{16}, \varepsilon_k)$. If T extends T_{k26}, then

$$(16) \qquad |m(T^t I \cap J) m(G^k_{25})/m(I) m(J) - 1| < \varepsilon_k, \quad t_{k2} \leq t \leq t_{k3},$$

where I and J are levels in G^k_{25}.

Now $r_{k3} > t_{k3}$ so we can choose n so large that if $G^{k+1}_{11} = S^n G^k_{16}$ and T extends $T_{k+1,1,1}$, then

$$(17) \qquad |m(T^t I \cap J) m(G^k_{16})/m(I) m(J) - 1| < \varepsilon_k, \quad t_{k3} \leq t \leq r_{k3},$$

where I and J are levels in G^k_{16}. Also $r_{k3} > t_{k2}$ so we can choose n so large that if $G^{k+1}_{21} = S^n G^k_{26}$ and T extends $T_{k+1,2,1}$, then

$$(18) \qquad |m(T^t I \cap J) m(G^k_{25})/m(I) m(J) - 1| < \varepsilon_k, \quad t_{k2} \leq t \leq r_{k3},$$

where I and J are levels in G^k_{25}.

Let $\varepsilon_{k+1} < \varepsilon_k$. Choose $t_{k+1,0} \geq \max\{t(G^{k+1}_{i1}, \varepsilon_{k+1}),\ i = 1, 2\}$ so that $t_{k+1,0} > r_{k,3}$. We now apply the induction step with $\varepsilon = \varepsilon_{k+1}$, $G_{i1} = G^{k+1}_{i1}$, $i = 1, 2$. We obtain $G^{k+1}_{1j} = G_{1j}$, $2 \leq j \leq 6$, $G^{k+1}_{2j} = G_{2j}$, $4 \leq j \leq 6$, and $t_{k+1,i}$, $1 \leq i \leq 3$. If I and J are levels in G^k_{11} or G^k_{21}, then $I/2$ and $J/2$ will be in G^{k+1}_{i1}, $i = 1, 2$. The induction step implies that if T extends $T_{k+1,i,6}$, $i = 1, 2$, then for levels I, J in G^k_{i1}, $i = 1, 2$,

$$(19) \quad m(T^t I \cap J) \geq (1 - \varepsilon_k) m(I) m(J)/4m(G^{k+1}_{15}), \quad t_{k+1,0} \leq t \leq t_{k+1,3}.$$

If I is a level in G_0, then $I/2$ is a union of levels in G^k_{11}. Hence (4) implies there exists $h = h_{k+1,1}$ such that

$$(20) \qquad m(T^h I \cap I) > (1 - \varepsilon_{k+1}) m(I)/2.$$

We now have $t_{k+1,3} > t_{k+1,0} > r_{k,3} > r_{k,2}$. The $(k + 1)$st stage of the con-

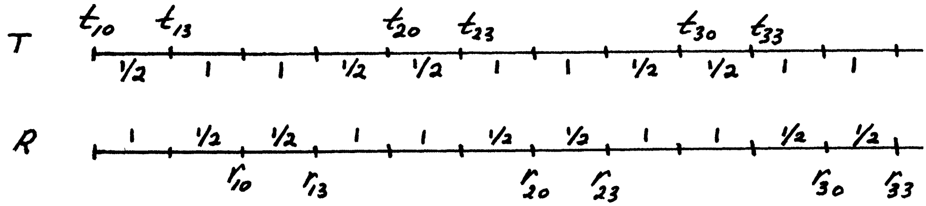

FIG. 2.

struction for R is now obtained by repeating the analogous kth stage of the construction for T, with G replaced by H, to obtain (16)–(20) with T replaced by R and $r_{k+1,i}$ replacing $t_{k,i}$.

If t is in $[t_{k3}, r_{k3}]$, then (17) and (18) imply

$$m(T^t I \cap J) \geq (1 - \varepsilon_k) m(I) m(J) / 2m\left(G_{15}^k\right), \tag{21}$$

where I and J are in G_{i1}^{k-1}, $i = 1, 2$. This is because $I/2$ and $J/2$ are in G_{i5}^k, $i = 1, 2$. Since we assume m is normalized so that $\lim_{k \to \infty} m(G_{15}^k) = \lim_{k \to \infty} m(H_{15}^k) = 1/2$, (21) implies T is essentially mixing for $t_{k3} \leq t \leq r_{k3}$ and levels in G_{i1}^{k-1}, $i = 1, 2$. These levels are arbitrarily small for k large. Hence finite unions of these levels are dense in $\mathscr{B}$. In particular, we conclude that T is essentially mixing for $t_{k3} \leq t \leq r_{k3}$ for all intervals, as $k \to \infty$.

Since R is constructed in the same way that T is, we conclude R is essentially mixing for $r_{k3} \leq t \leq t_{k+1,3}$ for all intervals, as $k \to \infty$.

From (19) we conclude that T is essentially $\frac{1}{2}$-mixing for $t_{k+1,0} \leq t \leq t_{k+1,3}$ and levels I, J in G_{i1}^k, $i = 1, 2$. By induction, this holds for all k. So we conclude that as $k \to \infty$, T will be essentially $\frac{1}{2}$-mixing for t in $[t_{k,0}, t_{k,3}]$ for all intervals.

Since R is constructed in the same way that T is, we conclude that as $k \to \infty$, R is essentially $\frac{1}{2}$-mixing for t in $[r_{k0}, r_{k3}]$ for all intervals. The following diagram is helpful. The $\frac{1}{2}$'s and 1's indicate where T and R are each essentially $\frac{1}{2}$-mixing and mixing. In particular, (1) is satisfied so T and R have the desired properties.

4. Uniform sweeping out

Let k be a positive integer. A transformation T *uniformly k-sweeps out* if given A_i of positive measure, $1 \leq i \leq k$, and $\varepsilon > 0$ there exists $N = N(A^{(k)}, \varepsilon)$ such that for all increasing $d_n^{(k)}$, $1 \leq n \leq N$,

$$m\left(\bigcup_{n=1}^{N} I\left(T, d_n^{(k)}, A^{(k)}\right) \right) > 1 - \varepsilon. \tag{4.1}$$

In [3] it is shown that mixing implies uniform 1-sweeping out. This result is generalized as follows.

(4.2) THEOREM. *If T is $(2k-1)$-mixing, then T uniformly k-sweeps out, $k \geq 1$.*

The proof of (4.2) depends on the following lemma.

(4.3) LEMMA. *If T is $(2k-1)$-mixing, then given $\varepsilon > 0$ and A_i, $1 \leq i \leq k$, of positive measure there exists $N = N(A^{(k)}, \varepsilon)$ such that $n \geq N$ implies*

$$(4.4)\quad \frac{1}{n^2} \sum_{i,j=1} \left| m\left(I\left(T, d_i^{(k)}, A^{(k)}\right) \cap I\left(T, d_j^{(k)}, A^{(k)}\right)\right) - \left(\prod_{i=1}^{k} m(A_i)\right)^2 \right| < \varepsilon,$$

for all increasing $d_i^{(k)}$, $1 \leq i \leq n$.

Proof. Since T is $(2k-1)$-mixing, there exists $M = M(A^{(k)}, \varepsilon)$ such that if a_i, $1 \leq i \leq 2k$, satisfy $|a_i - a_j| \geq M$, $i \neq j$, then

$$(1)\quad \left| m\left(\bigcap_{i=1}^{k} T^{a_i} A_i \cap \bigcap_{i=1}^{k} T^{a_{i+k}} A_i\right) - \left(\prod_{i=1}^{k} m(A_i)\right)^2 \right| < \varepsilon/2.$$

(This is a trivial reformulation of the definition of higher order mixing in §2.)

Now choose N so large that

$$(2)\quad \left(M^2 + k^2(2M+1)\right)/N < \varepsilon/2.$$

Consider increasing $d_i^{(k)}$, $1 \leq i \leq n$, where $n \geq N$. Denote the $(i,j)^{\text{th}}$ term in (4.4) by $\Delta(i,j)$. Call $\Delta(i,j)$ *bad* if $\Delta(i,j) > \varepsilon/2$.

If $\Delta(i,j)$ is bad, then (1) implies some pair of the exponents $d_{i,1}$, $(d_{i,1} + d_{i,2}), \ldots, (d_{i,1} + \cdots + d_{i,k}), d_{j,1}, \ldots, (d_{j,1} + \cdots + d_{j,k})$ must differ by less than M. Since the $d_i^{(k)}$ are increasing and since the entries are positive integers, all entries of $d_r^{(k)}$ are at least r. In particular, if $i > M$ and $j > M$, then the only way $\Delta(i,j)$ can be bad is that one of the first k exponents is within M of one of the last k exponents.

Thus we fix $i = a > M$ and count the number of possible $j > M$ such that $\Delta(i,j)$ could be bad. Since $d_{j,1}$ increases as j increases, there are at most $2M+1$ values of j such that $|d_{j,1} - d_{a,1}| \leq M$. Iterating this argument, we see that there are at most $k(2M+1)$ values of j for which $d_{j,1}$ can be within M of any of the first k exponents. The same estimate applies to each of the last k exponents. Thus for fixed $a > M$, there are at most $k^2(2M+1)$ values of $j > M$ such that $\Delta(a,j)$ is bad. Summing over a, we see that there are at most $(n-M)k^2(2M+1)$ values of $i > M$ and $j > M$ such that $\Delta(i,j)$ is bad.

Thus if E is the number of bad $\Delta(i, j)$'s, $1 \leq i, j \leq n$, then

$$E \leq M^2 + (n - M)k^2(2M + 1). \tag{3}$$

Now $\Delta(i, j) \leq 1$ for all i, j. Hence we can use (2) and (3) to estimate the sum in (4.4) as follows.

$$\begin{aligned} \frac{1}{n^2} \sum_{i,j=1}^{n} \Delta(i, j) &\leq \frac{1}{n^2}\left(E + (n^2 - E)\varepsilon/2\right) \\ &\leq \frac{M^2 + (n - M)k^2(2M + 1)}{n^2} + \varepsilon/2 \\ &< \frac{M^2 + k^2(2M + 1)}{N} + \varepsilon/2 \\ &< \varepsilon. \end{aligned} \tag{4}$$

Now Lemma (4.3) will be used to prove Theorem (4.2). Given $A^{(k)}$, we let $\Pi = \prod_{i=1}^{k} m(A_i)$ for convenience and let $\eta = (\varepsilon\Pi)^2/2$. Let $N = N(A^{(k)}, \eta)$ as in Lemma (4.3). Let

$$B_i = I\left(T, d_i^{(k)}, A^{(k)}\right) \quad \text{and} \quad C = \left(\bigcup_{i=1}^{N} B_i\right)^c.$$

We need to show $m(C) < \varepsilon$.

Let b_i denote the characteristic function of B_i and let $f = (\sum_{i=1}^{N} b_i)/N$. Since $C \cap B_i = \emptyset$, $1 \leq i \leq N$, Hölder's inequality implies

$$\Pi m(C) = \left| \int_C (f - \Pi)\, dm \right| \leq \|f - \Pi\|_1 \leq \|f - \Pi\|_2. \tag{1}$$

Now we have

$$\|f - \Pi\|_2^2 = \int \left(\frac{1}{N^2} \sum_{i,j=1}^{N} b_i b_j - \frac{2}{N}\Pi \sum_{i=1}^{N} b_i + \Pi^2 \right) dm. \tag{2}$$

By the choice of N, the first integral on the right in (2) is within η of Π^2. Hence (2) implies

$$\begin{aligned} \|f - \Pi\|_2^2 &\leq 2\Pi^2 - \frac{2}{N}\Pi \int \left(\sum_{i=1}^{N} b_i \right) dm + \eta \\ &\leq \frac{2\Pi}{N} \sum_{i=1}^{N} |m(B_i) - \Pi| + \eta. \end{aligned} \tag{3}$$

Since T is $(2k-1)$-mixing, T is also $(k-1)$-mixing. Hence, taking $\delta = \varepsilon^2\Pi/8$, there exists $P = P(A^{(k)}, \delta)$ such that, if $d_{i,j} \geq P$ for $j = 1, \ldots, k$, then $|m(B_i) - \Pi| < \delta$. Since the $d_i^{(k)}$ are increasing and all $d_{i,j}$ are positive, the above condition will be satisfied if $i \geq P$. We assume that N has also been chosen large enough so that $P(1-\delta)/N < \delta$. Let K denote the number of terms $|m(B_i) - \Pi|$ which exceed δ. We continue the estimate (3):

$$\begin{aligned}
&\leq \frac{2\Pi}{N}(K + (N-K)\delta) + \eta \\
&< \frac{2\Pi}{N}(P(1-\delta) + N \cdot \delta) + \eta \\
&< 4\Pi\delta + \eta \\
&= \varepsilon^2\Pi^2.
\end{aligned}$$

Taking square roots yields $\|f - \Pi\|_2 < \varepsilon\Pi$. Combining this with (1), we see that $\Pi m(C) < \varepsilon\Pi$, hence $m(C) < \varepsilon$. This was the desired inequality, and the proof of the theorem is complete.

(4.4) COROLLARY. *If T is mixing of all orders, then it is uniformly sweeping out of all orders.*

The converse of Theorem 4.2 is open. In fact, we do not know if uniform sweeping out of all orders implies mixing.

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