ILLINOIS JOURNAL OF MATHEMATICS
Volume 29, Number 3, Fall 1985

REFLECTIVE SUBCATEGORIES

BY

J. MARTIN HARVEY

This paper is concerned with the three major problems relating to reflective subcategories, namely, characterization, existence of reflective hulls and the preservation of reflectiveness under intersection (cf. Herrlich [5]). Using factorization techniques, we provide solutions to these problems under relatively mild conditions, generalising the corresponding results on epi-reflective subcategories.

Throughout the discussion, we consider a well-powered, cowell-powered category **A** with products. Further, we assume that **A** is either complete or admits all pushouts and coequalizers. All subcategories under discussion are assumed to be full and iso-closed. For terminology and standard results, we refer to Herrlich and Strecker [7].

Given a subcategory **B** and a class **E** of morphisms of **A**, we will say that **E** is **B**-*generating* and that **B** is **E**-*generated* iff for each morphism $e\colon X \to Y$ in **E**, $re = se$ and $\operatorname{cod}(r) \in \mathbf{B}$ implies $r = s$. This terminology will be abused in the obvious way in respect of singletons. Further, any **B**-generating morphism with codomain in **B** will be called a **B**-*epi*; and **B** will be called a *cowell-powering subcategory* whenever each **A**-object is the domain of a representative set of **B**-epis.

We let $\mathbf{B}_0$ (resp. $\mathbf{B}_1$) denote the subcategory comprising all (**B**-epi)-generated (resp. (**B**-generating)-generated) objects, and call $\mathbf{B}_0$ (resp. $\mathbf{B}_1$) the *point-separation axiom epi-generated* (resp. *generated*) *by* **B**. We also let $\mathbf{M}_0 = \mathbf{M}_0(\mathbf{B})$ (resp. $\mathbf{M}_1 = \mathbf{M}_1(\mathbf{B})$) denote the class of all morphisms $m\colon A \to B$ such that if $mh = ge$, with $e\colon X \to Y$ a **B**-epi (resp. a **B**-generating morphism), then there exists a (unique) morphism $d\colon Y \to A$ with $md = g$ and $de = h$.

We note that **A** has (epi, extremal mono)-factorization and (extremal epi, mono)-factorization (of morphisms). Hence, in the above, $\mathbf{B}_0$ (resp. $\mathbf{B}_1$) is an (extremal epi)-reflective subcategory, since it is closed under the formation of products and subobjects (cf. [4], [6], [7], [8]). Further, every subcategory **C** of $\mathbf{B}_0$ (resp. $\mathbf{B}_1$) containing **B** has the property that **B**-epis (resp. **B**-generating morphisms) are precisely the **C**-epis with codomain in **B** (resp. **C**-generating morphisms). The subcategory $\mathbf{B}_0$ (resp. $\mathbf{B}_1$) is the largest such that **B**-epis (resp. **B**-generating morphisms) are precisely the $\mathbf{B}_0$-epis with codomain in **B** (resp.

Received February 17, 1983.

B-generating morphisms). Hence, $\mathbf{B}_0 = (\mathbf{B}_0)_1 = (\mathbf{B}_1)_0$ and $\mathbf{B}_1 = (\mathbf{B}_1)_1$. Further, a morphism in $\mathbf{B}_0$ (resp. $\mathbf{B}_1$) is an extremal mono in $\mathbf{B}_0$ (resp. $\mathbf{B}_1$) iff it belongs to $\mathbf{M}_0$ (resp. $\mathbf{M}_1$). Finally, a reflective subcategory of **A** is cowell-powering iff it is cowell-powered.

These observations facilitate subsequent discussion and may be illustrated in **Top**, the usual category of continuous maps between topological spaces. Here, we observe the following, where we let $\mathbf{Top}_n$ denote the full subcategory of **Top** comprising T_n-spaces, for $n = 0, 1, 2, \dots,$ and assume all subcategories to be non-empty:

(1) If $\mathbf{B} \subset \mathbf{Top}$ and $\mathbf{B} \not\subset \mathbf{Top}_0$, then the **B**-generating morphisms are the onto maps, $\mathbf{B}_0 = \mathbf{B}_1 = \mathbf{Top}$ and **B** is cowell-powering.

(2) If $\mathbf{B} \subset \mathbf{Top}_0$ and $\mathbf{B} \not\subset \mathbf{Top}_1$, then the **B**-generating morphisms are the front-dense maps, $\mathbf{B}_0 = \mathbf{B}_1 = \mathbf{Top}_0$ and **B** is cowell-powering.

(3) If $\mathbf{B} \subset \mathbf{Top}_1$, then all morphisms $e\colon X \to Y$ satisfying the following density condition are **B**-generating: For each $y \in Y$, each open nbd G of y contains a point in the set

$$e(X) \cap \operatorname{cl}(\cap \{0 \subset Y: 0 \text{ is an open nbd of } y\}).$$

(4) If $\mathbf{B} \subset \mathbf{Top}_2$, then dense maps with codomain in **B** are **B**-epis, and this condition is characteristic if $\mathbf{B} = \mathbf{Top}_2$. More generally, dense maps are **B**-generating, though the converse does not appear to hold, even for $\mathbf{B} = \mathbf{Top}_2$.

These observations follow from the results of [1], [2] and, though incomplete, indicate that $\mathbf{B}_0$ (resp. $\mathbf{B}_1$) is relatively insensitive to changes in **B**. Further, it seems that **B**-epis are easier to characterise than **B**-generating morphisms. Granted these preliminary considerations, we turn to the main results, beginning with the following factorization result which is crucial to subsequent discussion:

LEMMA. *Let* **B** *be a subcategory of* **A**. *Then each* **A**-*morphism* $f\colon A \to B$ *with* B *in* $\mathbf{B}_1$ *has a unique factorization* $f = me$ *with* e **B**-*generating and* m *in* $\mathbf{M}_1$. *Further, if* **B** *is closed under the formation of* $\mathbf{M}_0$-*subobjects and* B *is in* **B**, *then we can choose* e *a* **B**-*epi and* m *in* $\mathbf{M}_0$.

Proof. The last statement follows from a modification of the following proof of the first statement. First, suppose **A** admits all pushouts and coequalizers. Then, from the dual of Dyckhoff [3, Theorem 1], **A** admits (**B**-generating, **M**)-factorization, for some class **M** of (extremal) monos, which is clearly $\mathbf{M}_1$.

Now, suppose **A** is complete. Let $\{(e_i, m_i)\}_{i \in I}$ be the class of all pairs with $f = m_i e_i$ and m_i in $\mathbf{M}_1$ for each $i \in I$, $m = m_i p_i$ $(i \in I)$ in a pullback (possible since **A** is well-powered) and $e\colon A \to C$ be the unique morphism with $me = f$ and $p_i e = e_i$ $(i \in I)$. Then, as $\mathbf{B}_1$ is epi-reflective, the above pullback is constructed within $\mathbf{B}_1$ [7] and, evidently, m belongs to $\mathbf{M}_1$. Now, suppose r, s: $C \to D$ is a pair of morphisms with $re = se$ and D in $\mathbf{B}$ and let n: $X \to C$ be the equalizer of r, s and h: $A \to X$ be the unique morphism with $e = nh$. Then, the equalizer has been constructed within $\mathbf{B}_1$; i.e., n belongs to $\mathbf{M}_1$. Hence, mn belongs to $\mathbf{M}_1$ and, therefore, there exists $i \in I$ with $m_i = mn$ and $e_i = h$. Hence, $(mn)p_i = m$, so that, as m is a mono, $np_i = 1$ and, therefore, n is an iso and so $r = s$; i.e., e is $\mathbf{B}$-generating.

PROPOSITION 1. *Let* $\mathbf{B}$ *be a cowell-powering subcategory of* $\mathbf{A}$. *Then* $\mathbf{B}$ *is a reflective subcategory of* $\mathbf{A}$ *iff* $\mathbf{B}$ *is closed under the formation of products and extremal subobjects in* $\mathbf{B}_0$ *(resp.* $\mathbf{B}_1$*).*

Proof. Suppose $\mathbf{B}$ is reflective. Then $\mathbf{B}$ is closed under the formation of products [7]. Now, let m: $A \to B$ be an extremal mono in $\mathbf{B}_0$ with B in $\mathbf{B}$, e: $A \to C$ be a $\mathbf{B}$-reflection and n: $C \to B$ be the unique morphism with $m = ne$. Then, it easily follows from the lemma above that, as e is a $\mathbf{B}$-epi, it is an iso, so that A is in $\mathbf{B}$, as required.

Conversely, suppose $\mathbf{B}$ is closed under the formation of products and extremal subobjects in $\mathbf{B}_0$. Let A be an $\mathbf{A}$-object and $(e_i$: $A \to B_i)_I$ be a representative set of $\mathbf{B}$-epis with domain A. Let me: $A \to \prod B_i$ be the unique morphism with $e_i = \pi_i(me)$ $(i \in I)$, e: $A \to A_0$ a $\mathbf{B}$-epi and m: $A_0 \to \prod B_i$ an extremal mono in $\mathbf{B}_0$. Then e: $A \to A_0$ is the required $\mathbf{B}$-reflection. For, suppose f: $A \to B$ is an $\mathbf{A}$-morphism with B in $\mathbf{B}$. Let $f = wu$, where u: $A \to X$ is a $\mathbf{B}$-epi and w: $X \to B$ is an extremal mono in $\mathbf{B}_0$. Then, there exists $i \in I$ and an iso v: $B_i \to X$ with $u = ve_i$. Hence, $f = f_0 e$, where $f_0 = wv\pi_i m$, which suffices.

The proof in the case of $\mathbf{B}_1$ is a straightforward modification of the above.

The following result on intersections immediately follows:

COROLLARY. *If the intersection of a family of reflective subcategories of* $\mathbf{A}$ *is cowell-powering, then it is reflective.*

Proof. Let $(\mathbf{B}_i)_I$ be a family of reflective subcategories of $\mathbf{A}$ with intersection $\mathbf{B}$. Then, from the above, $\mathbf{B}$ is closed under the formation of products. Further, as $\mathbf{M}_1(\mathbf{B}) \subset \mathbf{M}_1(\mathbf{B}_i)$ $(i \in I)$, $\mathbf{B}$ is closed under the formation of $\mathbf{M}_1(\mathbf{B})$-subobjects and, therefore, $\mathbf{B}$ is reflective.

We also have the following result on intersections:

PROPOSITION 2. *The intersection of a family of cowell-powering reflective subcategories of* $\mathbf{A}$ *which (epi-) generate the same point-separation axiom is reflective.*

Proof. Let $(\mathbf{B}_i)_I$ be a family of cowell-powering reflective subcategories of $\mathbf{A}$, $\mathbf{B}$ be the intersection of this family and $\mathbf{C}$ be the (epi-) generated point-separation axiom. Let $i \in I$. Then, $\mathbf{B}_i$ is cowell-powered, since it is cowell-powering, and is well-powered, since every mono in $\mathbf{B}_i$ is a mono in $\mathbf{A}$. Further, $\mathbf{B}_i$, being a reflective subcategory, has the same (co) completeness properties as $\mathbf{A}$. Hence, in particular, $\mathbf{B}_i$ admits (epi, extremal mono)-factorization. Now, $\mathbf{B}$ is closed under the formation of products and extremal subobjects in $\mathbf{B}_i$, since the products coincide with those in $\mathbf{A}$ and the extremal subobjects with those in $\mathbf{C}$. Hence, $\mathbf{B}$ is epi-reflective in $\mathbf{B}_i$.

Now, let A be an $\mathbf{A}$-object and, for each $i \in I$, let e_i: $A \to B_i$ be a $\mathbf{B}_i$-reflection and s_i: $B_i \to C_i$ be a $\mathbf{B}$-reflection. Further, let me: $A \to \prod C_i$ be the unique morphism with $\pi_i(me) = s_i e_i$ $(i \in I)$, where e: $A \to A_0$ is a $\mathbf{C}$-epi and m is in $\mathbf{M}_0(\mathbf{C})$. Then, m is an extremal mono in $\mathbf{C}$, since $\mathbf{C} = \mathbf{C}_0$; so that A_0 is in $\mathbf{B}$ and e: $A \to A_0$ is a $\mathbf{B}$-epi. To show that e is a $\mathbf{B}$-reflection, let f: $A \to B$ be a morphism with B in $\mathbf{B}$. Then, for each $i \in I$, there exists f_i: $B_i \to B$ and, hence, g_i: $C_i \to B$ with $f = f_i e_i$ and $f_i = g_i s_i$. Hence, there exists (unique) f_0: $A_0 \to B$ with $f = f_0 e$, where $f_0 = g_i \pi_i m$ for any $i \in I$, showing that $\mathbf{B}$ is reflective.

Finally, we have the following on reflective hulls:

PROPOSITION 3. *Let* $\mathbf{B}$ *be a cowell-powering subcategory of* $\mathbf{A}$ *which is closed under the formation of extremal subobjects in* $\mathbf{B}_1$. *Then,* $\mathbf{B}$ *has a reflective hull, which comprises the extremal subobjects in* $\mathbf{B}_1$ *of products of* $\mathbf{B}$*-objects.*

Proof. Let $\mathbf{C}$ denote the subcategory of $\mathbf{A}$ comprising all extremal subobjects in $\mathbf{B}_1$ of products of $\mathbf{B}$-objects. Further, let A be an A-object and $(e_i$: $A \to B_i)_I$ be a representative set of $\mathbf{B}$-epis with domain A. Furthermore, let e: $A \to A_0$ be constructed as in Prop. 1 with $\mathbf{C}$ and $\mathbf{B}_1$ in place of $\mathbf{B}$ and $\mathbf{B}_0$, respectively. Then, by a similar argument, e is a $\mathbf{C}$-reflection. Further, if $\mathbf{D}$ is a reflective subcategory of $\mathbf{A}$ containing $\mathbf{B}$, then every $\mathbf{D}$-generating morphism is $\mathbf{B}$-generating, so that $\mathbf{M}_1(\mathbf{B}) \subset \mathbf{M}_1(\mathbf{D})$. Thus, in view of the first part of the proof of Prop. 1 and the definition of $\mathbf{C}$, $\mathbf{D}$ contains $\mathbf{C}$; i.e., $\mathbf{C}$ is the reflective hull of $\mathbf{B}$.

The author would like to express his gratitude to Prof. B. Banaschewski for a Post-Doctoral Fellowship at McMaster University during which this research was brought to fruition.

REFERENCES

1. S. BARON, *Note on epi in* T_0, Canad. Math. Bull., vol. 11 (968), pp. 503–504.
2. W. BURGESS, *The meaning of mono and epi in some familiar categories*, Canad. Math. Bull., vol. 8 (1965), pp. 759–769.
3. R. DYCKHOFF, *Factorization theorems and projective spaces in topology*, Math Zeitschr., vol. 127 (1972), pp. 256–264.

4. H. Herrlich, *Topologische reflexionen und coreflexionen*, Lecture Notes in Math. 78, Springer, Berlin, 1968.
5. ______, *On the concept of reflections in general topology*, Proc. Sympos. contributions to the extension theory of topological structures, Springer, Berlin, 1969, pp. 105–114.
6. H. Herrlich and G.E. Strecker, *Coreflective subcategories*, Trans. Amer. Math. Soc., vol. 157 (1971), pp. 205–226.
7. ______, *Category theory*, 2nd ed., Heldermann Verlag, Berlin, 1979.
8. J.F. Kennison, *Full reflective subcategories and generalised covering spaces*, Illinois J. Math., vol. 12 (1968), pp. 353–365.

University of Zimbabwe
Harare, Zimbabwe