ILLINOIS JOURNAL OF MATHEMATICS
Volume 29, Number 4, Winter 1985

INDEX OF HECKE OPERATORS

BY

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1. Introduction

Let M be a complete Riemannian manifold. Suppose that the discrete group Γ acts isometrically and properly discontinuously on M with compact quotient $\overline{M} = \Gamma \setminus M$. Even though $\overline{M}$ need not be a manifold, the heat equation method [5] may be used to study the spectral theory of $\overline{M}$. Suppose S is a set of isometries satisfying (2.1). Associated to S is the Hecke operator T_S, and we will study the asymptotic behavior of its trace on the eigenspaces of the Laplacian.

Now suppose in addition that M is oriented and that Γ and S are orientation preserving. One may define the signature complex of $\overline{M}$ and consider the signature, $\operatorname{Sign}(T_S)$, of the Hecke operator T_S. We will give an explicit formula for $\operatorname{Sign}(T_S)$ in Theorem 4.1. Our approach is a natural extension of the technique used in [8] to prove the equivariant signature theorem. Since M need not be compact, it appears that the original proof of the equivariant signature theorem by Atiyah and Singer [2] does not generalize to compute $\operatorname{Sign}(T_S)$. In particular, Atiyah and Singer relied upon the representation theory of compact groups.

If $M = G/K$ is a globally symmetric space, then the Hecke operators associated to certain sets S of isometries have been studied by several authors [9], [11], [12], [13]. The most effective technique has been the Selberg trace formula. Of course, the trace formula can only be used when M admits a transitive group of isometries, so our results are more general. Furthermore, even in the case of symmetric spaces, Selberg's work does not immediately give an explicit formula for $\operatorname{Sign}(T_S)$. To derive our results from the Selberg trace formula, certain complicated orbital integrals must be simplified. Apparently, this has not been carried out except in special cases. On the other hand, the trace formula can give an expression for the individual traces of T_S on each harmonic piece of the signature complex, $\operatorname{Tr}_+(T_S)$ and $\operatorname{Tr}_-(T_S)$, rather than just the difference

$$\operatorname{Sign}(T_S) = \operatorname{Tr}_+(T_S) - \operatorname{Tr}_-(T_S).$$

Received May 27, 1983.

Unless a suitable vanishing theorem applies, the trace formula has a definite advantage here.

The results of this paper generalize to the signature complex with coefficients in a bundle. Using this generalized result, one can deduce the analogous theorems for the other classical elliptic complexes [1].

Note added in proof. There is some overlap of this paper with the recent work of H. Moscovici [14]. He applies the Selberg trace formula to give an expression for the index of Hecke operators on compact locally symmetric spaces $\Gamma \setminus GK$.

2. Hecke Operators

Let M be a connected manifold and Γ a group acting smoothly on M with compact quotient $\overline{M} = \Gamma \setminus M$. We assume that Γ acts properly discontinuously on M, meaning that each compact set in M intersects a finite number of its Γ-translates. Under these circumstances, one may construct a Γ-invariant metric on M, which is necessarily complete [5]. We assume henceforth that M is endowed with a complete Γ-invariant metric. If Γ acts freely, then $\overline{M}$ is a Riemannian manifold covered by M. In general, $\overline{M}$ is a space with singularities, sometimes called a V-manifold [10].

Suppose $\pi\colon M \to \overline{M}$ is the projection onto the orbit space of the Γ-action. A function f defined on $\overline{M}$ is said to be of class $C^l(\overline{M})$ if and only if $f \circ \pi \in C^l(M)$, the set of l times continuously differentiable functions on M. Also, one may define a measure on M by using a partition of unity relative to Γ. A non-negative function $\phi \in C_0^\infty(M)$ satisfying $\sum_{\gamma \in \Gamma} \phi(\gamma x) = 1$, for all $x \in M$, is by definition a partition of unity relative to Γ. Such ϕ always exist [5] and we assume that one is chosen. The measure $d\bar{x}$ on $\overline{M}$ may be defined by

$$\int_{\overline{M}} f(\bar{x})\, d\bar{x} = \int_M \phi(x) f(\pi x)\, dx$$

for any continuous function $f \in C(\overline{M})$. The measure $d\bar{x}$ is independent of the choice of partition of unity ϕ.

We proceed to define the Hecke operators. Let S be a set of isometries satisfying the following properties:

(i) $S = \Gamma S = S \Gamma$. (2.1)

(ii) The orbit space $\Gamma \setminus S$ is a finite set.

Note that S is not assumed to be a group. Suppose α_i, $i = 1, 2, \ldots, m$, are a set of representatives for the orbits of the Γ-action on S, so that $S = \bigcup_{i=1}^m \Gamma \alpha_i$. If f is a continuous function on $\overline{M}$, then identify f with a Γ-invariant function on M. For $z \in M$, the Hecke operator T_S is defined by

$$T_S f(z) = \sum_{i=1}^m f(\alpha_i z). \tag{2.2}$$

This definition is justified by the following elementary lemma:

LEMMA 2.3. *$T_S f$ is a Γ-invariant function which is independent of the choice of coset representatives α_i. Thus T_S maps $C(\overline{M})$ to $C(\overline{M})$.*

Proof. (i) If $\beta_j = \gamma_j \alpha_j$, $j = 1, 2, \dots, m$, is another choice of coset representatives, then

$$\sum_{i=1}^{m} f(\beta_i z) = \sum_{i=1}^{m} f(\gamma_i \alpha_i z) = \sum_{i=1}^{m} f(\alpha_i z)$$

by Γ-invariance of f. So T_S does not depend upon the choice of coset representatives.

(ii) Note that for any $\gamma \in \Gamma$, $\alpha_i \gamma$, $i = 1, 2, \dots, m$, is a set of coset representatives for $\Gamma \setminus S$. So, by (i) of the proof,

$$T_S f(\gamma z) = \sum_{i=1}^{m} f(\alpha_i \gamma z) = \sum_{i=1}^{m} f(\alpha_i z) = T_S f(z).$$

Thus, $T_S f$ is Γ-invariant.

If $M = G/K$ is a globally symmetric space then a subgroup $\Gamma \subset G$ acts properly discontinuously on M if and only if Γ is discrete in G [4]. Sets of isometries $S \subset G$ satisfying (2.1) occur naturally and the corresponding Hecke operators T_S have been studied by many authors [9], [11], [12], and [13]. In some of the most important examples, S is not a group.

3. Asymptotic expansions

Suppose that the group Γ acts properly discontinuously and isometrically on the complete Riemannian manifold M with compact quotient $\overline{M} = \Gamma \setminus M$. Since Γ acts isometrically on M, the Laplace operator Δ is Γ-invariant. Thus, Δ induces an operator $\overline{\Delta}$ on $C^2(\overline{M})$, the Γ-invariant functions in $C^2(M)$. In [5], it was shown that $\overline{\Delta}$ has a unique extension to a self-adjoint unbounded operator on $L^2(\overline{M})$. The extended operator $\overline{\Delta}$ has pure point spectrum. That is, there is an orthonormal basis ϕ_i of $L^2(\overline{M})$ consisting of eigenfunctions of $\overline{\Delta}$ with corresponding eigenvalues μ_i.

If $t > 0$, the bounded operator $\exp(-t\overline{\Delta})$: $L^2(\overline{M}) \to L^2(\overline{M})$ is well-defined by the functional calculus for self-adjoint operators. In fact, $\exp(-t\overline{\Delta})$ is represented by a smoothing kernel $\overline{E}(t, x, y)$. It follows that $\exp(-t\overline{\Delta})$ is Hilbert-Schmidt. Using the semigroup property, $\exp(-(t+s)\overline{\Delta}) = \exp(-t\overline{\Delta})\exp(-s\overline{\Delta})$, one sees that $\exp(-t\overline{\Delta})$ is trace class. Suppose $E(t, x, y)$ is the fundamental solution of the heat equation on M. One has, according to [5],

$$\overline{E}(t, x, y) = \sum_{\gamma \in \Gamma} E(t, x, \gamma y). \tag{3.1}$$

Here $x, y \in M$ are identified with their projections to $\overline{M}$. The sum converges uniformly on compact subsets of $(0, \infty) \times M \times M$.

We now consider the Hecke operators. Let S be a set of isometries satisfying (2.1) and T_S the associated Hecke operator. From the definition (2.2), one checks that T_S extends to a bounded operator on $L^2(\overline{M})$ which commutes with $\overline{\Delta}$. In particular, T_S induces linear maps on the eigenspaces of $\overline{\Delta}$ with eigenvalue μ. The trace of the induced map $T_{S,\mu}$ will be denoted by $\mathrm{Tr}(T_{S,\mu})$.

The composition $T_S \circ \exp(-t\overline{\Delta})$, of the bounded operator T_S with the trace class operator $\exp(-t\overline{\Delta})$ is necessarily trace class. One has

$$\mathrm{Tr}(T_S \circ \exp(-t\overline{\Delta})) = \sum_{\mu} \mathrm{Tr}(T_{S,\mu}) e^{-t\mu}$$

where μ is summed over the distinct eigenvalues of $\overline{\Delta}$. On the other hand, using (3.1) and the definition (2.2) of T_S, it follows that $T_S \circ \exp(-t\overline{\Delta})$ is represented by a smoothing kernel $\mathscr{L}$ where

$$\mathscr{L}(t, x, y) = \sum_{i=1}^{m} \sum_{\gamma \in \Gamma} E(t, \alpha_i x, \gamma y) = \sum_{s \in S} E(t, sx, y).$$

Here we used the isometry invariance of the heat kernel E and the decomposition $S = \cup_{i=1}^{m} \Gamma\alpha_i$. Consequently, one has

$$\sum_{\mu} \mathrm{Tr}(T_{S,\mu}) e^{-t\mu} = \int_M \sum_{s \in S} E(t, sx, x) \phi(x)\, dx \tag{3.2}$$

where ϕ is a partition of unity relative to Γ.

The main goal of this section is to investigate the asymptotic behavior of the left hand side of (3.2) at $t \downarrow 0$. In preparation, we record the elementary lemma:

LEMMA 3.3. *Suppose that C is a compact set with $\Gamma C = M$. Then only finitely many elements $s \in S$ have a fixed point in C.*

Proof. Recall that $S = \cup_{i=1}^{m} \Gamma\alpha_i$. If the lemma fails, there are infinitely many s in a single Γ-orbit, say $\Gamma\alpha_l$, which have a fixed point in C. Reasoning by contradiction, let $s_j = \gamma_j \alpha_l$ be a sequence of distinct elements each having a fixed point in C. The set $D = C \cup \alpha_l C$ is compact and $D \cap \gamma_j^{-1} D$ is non-empty, for all j. This contradicts the fact that Γ acts properly discontinuously.

One uses Lemma 3.3 with C equal to the support of ϕ, where ϕ is a partition of unity relative to Γ. Let F be the finite set of distinct elements having a fixed point in C. It follows from (3.2) and the estimates of [5, p. 491] that

$$\sum_{\mu} \mathrm{Tr}(T_{S,\mu}) e^{-t\mu} = \sum_{s \in F} \int_M E(t, sx, x) \phi(x)\, dx + O(e^{-C_1/t}) \tag{3.4}$$

for some constant $C_1 > 0$. Thus, only those s in F contribute to the asymptotic expansion of $\mathrm{Tr}(T_S \circ \exp(-t\bar{\Delta}))$, as $t \downarrow 0$.

Fix some $s \in F$. Since s acts isometrically its fixed point set is the disjoint union of closed connected submanifolds N of dimension n. For each $N \in \Omega$ and $z \in N$, the isometry s induces an $O(d-n)$ action A on the fiber of the normal bundle $(T_z N)^{\perp}$ at z. Here d is the dimension of M. The linear transformation $I - A$ is invertible and we denote $B = (I-A)^{-1}$. Let us state:

PROPOSITION 3.5. *For each fixed $s \in F$, there is an asymptotic expansion as $t \downarrow 0$:*

$$\int_M E(t, sx, x)\phi(x)\,dx \sim \sum_{N \in \Omega} (4\pi t)^{-n/2} \sum_{k=0}^{\infty} t^k \int_N b_k(s, \phi, z)\,dz.$$

The $b_k(s, \phi, z)$ are compactly supported smooth functions. Moreover,

$$b_k = |\det B| b'_k,$$

where b'_k is an $O(n) \times O(d-n)$ invariant polynomial in the components of B, the curvature tensor R of M and its covariant derivatives, and ϕ along with its covariant derivatives.

Proof. This result was proved in [7] for M compact and $\phi = 1$. The general result follows by a similar argument, using the compact support of ϕ.

It is immediate that one has:

PROPOSITION 3.6. *There is an asymptotic expansion as $t \downarrow 0$:*

$$\sum_{\mu} \mathrm{Tr}(T_{S,\mu}) e^{-t\mu} \sim (4\pi t)^{-d/2} \sum_{k=0}^{\infty} e_k t^k,$$

where d is the dimension of M and e_k are constants independent of t.

Proof. This follows from (3.4) by summing the expansions of Proposition 3.5 over $s \in F$.

4. Signature theorem

Suppose that M is a complete oriented Riemannian manifold of even dimension $2l$. Let Γ be a group of orientation preserving isometries of M which act properly discontinuously with compact quotient $\overline{M} = \Gamma \backslash M$. We denote by d the exterior derivative

$$d\colon \Lambda^i(\overline{M}) \otimes C \to \Lambda^{i+1}(\overline{M}) \otimes C,$$

where $\Lambda^i(\overline{M})$ are Γ-invariant i-forms on M. If d^* is the adjoint of d with respect to the induced inner products, then $\Delta = dd^* + d^*d$ is the Hodge Laplacian. Since M is compact, Δ has pure point spectrum on the space of Γ-invariant forms. In particular, the kernel $H^*(\overline{M}, C)$ of Δ is finite dimensional.

Now consider the first order operator $D = d + d^*$. It is formally self-adjoint and $\Delta = D^*D = D^2$. In particular, the solutions of $Du = 0$ coincide with those of $\Delta u = 0$. We introduce an operator $\tau(\omega) = (\sqrt{-1})^{p(p-1)+l} * \omega$ for $\omega \in \Lambda^p(\overline{M}) \otimes C$, where $*$ is the Hodge star operator. Since $\tau^2 = 1$, we may decompose $\Lambda = \Lambda^+ \oplus \Lambda^-$ into the ± 1 eigenspaces for τ. The signature complex is the elliptic complex of Γ-invariant forms

$$0 \to \Lambda^+ \xrightarrow{D} \Lambda^- \to 0.$$

Suppose now that S is an orientation preserving set of isometries satisfying (2.1). If $S = \bigcup_{i=1}^m \Gamma\alpha_i$, then define the Hecke operator T_S by

$$T_S\omega(z) = \sum_{i=1}^{m} \alpha_i^*\omega(\alpha_i z),$$

where ω is a differential form and α_i^* denotes the pull-back on forms. Clearly, T_S preserves the decomposition $\Lambda = \Lambda^+ \oplus \Lambda^-$ and commutes with Δ. We define the signature of the Hecke operator by

$$\mathrm{Sign}(T_S) = \mathrm{Tr}(T_S|H^+) - \mathrm{Tr}(T_S|H^-).$$

Our goal is to give an explicit formula for the signature of the Hecke operator. The method, via heat equation asymptotics, will be similar to the proof of the equivariant signature theorem given in [8].

Let $s \in S$ be an orientation preserving isometry of M. The fixed point set Ω of s is the disjoint union of connected totally geodesic submanifolds N. If N is any component of Ω, we decompose $TM|N = TN \oplus TN^\perp$ into the tangent and normal bundles. Suppose that A: $TN^\perp \to TN^\perp$ is the endomorphism induced by the differential of s. The eigenvalues of A are constant and one may decompose the normal bundle

$$TN^\perp = TN^\perp(-1) \oplus TN^\perp(\Theta_1) \oplus \cdots \oplus TN^\perp(\Theta_s), \quad \Theta_i \neq \pi.$$

Here A acts on $TN^\perp(-1)$ as multiplication by -1. The spaces $TN^\perp(\Theta_i)$ are even dimensional and A reduces on these to a direct sum of the rotations

$$\begin{pmatrix} \cos\Theta_i & -\sin\Theta_i \\ \sin\Theta_i & \cos\Theta_i \end{pmatrix}$$

This gives $TN^\perp(\Theta_i)$ a natural complex structure.

One may define various characteristic forms on N associated to the bundles TN, $TN^{\perp}(-1)$, and $TN^{\perp}(\Theta_i)$. These appear in the formula for the signature of T_S. We let

$$\mathscr{L}(N) = \prod_j \frac{x_j/2}{\tanh(x_j/2)}$$

where the Pontriagin forms of N are the elementary symmetric functions in the x_j^2. Let m be the dimension of $TN^{\perp}(-1)$. Since s is orientation preserving, m is necessarily even. One may write

$$2^{-m/2}\mathscr{L}(TN^{\perp}(-1))^{-1} e(TN^{\perp}(-1)) = \prod_j \tanh(x_j/2)$$

where the Pontriagin forms of $TN^{\perp}(-1)$ are the elementary symmetric functions in the x_j^2 and the Euler form is the product of the x_j's. Since $TN^{\perp}(-1)$ may not be globally orientable, $e(TN^{\perp}(-1))$ must be interpreted as the Euler form relative to some local choice of orientation. Finally, we set

$$\mathscr{M}^{\Theta_i} = \prod_j \frac{\tanh(\sqrt{-1}\,\Theta_i/2)}{\tanh\left(\dfrac{x_j + \sqrt{-1}\,\Theta_i}{2}\right)}$$

where the elementary symmetric functions of the x_j's are the Chern forms of $TN^{\perp}(\Theta_i)$. Let $c(\Theta_i)$ be the complex dimension of $TN^{\perp}(\Theta_i)$.

Suppose ϕ is a partition of unity relative to Γ. According to Lemma 3.3, the set of $s \in S$ having a fixed point in the support of ϕ is a finite set F.

The main result of this paper is:

THEOREM 4.1. *For any partition of unity ϕ relative to Γ,*

$$\operatorname{sign}(T_S) = \sum_{s \in F} \sum_{N \in \Omega} \int_N 2^{(n-m)/2} *_N$$
$$\left[\left(\prod_i (\sqrt{-1}\tan(\Theta_i/2))^{-c(\Theta_i)/2}\right)\mathscr{L}(N)\mathscr{L}(TN^{\perp}(-1))^{-1}\right.$$
$$\left. e(TN^{\perp}(-1))\prod_i \mathscr{M}^{\Theta_i}(TN^{\perp}(\theta_i))\right](z)\phi(z)\,dz$$

Here $*_N$ is the Hodge star operator relative to a local orientation of TN. Similarly, $e(TN^{\perp}(-1))$ is the Euler form of $TN^{\perp}(-1)$ relative to a local choice of orientation for $TN^{\perp}(-1)$. These orientations are chosen to be compatible with the orientation of $TM|N$.

One uses the heat equation asymptotics to prove Theorem 4.1. The method is similar to the proof of the equivariant signature theorem given in [8]. However, there is a new technical difficulty which arises.

First of all, Proposition 3.5 may be extended to differential forms and, in a standard way, one obtains a local expression ξ which integrates to give

Sign(T_S). Choosing a local orientation for N, ξ may be regarded as a differential n-form. Moreover, ξ has the following properties:

(i) ξ is an $O(n) \times SO(m) \times U(c_1) \times \cdots \times U(c_s)$ invariant polynomial n-form in the components of the curvature tensor R of M, the partition of unity ϕ, and their covariant derivatives. The coefficients of this polynomial depend rationally upon the eigenvalues of A.

(ii) ξ is invariant under scaling of the metric on M.

(iii) ξ depends linearly upon ϕ and coincides with the integrand of Theorem 4.1 when ϕ is locally equal to one.

To complete the proof of Theorem 4.1, it needs to be shown that the covariant derivatives of ϕ never contribute to the local formula ξ. This will be proved in the next section using invariant theory.

5. Invariant theory

Let γ be an $O(n) \times SO(m) \times U(c_1) \times \cdots \times U(c_s)$ invariant polynomial n-form in the components of the curvature tensor R of M, the function ϕ, and their covariant derivatives. Suppose that γ depends linearly upon ϕ. We say that γ is of weight k if under the scaling of metrics $g \to c^2 g$ on M, one has $\gamma \to c^k \gamma$. Thus γ is invariant under scaling if and only if γ has weight zero.

Classical invariant theory [8] implies that any such γ lies in the space spanned by the elementary monomial invariants,

$$\operatorname{mon}(R, \phi) = \sum R_{F_1} R_{F_2} \cdots R_{F_p} \phi_H, \tag{5.1}$$

where $F_1, \dots, F_p, H$ are multi-indices containing both tangential and normal indices. It is understood that n of the tangential indices are to be alternated, det may be applied to the normal indices in $TN^{\perp}(-1)$, and the remaining indices must be contracted pairwise. Only one term, ϕ_H, occurs since γ depends linearly upon ϕ.

We analyze the elementary monomial invariants in a series of lemmas:

Lemma 5.2. *The weight of an elementary monomial invariant* mon(R) *is* $2p + n - \Sigma f_i - h$, *where* f_i *is the total number of indices in* F_i *and* h *is the total number of indices in* H.

Proof. Similar to the proof of Lemma 5.2 in [8, p. 8]. Note that ϕ is invariant under scaling of the metric.

Suppose ε_R denotes the total number of covariant derivatives in the R's. Then $\Sigma f_i = 4p + \varepsilon_R$. Thus one obtains the formula

$$n = \operatorname{weight}(\operatorname{mon}(R)) + 2p + \varepsilon_R + h.$$

Before proceeding further, we recall some elementary identities:

Lemma 5.3. (i) $\phi_{ij} = \phi_{ji}$.

(ii) $R_{i\underbrace{jkl}} = 0, \qquad R_{ij\underbrace{kl,r}} = 0,$

where the bow denotes alternation.

(iii) $R_{ijkl} = -R_{jikl}$, $R_{ijkl} = -R_{ijlk}$.

The identities 5.3 (ii), (iii) imply that for the tensors R_{F_i} we may alternate over at most two of the first five indices, else $\operatorname{mon}(R) = 0$. According to 5.3 (i), we may alternative over at most one of the first two indices in ϕ_H. Thus, as n is the total number of alternations in the R's one has $n \leq 2p + \varepsilon_R + h$, with strict inequality if $\varepsilon_R > 0$ or if $h > 1$. However,

$$2p + \varepsilon_R + h = n - \operatorname{weight}(\operatorname{mon}(R)).$$

We deduce:

LEMMA 5.4. *If* $\operatorname{weight}(\operatorname{mon}(R)) \geq 0$, *then* $\operatorname{weight}(\operatorname{mon}(R)) = 0$, $\varepsilon_R = 0$, *and* $h \leq 1$.

The total number of indices in $\operatorname{mon}(R)$ is even since m and n are both even and contractions occur in pairs. If $\varepsilon_R = 0$ and $h \leq 1$, this forces $h = 0$. So one has:

LEMMA 5.5. *If* $\operatorname{weight}(\operatorname{mon}(R)) \geq 0$, *then* $\varepsilon_R = h = \operatorname{weight}(\operatorname{mon}(R)) = 0$.

In particular, if $\operatorname{weight}(\operatorname{mon}(R)) = 0$ then $h = 0$. This fact is precisely what is needed to finish the proof of Theorem 4.1. Its proof is now complete.

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