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A NOTE ON FUCHSIAN GROUPS

BY

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Introduction

The origin of this note was in a query to the author from J. Lehner and M. Sheingorn, asking for a proof of Theorem 2 below. This theorem follows directly from Theorem 1 below, which is itself an easy consequence of known theorems, such as the Frobenius induced representation theorem, but does not seem to have been noticed previously. In the author's opinion this theorem is of genuine interest, since integral matrix groups may be treated arithmetically, and so may cast light on the more complicated fuchsian groups.

I am indebted to M. Tretkoff for supplying many references to the literature on theorems of the type of Theorem 2, and for numerous interesting remarks. In particular paper [6] by P. Scott contains a proof of Theorem 2 for surface groups, using hyperbolic geometry as the tool; and M. and C. Tretkoff have given a proof using covering spaces.

I am also indebted to W. Magnus for numerous valuable comments and suggestions. Most of the necessary background material for this note may be found in his book on the noneuclidean tessellations [4].

Finally, I am indebted to R. Lyndon for reading a preliminary version of this note and for suggesting a number of additions and improvements which have been incorporated into the text.

The theorem of Frobenius referred to above is as follows: Let G, H be groups such that $G \supset H$, $(G : H) = \mu < \infty$. Let α be a faithful representation of H of degree n. Then α induces a faithful representation β of G of degree μn, and β is integral if α is integral. A convenient reference for this theorem is Boerner's book [1].

Now suppose that G is a finitely generated fuchsian group. Then as an abstract group G is generated by elements

$$E_1, E_2, \ldots, E_s;\ P_1, P_2, \ldots, P_t;\ A_1, B_1, A_2, B_2, \ldots, A_g, B_g.$$

Any one of s, t, g may be 0, but to avoid degeneracy, it is assumed that

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$s + t + g > 0$. If $s > 0$, then there are integers $m_i \geqq 2$ such that E_i is of order m_i, $1 \leqq i \leqq s$. These generators are referred to as "elliptic". The P_i are referred to as "parabolic", and the A_i and B_i as "hyperbolic". The defining relations for G are

(1) $E_i^{m_i} = 1,\ 1 \leqq i \leqq s,$
(2) $E_1 E_2 \cdots E_s P_1 P_2 \cdots P_t [A_1, B_1][A_2, B_2] \cdots [A_g, B_g] = 1,$

$$[A_i, B_i] = A_i B_i A_i^{-1} B_i^{-1}, \quad 1 \leqq i \leqq g.$$

If $s = 0$, then G is torsion-free. If $t = 0$, then G is said to be *compact*. If $t > 0$, then the relationship (2) may be eliminated (by eliminating P_t, say) and G becomes the free product of $2g + t - 1$ infinite cyclic groups and s finite cyclic groups. If $s = 0$ as well, then G is just F_{2g+t-1}, the free group of rank $2g + t - 1$. If $s = t = 0$, then G is ϕ_g, the fundamental group of genus g. G may always be realized as a discontinuous subgroup of $PSL(2, R)$, provided that it has positive hyperbolic area. This condition is equivalent to the requirement that

$$2g - 2 + t + \sum_{i=1}^{s} \left(1 - \frac{1}{m_i}\right) > 0.$$

The theorems

The theorems we will prove are the following:

THEOREM 1. *Let G be a finitely generated fuchsian group. Then G has a faithful representation as a subgroup of $GL(n, Z)$, where n depends on G.*

THEOREM 2. *Let G be a fuchsian group, $S = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ a finite subset of G, γ an element of G such that $\gamma\alpha_i \neq \alpha_i\gamma$, $1 \leqq i \leqq k$. Then a subgroup H of G exists such that $(G\colon H) < \infty$, $\gamma \in H$, but $\alpha_i \notin H$, $1 \leqq i \leqq k$.*

These are the principal results. Some additional remarks will be made in the next section.

Proofs

As was indicated in the introduction, F_r, ϕ_g will stand for the free group of rank r and the fundamental group of genus g, respectively.

LEMMA 1. *Suppose that $r \geqq 1$, $g \geqq 2$. Then F_r has a faithful representation as a subgroup of $SL(2, Z)$, and ϕ_g has a faithful representation as a subgroup of $SL(8, Z)$. In addition, ϕ_1 has a faithful representation as a subgroup of $SL(3, Z)$.*

Proof. If $r = 1$, we may choose

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

as a generator of F_1. Suppose that $r \geqq 2$. It is well known that F_r occurs as a subgroup of F_2. (This is an easy consequence of the fact that F_2 contains subgroups of any index $s \geqq 1$, and that the rank of a subgroup of index s is $1 + s$, a consequence of the Reidemeister-Schreier algorithm. If F_2 is freely generated by X, Y, then the subgroup consisting of all words in the generators such that the exponent sum in X is divisible by s, is an example of a normal subgroup of index s.) Thus it is only necessary to show that F_2 has a faithful representation as a subgroup of $SL(2, Z)$. One such representation (among the many possible) is obtained by choosing

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

as generators of F_2. This result goes back to I.N. Sanov (see [5]).

If $g = 1$, then the matrices

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are generators of ϕ_1 (the free abelian group of rank 2). Suppose that $g \geqq 2$. It is also well known that ϕ_g occurs as a subgroup of ϕ_2. (This depends on the fact that ϕ_2 contains subgroups of any index $s \geqq 1$, and that the genus of a subgroup of index s is $1 + s$, which is a consequence of the fact that the hyperbolic area of a subgroup is equal to the hyperbolic area of the group, multiplied by the index of the subgroup. If ϕ_2 is the group generated by A_1, B_1, A_2, B_2, with defining relation $[A_1, B_1][A_2, B_2] = 1$, then the subgroup consisting of all words in the generators such that the exponent sum in A_1 is divisible by s, is an example of a normal subgroup of index s.) In his book on the noneuclidean tessellations [4], W. Magnus shows that ϕ_2 occurs as a subgroup of the group generated by B, ABA^{-1}, where

$$A = i\begin{bmatrix} \sqrt{2} & -1 \\ 1 & -\sqrt{2} \end{bmatrix},$$

$$B = \begin{bmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{bmatrix}, \quad \alpha = \frac{\sqrt{3} + i}{2},$$

$$ABA^{-1} = \begin{bmatrix} 2\alpha - \bar{\alpha} & -i\sqrt{2} \\ i\sqrt{2} & -\alpha + 2\bar{\alpha} \end{bmatrix}.$$

We note that α is a primitive $12th$ root of unity.

After a suitable diagonal conjugacy, we may conclude that ϕ_2 occurs as a subgroup of the group generated by B, C, where

$$C = \begin{bmatrix} 2\alpha - \bar{\alpha} & 2 \\ 1 & -\alpha + 2\bar{\alpha} \end{bmatrix}.$$

Now B and C are of determinant 1 and have entries which are integers of the algebraic number field $Q(\alpha)$. Since $Q(\alpha)$ is of degree 4 over the rational field Q, this implies that ϕ_2 has a faithful representation as a subgroup of $SL(8, Z)$. This concludes the proof of the lemma.

LEMMA 2. *Let G be a finitely generated fuchsian group. Then G contains a subgroup H of finite index (so that H is also a finitely generated fuchsian group) with no elements of finite order.*

Proof. This result is well known: It is the Nielsen-Fenchel-Fox theorem (see [3]), and may also be derived from a general theorem of Selberg on matrix groups (see [7]).

The set of values assumed by the index $(G: H)$ has been exactly described in a recent paper of A.L. Edmonds, J.H. Ewing, and R.S. Kulkarni (see [2]).

We are now in a position to prove Theorem 1. Let H be the subgroup of G whose existence is guaranteed by Lemma 2. Then H, as a finitely generated torsion-free fuchsian group, is either a free group of finite rank or a fundamental group of finite genus, depending on whether or not it is compact. In either case, Lemma 1 guarantees the existence of a faithful representation of H as a subgroup of $SL(k, Z)$, where k may be taken as 2, 3, or 8. But then the Frobenius theorem implies the existence of a faithful representation of G as a subgroup of $GL(n, Z)$, where $n = \mu k$, and $\mu = (G: H)$. This concludes the proof of Theorem 1.

The proof of Theorem 2 is now quite simple. We may replace G by a subgroup of $GL(n, Z)$. Choose m so that $\gamma\alpha_i \not\equiv \alpha_i\gamma \bmod m$, $1 \leqq i \leqq k$ (all sufficiently large integers m satisfy this condition). Take H as the congruence subgroup of G consisting of all $g \in G$ such that $g \equiv \gamma^t \bmod m$ for some integer t (positive, negative, or zero). Since every element of G commutes modulo m with γ, and no α_i commutes modulo m with γ, it follows that no α_i can belong to G, $1 \leqq i \leqq k$.

In conclusion, we give an explicit construction for the subgroup H of Lemma 2, in the case when G is not compact. We retain the notation of the introduction.

Assume that G has parabolic and elliptic elements (so that $t \geqq 1$, $s \geqq 1$). Then G has a torsion-free normal subgroup H of index $\mu = m_1 m_2 \dots m_s$. The proof is as follows:

Since G has a parabolic element, G is the free product

$$G = C_{m_1} * C_{m_2} * \cdots * C_{m_s} * F,$$

where $C_{m_i} = \{E_i\}$ is a cyclic group of order m_i, and F is a free group of finite rank. Define H as the subgroup of G consisting of all words $w \in G$ such that the exponent sum of w in the generator E_i is divisible by m_i, $1 \leqq i \leqq s$. Then H is well defined,

$$G = \sum_{\substack{0 \leqq r_i \leqq m_i - 1 \\ 1 \leqq i \leqq s}} E_1^{r_1} E_2^{r_2} \dots E_s^{r_s} H$$

is a left coset decomposition for G modulo H, so that

$$(G : H) = m_1 m_2 \dots m_s,$$

and H is clearly a normal subgroup of G. Furthermore, any element of finite order must be of the form

$$A E_i^{k_i} A^{-1}, \quad 1 \leqq k_i \leqq m_i - 1, \quad 1 \leqq 1 \leqq s, \quad A \in G$$

(by the Kurosch subgroup theorem) and none of these belongs to H. Thus H is torsion-free as well, and the proof is complete.

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