

STABLE RANK IN HOLOMORPHIC FUNCTION ALGEBRAS

BY

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Introduction

The concept of the stable rank of a ring, introduced by H. Bass [1], has been very useful in treating some problems in algebraic K -theory. In this paper we show how this concept is related to the structure of a commutative Banach algebra A . For example, we show that A has a finite stable rank if its spectrum $X(A)$ has finite dimension.

First, we prove, in an analytical way, that the algebra of continuous functions on the disc $\bar{\Delta} = \{z \in \mathbf{C}: |z| \leq 1\}$ which are holomorphic on its interior Δ , has stable rank 1. In Sections 2 and 3 we give other proofs of this fact, but it is convenient to have a classical proof. Furthermore, some ideas contained in it lead to the notion of punctual stability, to be developed and applied in Section 3.

In Section 2 we mention some results, taken from [4], and we apply them to the study of the stable rank of some algebras of holomorphic functions.

In Section 3 we introduce the concept of stability of a ring A at a point $g \in A$. In the case of a commutative Banach algebra A we relate this concept to the topological structure of some subsets of A^n . As an application we prove that $\mathcal{P}(X)$ and $\mathcal{R}(X)$ (see definitions below) have stable rank one. Finally we give a list of some open problems.

Most results about Banach algebras we prove here may be stated for topological algebras A whose unit groups A^\times are open and the inversion is a homeomorphism of A^\times . The proofs are analogous to those given here.

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Section 1

This section considers notations and preliminary results. In this paper, rings and algebras have identity. The group of units of A is denoted by A^\times . Given a ring A , $a \in A^n$ is *unimodular* if there exists $b \in A^n$ such that $\langle b, a \rangle = \sum_{i=1}^n b_i a_i = 1$. We denote by $U_n(A)$ the unimodular elements of A^n . We say

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that $a \in U_n(A)$ is *reducible* if there exist x_1, \dots, x_{n-1} in A such that

$$(a_1 + x_1 a_n, a_2 + x_2 a_n, \dots, a_{n-1} + x_{n-1} a_n) \in U_{n-1}(A).$$

A is said to have a stable rank at most $n - 1$ if every $a \in U_n(A)$ is reducible. The stable rank of A , denoted by $sr(A)$, is the least $n - 1$ with this property. The reader is referred to [1], [4] and [14] for some applications of this concept.

If X is a compact subset of \mathbf{C} , $\mathcal{P}(X)$ (resp. $\mathcal{R}(X)$) is the completion, with respect to the supreme norm, of the algebra of polynomial (resp. rational) functions restricted to X .

1.1. THEOREM. *Let \mathcal{E} be the algebra of entire functions on the complex plane \mathbf{C} . Then \mathcal{E} has stable rank 1.*

Proof. Let (f, g) be a unimodular pair. This means that $Z_f \cap Z_g = \emptyset$, where for a complex function h , $Z_h = \{z: h(z) = 0\}$. It suffices to find h, r in \mathcal{E} such that $f + hg = e^r$. Putting

$$h(z) = (e^{r(z)} - f(z))/g(z),$$

we only need to find r in \mathcal{E} such that h belongs to \mathcal{E} . Thus, the theorem will be proved if we solve the following interpolation problem:

$$(e^r - f)^{(j)}(a_k) = 0, \quad j = 1, \dots, m_k,$$

for each $a_k \in Z_g$ with multiplicity m_k . By a corollary of Mittag-Leffler's theorem [7] it follows that, given a sequence

$$(a_k, w_k^{(0)}, \dots, w_k^{(n_k)})$$

with $a_k \in \mathbf{C}, \lim_{k \rightarrow \infty} |a_k| = \infty, w_k^{(j)} \in \mathbf{C}$, there exists $d \in \mathcal{E}$ such that $d^{(j)}(a_k) = w_k^{(j)}$ for every $k \in \mathbf{N}, j = 0, \dots, m_k$. In our case, it suffices to apply this result when $\{a_k\}$ is the zero-set of g, m_k is the multiplicity of a_k as a zero of g , and $w_k^{(j)} (0 \leq j \leq m_k)$ is obtained, recursively, from the equation $(e^r - f)^{(j)}(a_k) = 0$. For example, $w_k^{(0)} = r(a_k) = \ln f(a_k)$ for some branch of the logarithm, w_k^1 is obtained from the formula

$$(r'e^r - f')(a_k) = 0,$$

that is,

$$w_k^{(1)} = r'(a_k) = f'(a_k)e^{-r(a_k)} = (f'/f)(a_k),$$

and so on.

1.2. THEOREM. *The disc algebra A has stable rank 1.*

Proof. Let (f, g) be an element of $U_2(A)$. If g is identically zero, it is clear that (f, g) is reducible; therefore, we may suppose, by multiplying by a constant if necessary, that $\|g\| = 1$. The proof is divided into five steps.

First step. There exists $z_1 \in S^1$ such that $(z - z_1, g)$ is reducible. Since the polynomials form a dense subalgebra of A (see [5], [9]), there exists a polynomial p such that $\|p - g\| < 1/4$. Then $\|p/\|p\| - g\| < 1/2$, for

$$\begin{aligned} \|p/\|p\| - g\| &\leq \|p/\|p\| - p\| + \|p - g\| \\ &= \|p\| |1/\|p\| - 1| + \|p - g\| \\ &= |1 - \|p\|| + \|p - g\| \\ &= | \|g\| - \|p\|| + \|p - g\| \\ &\leq 2\|p - g\| \\ &< 1/2. \end{aligned}$$

Define $q = p/\|p\|$. Then we can choose a z_1 in S^1 such that $q(z_1) = \|q\| = 1$. By a suitable rotation of the disc we can suppose $z_1 = 1$. We will show that $f_n(z) = ((z + 1)/2)^n q(z)$ peaks in 1 for all positive integers n (a function h is said to peak in x if $\|h\| = |h(x)|$ and $|h(y)| < |h(x)|$ for all $y \neq x$). If $z \neq 1$,

$$|((z + 1)/2)^n q(z)| < |q(z)| \leq \|q\| = 1 = |f_n(1)|.$$

Choose n such that $f'_n(1) = (n/2)q(1) + q'(1) \neq 0$. Now, we prove that $(z - 1, g)$ is reducible. Put

$$v(z) = \frac{((z + 1)/2)^n q(z) - q(1)}{1 - z}.$$

From the identity $(z - 1)v(z) + ((z + 1)/2)^n q(z) = q(1)$ and observing that $v \in A'$ and $|q(1)| = 1$, it is clear that $(z - 1, q)$ is reducible. Let

$$\begin{aligned} l(z) &= (z - 1)v(z) + ((z + 1)/2)^n q(z) \\ &= q(1) - ((z + 1)/2)^n (q(z) - g(z)). \end{aligned}$$

Then

$$\begin{aligned} \|l - q(1)\| &= \|((z + 1)/2)^n (q - g)\| \leq \|((z + 1)/2)^n\| \|q - g\| \\ &= \|g - g\| < 1/2, \end{aligned}$$

so $l \in A'$ for $|q(1)| = 1$. From the invertibility of v , we have $(z - 1, g)$ reducible.

Second step. There exists $z_0 \in \Delta \setminus Z_g$ such that $(z - z_0, g)$ is reducible. By the first step, there exist $h \in A, u \in A'$ such that $(z - 1) + hg = u$. As A' is open, for $n \geq n_0$,

$$(z - 1 + 1/n) + hg = u + 1/n \in A'.$$

It suffices to take $z_0 = 1 - 1/n_0$.

Third step. For every $\alpha \in \Delta \setminus Z_g$, $(z - \alpha, g)$ is reducible.

(i) If $\alpha \in \Delta \setminus Z_g$ there is a neighborhood U_α of α with the following property: if $w \in U_\alpha$ there exist $h \in A, u \in A'$ such that $(z - \alpha) + hg = (z - w)u$.

Given $0 < \varepsilon < 1$ such that the open ball centered at α with radius ε is contained in Δ , there exists $0 < \delta < \varepsilon$ such that

$$|w - \alpha| < \varepsilon |g(w)| \quad \text{if } |w - \alpha| < \delta.$$

If $|z - \alpha| \geq \varepsilon$, then $|z - \alpha| > |(w - \alpha)/g(w)| |g(z)|$, so the function

$$b(z) = (z - \alpha) - ((w - \alpha)/g(w))g(z)$$

has no zeroes in $|z - \alpha| \geq \varepsilon$, and in $|z - \alpha| < \varepsilon$ it has only one zero, by Rouché's theorem [7]. But $b(w) = 0$, so $b(z) = (z - w)u(z)$ where $u \in A'$.

(ii) Given $\alpha, \beta \in \Delta \setminus Z_g$ let $\alpha \sim \beta$ if there exist $h \in A, u \in A'$ such that $(z - \alpha) + hg = (z - \beta)u$.

It is easy to see that \sim is an equivalence relation on $\Delta \setminus Z_g$. By (i), \sim is open, so, as $\Delta \setminus Z_g$ is connected, there is only one equivalence class.

Now, by the second step, there is $z_0 \in \Delta \setminus Z_g$ such that $(z - z_0) + lg = v$ for some $l \in A, v \in A'$. By (ii) if $\alpha \in \Delta \setminus Z_g$, there are $h \in A, u \in A'$ such that

$$(z - \alpha) + gh = (z - z_0)u,$$

$$\text{so } (z - \alpha)u^{-1} + (u^{-1}h + l)g = (z - z_0) + lg = v \in A'$$

and $(z - \alpha, g)$ is reducible.

Fourth step. For every polynomial p such that $Z_p \cap (S^1 \cup Z_g) = \emptyset$, (p, g) is reducible. If $|\alpha| > 1$, $z - \alpha \in A'$; if $\alpha \notin Z_g$ and $|\alpha| < 1$, by the third step $(z - \alpha, g)$ is reducible. In other words, $(z - \alpha, g)$ is reducible if $\alpha \notin S^1 \cup Z_g$.

Let p be a polynomial as above; then $p(z) = \lambda \prod_{j=1}^n (z - \alpha_j)$ for some $\lambda \in \mathbb{C}$ and $\alpha_j \notin S^1 \cup Z_g$ ($j = 1, \dots, n$). Let $h_j \in A, u_j \in A'$ be such that

$$(z - \alpha_j) + h_j g = u_j \quad (j = 1, \dots, n).$$

Then

$$\lambda \prod_{j=1}^n u_j = \lambda \prod_{j=1}^n [(z - \alpha_j) + h_j g] = \lambda \prod_{j=1}^n (z - \alpha_j) + Hg = p + Hg,$$

with $H \in A$. This proves the assertion.

Fifth step. Every $(f, g) \in U_2(A)$ is reducible. Since the complement of Z_g is dense in \mathbf{C} the set B of polynomials p with $Z_p \cap (S^1 \cup Z_g) = \emptyset$ is dense in A . Thus, given $(f, g) \in U_2(A)$ there are $p, q \in B$ such that $pf + qg \in A^\cdot$. Now, (p, g) is reducible so there are $h \in A, u \in A^\cdot$ with $hg + p = u$ and, from this

$$uf + (q - hf)g = (hg + p)f + (q - hf)p = pf + qg \in A^\cdot,$$

so we have proved that (f, g) is reducible. This finishes the proof.

Section 2

Here we state some results on stable range (see [4] for the proofs) and we use them to estimate the stable range of some algebras of holomorphic functions.

2.1. PROPOSITION. *Let A be a commutative ring. The following conditions are equivalent.*

- (i) $sr(A) \leq n$;
- (ii) *For every ideal J of A , the natural mapping*

$$\pi: U_n(A) \rightarrow U_n(A/J)$$

is surjective;

- (iii) *For every element $a \in A$ the mapping $\pi: U_n(A) \rightarrow U_n(A/J)$ is surjective, where J is the ideal of A generated by a .*

When A is a Banach algebra it suffices to consider, in (ii) and (iii), only closed ideals.

Proof. See [4].

2.2. PROPOSITION. *Let A be a ring with unit. Then, an element $a \in A^n$ is unimodular if and only if the application*

$$t_a: M_n(A) \rightarrow A^n,$$

defined by $t_a(\sigma) = \sigma a$, is surjective.

2.3. COROLLARY. *Let A be a Banach algebra. Then, for every $a \in U_n(A)$,*

$$t_a: GL_n(A) \rightarrow U_n(A)$$

is an open mapping and it has the following property: if $x \in \text{im } t_a$ and x' belongs to the connected component of x in $U_n(A)$, then $x' \in \text{im } t_a$.

Proof. Consider the equivalence relation on $U_n(A)$ defined by $x \equiv y$ if and only if there is a $\sigma \in GL_n(A)$ such that $\sigma x = y$. Then, by 1(iii) and the open mapping theorem, \equiv is open and, consequently, closed.

2.4. PROPOSITION. *Let $f: A \rightarrow B$ be an epimorphism of Banach algebras. Then the induced mapping $f: U_n(A) \rightarrow U_n(B)$ is a Serre fibration. In particular, if $b \in \text{im } f$ and b' is connected to b by a curve in $U_n(B)$, then $b' \in \text{im } f$.*

2.5. COROLLARY. *Let $f: A \rightarrow B$ be an epimorphism of Banach algebras. Suppose that $U_n(B)$ is connected. Then $f: U_n(A) \rightarrow U_n(B)$ is surjective.*

2.6. THEOREM. *Let A be a commutative Banach algebra. Consider the following conditions:*

- (1) $sr(A) \leq n - 1$;
- (2) $sr(A/J) \leq n - 1$ for every ideal J of A ;
- (3) $U_n(A/J)$ is connected for every closed ideal J of A ;
- (4) $sr(A) \leq n$.

Then (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4).

Proof. It is clear that (1) \Rightarrow (2).

(2) \Rightarrow (3). We shall prove that, for a Banach algebra B , $U_n(B)$ is connected if $sr(B) \leq n - 1$. Given $b = (b_1, \dots, b_n) \in U_n(B)$ there exist x_1, \dots, x_{n-1} in B such that

$$b' = (b_1 + x_1 b_n, \dots, b_{n-1} + x_{n-1} b_n) \in U_{n-1}(B).$$

Therefore, there exist $z_1, \dots, z_{n-1} \in B$ such that

$$\sum_{i=1}^{n-1} z_i b'_i = \sum_{i=1}^{n-1} z_i (b_i + x_i b_n) = 1.$$

Consider the $n \times n$ -matrix

$$\sigma_t = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & \ddots & & \\ -tb_n z_1 & \dots & -tb_n z_{n-1} & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & tx_1 \\ & \ddots & & \vdots \\ & & \ddots & tx_{n-1} \\ & & & 1 \end{bmatrix}.$$

Now, the curve $\gamma: [0, 1] \rightarrow U_n(B)$ $\gamma(t) = \sigma_t b$ joins $b = \gamma(0)$ with $(b', 0) = \gamma(1)$. But it is clear, putting

$$\delta: [0, 1] \rightarrow U_n(B) \delta(t) = (tb', 1 - t),$$

that $(b', 0)$ is connected to $e_n = (0, \dots, 0, 1)$. This proves that $U_n(B)$ is connected.

(3) \Rightarrow (4). Let J be a closed ideal of A ; it suffices to prove that $U_n(A) \rightarrow U_n(A/J)$ is surjective. But, by hypothesis, $U_n(A)$ is connected, so the result follows from (2.4).

Let A be a commutative Banach algebra with spectrum $X(A)$ and Gelfand transform $g: A \rightarrow C(X(A))$. For $U \subset \mathbb{C}^n$ define

$$A_U = \{a \in A^n: sp(a) \subset U\},$$

where $sp(a) = \{(\phi(a_1), \dots, \phi(a_n)) \in \mathbb{C}^n: \phi \in X(A)\}$.

Following some results of Novodvorski [9] and Taylor [13], Raeburn [10] proved that g induces a bijection from the set $\pi_0(A_U)$ of connected components of A_U onto the set $[X(A), U]$ of homotopy classes of continuous mappings from $X(A)$ into U , when $U \in \mathbb{C}^n$ is open. Observe that, taking $U = \mathbb{C}_*^n = \mathbb{C}^n \setminus \{0\}$, we get $A_U = U_n(A)$. Since S^{2n-1} is homotopically equivalent to \mathbb{C}_*^n we get:

2.6. PROPOSITION. *Let A be a commutative Banach algebra. Then, the Gelfand transform induces a bijection from $\pi_0(U_n(A))$ onto $[X(A), S^{2n-1}]$, where $[X, Y]$ denotes the set of all homotopy classes of mappings from X into Y .*

2.7. Remarks. Given an ideal J of A the *hull* of J is defined by

$$\text{hull}(J) = \{\phi \in X(A): \phi|_J = 0\}.$$

It is clear that $\text{hull}(J)$ is homeomorphic to $X(A/J)$.

2.8. COROLLARY. *The disc algebra has stable rank 1.*

Proof. By (2.8), it is enough to prove that $U_1(A/J)$ is connected for every closed ideal J of A . Now, given J , it is clear that $F = \text{hull}(J)$ is \bar{D} or totally disconnected. In both cases $\pi_0(U_1(A/J)) = [\text{hull}(J), S^1]$ is trivial.

2.9. THEOREM. *Let A be a commutative Banach algebra. Let d be the covering dimension of $X(A)$. Then $sr(A) \leq (d/2) + 1$.*

Proof. Recall that the covering dimension $\text{dim}(X)$ of a compact space X has the following property (see [8]): if $\text{dim}(X) \leq n$, $[F, S^{n+1}]$ is trivial for every closed subset F of X . Suppose that $d \leq 2n$. Then $[F, S^{2n+1}]$ is trivial for every closed subset F of X . In particular $[\text{hull}(J), S^{2n+1}]$ is trivial for every closed ideal J of A . But this means that $U_{n+1}(A/J)$ is connected for every J , so $sr(A) \leq n + 1$.

2.10. COROLLARY. *If A has a system of n generators then $sr(A) \leq n + 1$.*

Proof. In this case, $X(A)$ is homeomorphic to a compact subset of \mathbf{C}^n , so $d \leq 2n$.

2.11. COROLLARY. *The polydisc algebra of \mathbf{C}^n has stable rank at most $n + 1$.*

3.1. *Definition.* Let A be a commutative ring with unity and let $g \in A$. In A^n we define the following equivalence relation: given $\bar{b}, \bar{h} \in A^n$, $\bar{b} \sim_g \bar{h}$ if and only if there exist $\bar{u} \in (A^\cdot)^n$ such that $\bar{b} - \bar{u}\bar{h} \in A^n \cdot g$, where the product in A^n is coordinate to coordinate. The equivalence relation depends on n and g , but it will be noted \sim when there is no risk of confusion.

3.2. LEMMA. *If $\bar{b}_1 \sim \bar{b}_2$ and $\bar{d}_1 \sim \bar{d}_2$, then $\bar{b}_1\bar{d}_1 \sim \bar{b}_2\bar{d}_2$.*

Proof. Since the relation in A^n is the product of the relation in A , it is sufficient to prove the lemma for the case $n = 1$. If $b_1 = ub_2 + hg$ and $d_1 = vd_2 + lg$, where $u, v \in A^\cdot$ and $l, h \in A$; then

$$b_1d_1 = (ub_2 + hg)(vd_2 + lg) = uvd_2d_2 + (ub_2l + hvd_2 + hlg)g$$

which proves the lemma.

3.3. *Definition.* Let A be a commutative ring with unit, and let $g \in A$. Let $H_{n,g}(A) = \{\bar{b} \in A^n / (\bar{b}, g) \in U_{n+1}(A)\}$. We say that A is n -stable at g if and only if for each $\bar{b} \in H_{n,g}(A)$, there exist $\bar{c} \in U_n(A)$ such that $\bar{b} \sim \bar{c}$; in other words, $H_{n,g}(A)$ is contained in

$$\bigcup_{\bar{c} \in U_n(A)} (\bar{c}) = U_n(A) + A^n g,$$

where (\bar{c}) denotes the equivalence class of \bar{c} . It is immediate that $U_n(A) + A^n g$ is contained in $H_{n,g}(A)$, without hypothesis.

3.4. *Remarks.* (1) For a non-commutative ring A it is possible to define the n -stability without the equivalence relation, by

$$H_{n,g}(A) \subset U_n(A) + A^n g;$$

but in this paper we do not consider non-commutative rings.

(2) It is easy to see that $sr(A) \leq n$ if and only if A is n -stable at g for all $g \in A$, but we do not know if the fact that A is n -stable at g implies that A is $(n + 1)$ -stable at g .

(3) From now on the ring A will be a complex (or real) Banach algebra. With this hypothesis it is easy to verify that $U_n(A)$ and $H_{n,g}(A)$ are open for all positive integer n and all $g \in A$. Furthermore if $L \subset A^n$ is open, then $\bigcup_{i \in L} (\bar{i})$ is open, for

$$\begin{aligned} \bigcup_{i \in L} (\bar{i}) &= (A^\cdot)^n \cdot L + A^n g \\ &= \bigcup_{\bar{u} \in (A^\cdot)^n} (\bar{u} \cdot L + A^n g) \\ &= \bigcup_{\bar{u} \in (A^\cdot)^n} \bigcup_{\bar{a} \in A^n} (\bar{u}L + \bar{a}g) \end{aligned}$$

which is the union of open sets.

(4) If S is a multiplicative set contained in A^n , and $g \in A$, then $\bigcup_{\bar{s} \in S} (\bar{s})$ is a multiplicative set. This fact, with the above remarks, shows that if $T \subset A$, and $b \sim_g 1$ for all $b \in T$ and furthermore the multiplicative set generated by $T, S(T)$ is dense, then A is 1-stable at g , for

$$H_{1,g}(A) \subset A = S(T) \subset \overline{A^\cdot + Ag}.$$

With these remarks in mind, we analyze the proof of Theorem 1.2 again. Let g be an element of $A, g \neq 0$. Then A is 1-stable at g . In fact, at the first step we proved that there exists $z_1 \in S^1$ such that $z - z_1 \sim_g 1$, but $(1)_g$ being open, there exists $z_0 \in \Delta$ such that $z - z_0 \sim_g 1$ (this was the second step); then it was proved that if $w_1, w_2 \in \Delta$ are such that

$$z - w_1, z - w_2 \in H_{1,g}(A)$$

then $z - w_1 \sim_g z - w_2$, so $z - w_1 \sim_g 1$. At the fourth and fifth steps we saw that $z - \alpha \sim_g 1$ for all $\alpha \notin Z_g$ and $S(\{z - \alpha \in A : \alpha \notin Z_g\})$ is dense in A , so A is 1-stable at g , as claimed.

3.5. PROPOSITION. *The following conditions are equivalent.*

- (i) A is n -stable at g .
- (ii) $U_n(A) + A^n g \supset H_{n,g}(A)$.
- (iii) $\overline{U_n(A) + A^n g} \supset H_{n,g}(A)$.

Proof. (i) \Rightarrow (ii). By definition.

(ii) \Rightarrow (iii). Trivial.

(iii) \Rightarrow (i). Take $\bar{b} \in H_{n,g}(A)$, and choose $\bar{\lambda} \in A^n, \beta \in A$, such that $\langle \bar{\lambda}, \bar{b} \rangle + \beta g = 1$. But then $\bar{\lambda} \in H_{n,g}(A)$, so there exists a neighborhood V of $\bar{\lambda}$ such that $\langle \bar{u}, \bar{b} \rangle + \beta g \in A^\cdot$ if $\bar{u} \in (U_n(A) + A^n g) \cap V$ (by hypothesis this set is non-void). Now $\bar{u} = \bar{d} + \bar{l}g$ for some $\bar{d} \in U_n(A)$ and $\bar{l} \in A^n$, and choosing

$\bar{r} \in A^n$ with $\langle \bar{d}, \bar{r} \rangle = \langle \bar{l}, \bar{b} \rangle + \beta$ we get

$$\begin{aligned} \langle \bar{d}, \bar{b} + \bar{r}g \rangle &= \langle \bar{d}, \bar{b} \rangle + \langle \bar{d}, \bar{r} \rangle g \\ &= \langle \bar{d}, \bar{b} \rangle + (\langle \bar{l}, \bar{b} \rangle + \beta) g \\ &= \langle \bar{d} + \bar{l}g, \bar{b} \rangle + \beta g \\ &= \langle \bar{u}, \bar{b} \rangle + \beta g \in A'. \end{aligned}$$

This proves that A is n -stable at g .

3.6. PROPOSITION. *If $A' + Ag$ is dense in A , then for every $\bar{b} \in H_{n,g}(A)$ there exist $\bar{u} \in (A')^n, \beta \in A$ such that $\langle \bar{b}, \bar{u} \rangle + \beta g = 1$.*

Proof. Take a_1, \dots, a_n, h in A such that $\langle \bar{a}, \bar{b} \rangle + hg = 1$. By hypothesis there exist $c_j = u_j + r_j g \in A' + Ag$ ($1 \leq j \leq n$) such that

$$\langle \bar{c}, \bar{b} \rangle + hg \in A',$$

but

$$\langle \bar{c}, \bar{b} \rangle + hg = \langle \bar{u}, \bar{b} \rangle + (\langle \bar{r}, \bar{b} \rangle + h)g$$

as claimed.

3.7. LEMMA. *$U_n(A) + A^n g$ is open and closed in $H_{n,g}(A)$.*

Proof. $U_n(A) + A^n g$ is open in A^n so it is open in $H_{n,g}(A)$. First, observe that if $\bar{b} \in A^n, \bar{a} \in U_n(A) + A^n g$ and $\beta \in A$ such that $\langle \bar{a}, \bar{b} \rangle + \beta g \in A'$, then $\bar{b} \in U_n(A) + A^n g$: in fact $\bar{a} = \bar{c} + \bar{h}g$, where $\bar{c} \in U_n(A)$ and $\bar{h} \in A^n$. Then

$$\langle \bar{c} + \bar{h}g, \bar{b} \rangle + \beta g = \langle \bar{c}, \bar{b} \rangle + (\langle \bar{h}, \bar{b} \rangle + \beta)g \in A'.$$

But since $\bar{c} \in U_n(A)$, there exists $\bar{r} \in A^n$ such that $\langle \bar{h}, \bar{b} \rangle + \beta = \langle \bar{c}, \bar{r} \rangle$, so

$$\langle \bar{c}, \bar{b} \rangle + \langle \bar{c}, \bar{r} \rangle g = \langle \bar{c}, \bar{b} + \bar{r}g \rangle \in A',$$

and it follows that $\bar{b} + \bar{r}g \in U_n(A)$.

Now, we will show that $U_n(A) + A^n g$ is closed in $H_{n,g}(A)$. Let $(\bar{\alpha}_m)$ be a sequence in $U_n(A) + A^n g$ with $\lim \bar{\alpha}_m = \bar{\alpha} \in H_{n,g}(A)$; as $(\bar{\alpha}, g)$ belongs to $U_{n+1}(A)$ there exist $\bar{\lambda} \in A^n, \beta \in A$ such that $\langle \bar{\lambda}, \bar{\alpha} \rangle + \beta g = 1$. So, as A' is open, $\langle \bar{\lambda}, \bar{\alpha}_m \rangle + \beta g \in A'$ for $m \geq m_0$ or, by the remark just proved, $\bar{\lambda} \in U_n(A) + A^n g$. But, $\bar{\alpha} \in U_n(A) + A^n g$ by the fact that $\bar{\lambda} \in U_n(A) + A^n g$ and the first remark.

3.8. THEOREM. Given $\bar{b} \in A^n$, the following conditions are equivalent.

- (i) There exist $\bar{c} \in U_n(A)$ such that $\bar{b} \sim_g \bar{c}$.
- (ii) There exists $\bar{d} \in U_n(A)$ and a curve $\gamma: [0, 1] \rightarrow H_{n,g}(A)$ such that $\gamma(0) = \bar{b}$ and $\gamma(1) = \bar{d}$.

Proof. (i) \Rightarrow (ii). $\bar{b} \sim_g \bar{c} \in U_n(A)$, so there exist $\bar{h} \in A^n, \bar{u} \in (A')^n$ such that $\bar{b} + \bar{h}g = \bar{u}\bar{c}$. Defining $\gamma(t) = \bar{b} + t\bar{h}g$, it is clear that

$$\gamma(0) = \bar{b}, \quad \gamma(1) = \bar{u}\bar{c} \in U_n(A) \quad \text{and} \quad \gamma(t) \in H_{n,g}(A),$$

since

$$\gamma(t) = \bar{b} + t\bar{h}g + (1-t)\bar{h}g = \bar{u}\bar{c} + (1-t)\bar{h}g \in U_n(A) + A^n g$$

for all $t \in [0, 1]$.

(ii) \Rightarrow (i). Given $\gamma: [0, 1] \rightarrow H_{n,g}(A)$ joining \bar{b} and $\bar{d} \in U_n(A)$, it is clear that \bar{b} and \bar{d} belong to the same connected component D of $H_{n,g}(A)$. By (3.7), $(U_n(A) + A^n g) \cap D$ is open and closed in D , but

$$\bar{d} \in (U_n(A) + A^n g) \cap D$$

so $(U_n(A) + A^n g) \cap D = D$, and $\bar{b} \in U_n(A) + A^n g$ as claimed.

3.9. COROLLARY. A is n -stable at g if and only if each connected component of $H_{n,g}(A)$ meets $U_n(A)$.

Proof. Let D be a connected component of $H_{n,g}(A)$ and $\bar{b} \in D$; then by the above theorem there exists $\bar{c} \in U_n(A)$ such that $\bar{b} \sim_g \bar{c}$ if and only if there exists $\bar{d} \in U_n(A)$ with $\bar{d} \in D$.

3.10. COROLLARY. Let A be a Banach algebra. Given elements g_1, \dots, g_s in A , there exist positive integers e_1, \dots, e_s such that A is n -stable at $g_1^{e_1} \cdots g_s^{e_s}$ if and only if for all positive integers p_1, \dots, p_s , A is n -stable at $g_1^{p_1} \cdots g_s^{p_s}$.

Proof. It suffices to consider the case of a semisimple Banach algebra, for, it is clear that A is n -stable at g if and only if $A/\text{rad } A$ is n -stable at \tilde{g} , the class of g in $A/\text{rad } A$. Note that if

$$g = g_1^{e_1} \cdot g_2^{e_2} \cdots g_s^{e_s} \quad \text{and} \quad g' = g_1^{p_1} \cdot g_2^{p_2} \cdots g_s^{p_s}$$

then $H_{n,g}(A) = H_{n,g'}(A)$, since

$$H_{n,g}(A) = \{a \in A^n: |\hat{a}_1| + \cdots + |\hat{a}_n| + |\hat{g}| > 0 \text{ on } X(A)\}.$$

The rest is clear from the corollary above.

3.11. THEOREM. *Let X be a compact subset of \mathbb{C} . Then:*

- (1) $sr(\mathcal{P}(X)) = sr(\mathcal{R}(X)) = 1$.
- (2) *If $A = \mathcal{P}(X)$ or $\mathcal{R}(X)$ then $A' + Ag$ is dense in A for every $g \neq 0$. In particular (3.6) holds.*

Proof. Consider first the case $A = \mathcal{P}(X)$ and $g \neq 0$. There is no loss of generality if we consider X polynomially convex. We shall prove that

$$T = \{(z - z_0) \in A : z_0 \in (X^0 \setminus Z_g) \cup X^c\} \subset A' + Ag.$$

Since $S(T)$ is dense, this implies that A is 1-stable at g by (3.4.4). If $z_0 \in X^c$, then $z - z_0 \in A'$. If $z_0 \in X^0 \setminus Z_g$, take D to be the connected component of z_0 in X^0 . There exists $z_1 \in \partial D$ such that $g(z_1) \neq 0$ and a curve $\gamma: [0, 1] \rightarrow \mathbb{C} \setminus Z_g$ such that $\gamma(0) = z_0$, and $\gamma(1) = z_1$. Thus $z - z_0$ is connected through $H_{1,g}(A)$ to $z - z_1$, by $z - \gamma(t)$. In an analogous way $z - z_1$ is connected through $H_{1,g}(A)$ to $z - z_2$ for some $z_2 \notin X$. Observe that $z - z_2$ is invertible. So by Theorem 3.8, $z - z_0 \sim_g z - z_2 \sim_g 1$; in other words, $z - z_0 \in A' + Ag$. Since $S(T)$ is dense and $A' + Ag \supset S(T)$, we obtain (2). The case $A = \mathcal{R}(X)$ is similar, taking

$$T = \{(z - z_0) \in A : z_0 \in (X^0 \setminus Z_g) \cup X^c\} \cup \{1/(z - z_1) \in A : z_1 \notin X\}.$$

3.12. Remark. When $X = \bar{\Delta}$, this gives another proof of 1.2.

3.13. Some open problems. (1) Given a compact X of \mathbb{C} , let $A(X)$ be a subalgebra of $C(X)$ of functions which are holomorphic in X^0 . From (2.11) we know that $sr(A(X)) \leq 2$. It would be interesting to calculate this number exactly. We conjecture that $sr(A(X)) = 1$.

(2) Is it true that A n -stable at g implies A $(n + 1)$ -stable at g ?

(3) Let A_n be the polydisc algebra of continuous functions on $\bar{\Delta}_n = \bar{\Delta} \times \dots \times \bar{\Delta}$, holomorphic on Δ_n . Calculate $sr(A_n)$. Observe that, from 2.11, $sr(A_n) \leq n + 1$.

(4) Calculate $sr(H^\infty)$. We believe that $sr(H^\infty) = 1$.

(5) Does there exist an algebra A such that $d = \dim(X(A)) = \infty$ and $sr(A) < \infty$?

Added in proof. (1) P. Jones, D. Marshall and T. Wolff (Proc. Amer. Math. Soc., to appear) proved that the disc algebra has stable rank one. However, their proof is completely different from ours.

(2) In a paper to appear in *Topology and Its Applications*, we prove that our conjecture 3.13(1) is true, that 3.13(2) is false and that

$$\frac{n}{2} + 1 \leq sn(A_n) \leq n.$$

Problems (3), (4) and (5) remain open.

(3) We have learned from Rieffel's paper, page 306, that L. A. Rubel (Amer. Math. Monthly, 1978, pp. 505–506) has proved Theorem 1.

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