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AN APPROACH TO NUMERICAL RANGES WITHOUT BANACH ALGEBRA THEORY

BY

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1. Introduction

The theory of numerical ranges in unital Banach algebras has been extensively developed in recent years. Its most remarkable results can be found in [6] and [7]. Numerical range techniques have been successfully applied to nonassociative normed algebras (for example, see [5], [15], [11], [19], [20]).

Since the concept of numerical range of an element of an (even nonassociative) unital Banach algebra does not depend on the product of the algebra but only on the underlying Banach space and on the unit, one is tempted to consider numerical ranges in an arbitrary Banach space X in which a norm-one element u has been selected. (We shall say that the pair (X, u) or simply X is a *numerical range space*). As in unital Banach algebras we can consider the *state space*

$$D(X) = \{f \in X' : \|f\| = f(u) = 1\}$$

and the *numerical range* of an element x in X, namely

$$V(x) = \{f(x) : f \in D(X)\}.$$

This idea appears implicitly in the classical paper by Bohnenblust and Karlin [4] and has been shown to be useful in obtaining relevant results on numerical ranges in certain Banach algebras (for example, see [8]).

In this paper we begin a methodical consideration of numerical range spaces. A number of results on numerical ranges in unital Banach algebras can be not only extended to our general context but even improved in their original context. This is the case for example with our Corollary 2.9 which improves Theorem 2.4 of [17] and whose proof uses essentially numerical ranges in Banach spaces which need not be Banach algebras.

The leitmotiv in our work is the problem considered by R.R. Smith in [17] and [18] in the context of unital complex Banach algebras. This problem can be posed as follows. Given an element F in the second dual A'' of a unital

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complex Banach algebra A, is it possible to find a net $\{a_\lambda\}$ of elements in A such that

$$F = w^* - \lim\{a_\lambda\} \quad \text{and} \quad V(a_\lambda) \subset V(F)$$

for all λ? The answer to this problem is affirmative whenever $V(F)$ has nonempty interior [18, Theorem 3.1]. In Section 2 we extend this result to arbitrary real or complex numerical range spaces. Arguments different from those in [18] have to be used for this. (The proof in [18] uses the fact that the numerical radius is an equivalent norm and this may fail even for unital real Banach algebras.)

If A, F are as before and $V(F)$ is a single point the answer to the problem is clearly affirmative. But if $V(F)$ is a segment but not a single point the answer is in general negative (see [17, Example 2.3]). Under an additional condition on A Smith [17, Theorem 2.4] shows that the set of hermitian elements in A is w^*-dense in the set of hermitian elements in A'', giving a partial answer to the problem in the case that $V(F)$ has empty interior. In Section 3 we give a complete affirmative answer to the problem for complex numerical range spaces satisfying Smith's condition when $V(F)$ is a segment but not a point. The residual case in arbitrary numerical range spaces where $V(F)$ is a point is also considered. The main tool used in this section is a discussion in the general context of numerical range spaces of the well-known theorem by Moore and Sinclair (see [7, Theorem 31.1]) on the linear span of the state space. We show that the above mentioned condition of Smith holds if and only if the real linear span of the state space is w^*-closed.

Section 4 is devoted to several applications. First we give a dual characterization of unital noncommutative JB^*-algebras. They are the unital complete normed complex nonassociative algebras A satisfying $H(A') \cap iH(A') = \{0\}$ where $H(A')$ denotes the real linear span of the state space of A. As a second application we show that every complex-valued bounded affine function f on a compact convex set is a pointwise limit of a net of continuous affine functions whose values lie in the closure of the range of f. Recall that if A is a unital Banach algebra and F is an element in the second dual A'' of A, then

$$(*) \qquad V(F) = \overline{\{F(f) : f \in D(A)\}} \quad [6, \text{Theorem } 12.2].$$

Using Theorem 3.1 of [10], numerical range spaces (X, u) for which $(*)$ holds are characterized as those whose duality mapping have a "good" behaviour at u. The paper concludes with the consideration of these special numerical range spaces. We apply the results of Sections 2 and 3 to these spaces and extend to them the assertion of [18, Corollary 3.3]. For the proof of this result we show previously that the numerical index of a numerical range space agrees with the one of its bidual. This last fact seems to be new even for unital Banach algebras.

2. Numerical ranges with nonempty interior

A numerical range space will be a pair (X, u) where X is a Banach space over the scalar field $\mathbf{K}$ ($\mathbf{R}$ or $\mathbf{C}$) and u is a fixed element in the unit sphere $S(X)$ of X. We shall usually keep the element u in mind and simply say that X is a numerical range space. The state space $D(X)$ of X is then defined as the nonempty convex and w^*-compact subset of X' (the dual space of X) given by

$$D(X) = \{f \in S(X'): f(u) = 1\} \tag{2.1}$$

For x in X we define the numerical range $V(X, x)$ of x by

$$V(X, x) = \{f(x): f \in D(X)\} \tag{2.2}$$

If X is clear from the context we shall write $V(x)$ instead of $V(X, x)$. $V(x)$ is a nonempty compact convex subset of $\mathbf{K}$, and simple properties as

$$V(x + y) \subset V(x) + V(y) \quad (x, y \in X) \tag{2.3}$$

$$V(\alpha x + \beta u) = \alpha V(x) + \beta \quad (x \in X;\ \alpha, \beta \in \mathbf{K}) \tag{2.4}$$

are easily verified. If Y is a Banach space containing X as a subspace, then Y will be considered as a new numerical range space with the same distinguished element u. $D(X)$ is the set of restrictions to X of the elements in $D(Y)$ so

$$V(Y, x) = V(X, x) \quad \text{for all } x \text{ in } X. \tag{2.5}$$

X will be always considered as a closed subspace of its second dual X'' so we can take $Y = X''$. However another numerical range can be considered for elements F in X'', namely $V_l(F)$ defined by

$$V_l(F) = \{F(f): f \in D(X)\} \quad (F \in X'') \tag{2.6}$$

Every element in $D(X)$ can be considered as an element in $D(X'')$ so we have

$$V_l(F) \subset V(X'', F) \quad (F \in X'') \tag{2.7}$$

and from (2.5) we deduce

$$V(X, x) = V_l(x) = V(X'', x) \quad (x \in X) \tag{2.8}$$

Our first lemma is a general result on duality. To state it we need some additional notation. If M, N are vector spaces in duality we denote $\sigma(M, N)$ the weak topology on M and if S is a subset of M the polar S° of S in N is defined by

$$S^\circ = \{ y \in N: \operatorname{Re}\langle x, y \rangle \leq 1 \text{ for all } x \in S \}$$

The polar of a subset of N is defined analogously. It is known that the bipolar $(S^\circ)^\circ \equiv S^{\circ\circ}$ of S is the weakly closed convex cover of $S \cup \{0\}$. This fact will be used without comment in the sequel.

2.1 Lemma. *Let M, N be vector spaces over $\mathbf{K}$ in duality, S a convex $\sigma(M, N)$-compact subset of M, Ω a convex compact subset of $\mathbf{K}$ containing 0 as an interior point and τ any topology on M compatible with its linear structure and stronger than $\sigma(M, N)$. Then the τ-closure and the $\sigma(M, N)$-closure of* $\operatorname{co}(\Omega S)$ *agree* (co *is the convex cover*).

Proof. Let δ be a fixed positive real number. Since 0 is interior to Ω we can find $z_1, z_2, \dots, z_n \in \mathbf{K}$ such that the set

$$\Omega_\delta = \operatorname{co}\{ z_1, \dots, z_n \}$$

satisfies $\Omega \subset \Omega_\delta \subset (1 + \delta)\Omega$, hence

$$\operatorname{co}(\Omega S) \subset \operatorname{co}(\Omega_\delta S) \subset (1 + \delta)\operatorname{co}(\Omega S).$$

Since $\operatorname{co}(\Omega_\delta S) = \operatorname{co}(\bigcup_{j=1}^n z_j S)$, $\operatorname{co}(\Omega_\delta S)$ is $\sigma(M, N)$-compact so $\sigma(M, N)$-closed and τ-closed, and from the above inclusion we obtain

$$\sigma(M, N) - \operatorname{cl}(\operatorname{co}(\Omega S)) \subset \operatorname{co}(\Omega_\delta S) \subset (1 + \delta)(\tau\text{-cl}(\operatorname{co}(\Omega S))).$$

Now let $\delta \to 0$ to obtain

$$\sigma(M, N) - \operatorname{cl}(\operatorname{co}(\Omega S)) \subset \tau\text{-cl}(\operatorname{co}(\Omega S))$$

and the reverse inclusion follows from the assumption that τ is stronger than $\sigma(M, N)$.

2.2 Lemma. *Let X be a numerical range space and let Ω be a convex compact subset of $\mathbf{K}$ containing 0 as an interior point. Then in the canonical duality (X', X),*

$$\{ x \in X: V(x) \subset \Omega \}^\circ = \tau\text{-cl}(\operatorname{co}(\Omega^\circ D(X)))$$

where τ is any topology on X' compatible with its linear structure and stronger than the w^*-topology and

$$\Omega^\circ = \{ w \in \mathbf{K}: \operatorname{Re}(wz) \leq 1 \textit{ for all } z \in \Omega \}$$

Proof. Clearly $(\Omega^{\circ}D(X))^{\circ} = \{x \in X\colon V(x) \subset \Omega\}$ and 0 is an interior point in Ω°. Therefore

$$\{x \in X\colon V(x) \subset \Omega\}^{\circ} = (\Omega^{\circ}D(X))^{\circ\circ} = w^*\text{-cl}(\text{co}(\Omega^{\circ}D(X)))$$

and the result follows by Lemma 2.1.

2.3 THEOREM. *Let X be a numerical range space and Ω a convex closed subset of $\mathbf{K}$ with nonempty interior. Then*

$$w^*\text{-cl}\{x \in X\colon V(x) \subset \Omega\} = \{F \in X''\colon V_l(F) \subset \Omega\}$$

Proof. Since $\{F \in X''\colon V_l(F) \subset \Omega\}$ is w^*-closed in X'', by (2.8) we have

$$w^*\text{-cl}\{x \in X\colon V(x) \subset \Omega\} \subset \{F \in X''\colon V_l(F) \subset \Omega\}.$$

In view of (2.4) we can assume that Ω contains 0 as an interior point. First we assume also that Ω is compact. Then by Lemma 2.2 if τ denotes the norm topology on X' we have

$$\{x \in X\colon V(x) \subset \Omega\}^{\circ} = \tau\text{-cl}(\text{co}(\Omega^{\circ}D(X)))$$

so

$$\begin{aligned} w^*\text{-cl}\{x \in X\colon V(x) \subset \Omega\} &= [\tau\text{-cl}(\text{co}(\Omega^{\circ}D(X)))]^{\circ} \\ &= (\Omega^{\circ}D(X))^{\circ} \\ &= \{F \in X''\colon V_l(F) \subset \Omega\}. \end{aligned}$$

If Ω is not compact let $G \in X''$ be such that $V_l(G) \subset \Omega$ and let Δ be a closed disk contained in Ω with center at 0 and positive radius. Then

$$\Omega_1 = \text{co}\left(\Delta \cup \overline{V_l(G)}\right) \subset \Omega$$

is a compact convex set containing 0 as an interior point so by the first part of the proof

$$G \in w^*\text{-cl}\{x \in X\colon V(x) \subset \Omega_1\} \subset w^*\text{-cl}\{x \in X\colon V(x) \subset \Omega\}$$

2.4 COROLLARY. *Let X be a numerical range space and Ω a convex closed subset of $\mathbf{K}$ with nonempty interior. For all F in X'' with $V(F) \subset \Omega$ we can find a net $\{x_\lambda\}_{\lambda \in \Lambda}$ in X such that*

(i) *$V(x_\lambda) \subset \Omega$ for all $\lambda \in \Lambda$,*

(ii) *$\{x_\lambda\} \to F$ in the w^*-topology.*

Proof. By (2.7), $V_l(F) \subset \Omega$. Now apply the above theorem.

If in the above corollary $V(F)$ has nonempty interior then Ω may be chosen to be $V(F)$. For the case of empty interior Theorem 2.3 gives the following approximation.

2.5 COROLLARY. *Let X be a numerical range space, Ω a convex closed subset of $\mathbf{K}$ and let Δ denote the closed unit ball of $\mathbf{K}$. Then*

$$\{F \in X'': V_l(F) \subset \Omega\} = \bigcap_{\delta > 0} w^*\text{-cl}\{x \in X: V(x) \subset \Omega + \delta\Delta\}.$$

Proof. The condition $V_l(F) \subset \Omega$ is equivalent to $V_l(F) \subset \Omega + \delta\Delta$ for all $\delta > 0$ so the result follows from Theorem 2.3.

2.6 *Remark.* In the particular case of unital complex Banach algebras Corollary 2.4 has been obtained by R. R. Smith with different techniques [18, Theorem 3.1]. See also [18, Remark 3.2] in connection with our Corollary 2.5.

3. The empty interior case

As stated in [18, Example 2.3], Theorem 2.3 may fail if Ω has empty interior. In this section we will consider natural conditions on X which imply the conclusion of Theorem 2.3 when Ω has empty interior.

Let (X, u) be a real numerical range space and suppose that $V(x)$ has empty interior, that is $V(x) = \{\lambda\}$. Then by (2.4), $V(x - \lambda u) = \{0\}$. We define the *radical* of a (real or complex) numerical range space X by

$$\text{Rad}(X) = \{x \in X: V(x) = \{0\}\}. \tag{3.1}$$

The radical is clearly a closed subspace of X. Now consider a complex numerical range space (X, u) and let $x \in X$ be such that $V(x)$ has empty interior. Then $V(x)$ is a segment or possibly a single point. Using (2.4) again we can write $x = \alpha y + \beta u$ with $\alpha, \beta \in \mathbf{C}$ and $V(y) \subset \mathbf{R}$. We define the set of *hermitian* elements in X, $H(X)$ by

$$H(X) = \{x \in X: V(x) \subset \mathbf{R}\} \tag{3.2}$$

It is clear that (for complex X),

$$\text{Rad}(X) = H(X) \cap iH(X) \tag{3.3}$$

If (X, u) is a real or complex numerical range space it is clear that

$$d(u, \text{Rad}(X)) = 1$$

so $(X/\mathrm{Rad}(X), u + \mathrm{Rad}(X))$ is a new numerical range space. We have clearly $D(X) \subset \mathrm{Rad}(X)^\circ$ so with the usual identification between $(X/\mathrm{Rad}(X))'$ and $\mathrm{Rad}(X)^\circ$ in mind we can write

$$D(X/\mathrm{Rad}(X)) = D(X) \tag{3.4}$$

and from this we obtain

$$V(x + \mathrm{Rad}(X)) = V(x) \quad (x \in X) \tag{3.5}$$

and

$$\mathrm{Rad}(X/\mathrm{Rad}(X)) = \{0\}. \tag{3.6}$$

For x in X we define the *numerical radius* $v(x)$ of x by

$$v(x) = \mathrm{Max}\{|\lambda|: \lambda \in V(x)\}. \tag{3.7}$$

It is clear from (2.3) and (2.4) that $v(\cdot)$ is a seminorm on X whose annihilator subspace is $\mathrm{Rad}(X)$. We define the *numerical index* $n(X)$ of X by

$$n(X) = \mathrm{Inf}\{v(x): x \in S(X)\} \tag{3.8}$$

or equivalently

$$n(X) = \mathrm{Max}\{\alpha \geq 0: \alpha\|x\| \leq v(x) \text{ for all } x \in X\}. \tag{3.9}$$

Since $v(x) \leq \|x\|$ for all x in X we have that $0 \leq n(X) \leq 1$ and $n(X) > 0$ if and only if $v(\cdot)$ is an equivalent norm on X. The assertion $n(X) > 0$ for a numerical range space X must be understood as a desirable property of X. For example if A is a unital complex Banach algebra it is known that $n(A) \geq 1/e$ (see [6, Theorem 4.1]). This fact is essential in the proof of some known results on numerical ranges in complex Banach algebras. One of these results is the Moore-Sinclair theorem (see [7, Theorem 31.1]) asserting that if A is a complex Banach algebra, then A' is the linear span of $D(A)$. The next theorem will allow us to discuss the Moore-Sinclair theorem for arbitrary numerical range spaces. As a particular case we obtain an extension of the Moore-Sinclair theorem to nonassociative algebras.

3.1 THEOREM. *Let X be a numerical range space. Then*

$$\{f \in X': |f(x)| \leq v(x) \text{ for all } x \in X\} = \tau\text{-cl}(|\mathrm{co}|D(X))$$

($|\mathrm{co}|$ is the absolutely convex hull) where τ is any topology on X' compatible with its linear structure and stronger than the w^*-topology.

Proof. If Δ denotes the closed unit ball of $\mathbf{K}$ we have clearly that in the duality (X', X),

$$\{x \in X \colon V(x) \subset \Delta\}^\circ = \{f \in X' \colon |f(x)| \leq v(x) \text{ for all } x \in X\}$$

Now apply Lemma 2.2 and note that $\operatorname{co}(\Delta D(X)) = |\operatorname{co}| D(X)$.

In what follows $\operatorname{Lin} D(X)$ will denote the linear span of $D(X)$.

3.2 THEOREM. *Let X be a numerical range space. The following statements are equivalent*:

(i) $n(X) > 0$,

(ii) $\operatorname{Lin} D(X) = X'$.

Moreover if (i) *or* (ii) *hold, then*:

(a) *If X is real, for all f in X' there are $\alpha_1, \alpha_2 \geq 0$ and $f_1, f_2 \in D(X)$ such that*

$$f = \alpha_1 f_1 - \alpha_2 f_2 \quad \textit{and} \quad \alpha_1 + \alpha_2 \leq \frac{\|f\|}{n(X)}.$$

(b) *If X is complex, for all f in X' there are $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \geq 0$ and $f_1, f_2, f_3, f_4 \in D(X)$ such that*

$$f = \alpha_1 f_1 - \alpha_2 f_2 + i(\alpha_3 f_3 - \alpha_4 f_4)$$

and

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leq \frac{\sqrt{2}\,\|f\|}{n(X)}.$$

Proof. First we show that (i) implies (a) or (b) so in particular (i) $\Rightarrow$ (ii). In view of (i), v and $\|\cdot\|$ are equivalent norms on X so they give the same continuous linear functionals and the dual norms v' of v and $\|\cdot\|$ of $\|\cdot\|$ are equivalent norms on X'. In fact we have

$$n(X) v'(f) \leq \|f\| \leq v'(f) \quad (f \in X').$$

Fix f in X' $(f \neq 0)$. Then by Theorem 3.1 we have

$$\frac{f}{v'(f)} \in w^*\text{-cl}(|\operatorname{co}| D(X)).$$

Now if X is real $|\operatorname{co}| D(X) = \operatorname{co}(D(X) \cup -D(X))$ is w^*-closed so we can write

$$\frac{f}{v'(f)} = \beta_1 f_1 - \beta_2 f_2$$

with $\beta_1, \beta_2 \geq 0$, $\beta_1 + \beta_2 = 1$, $f_1, f_2 \in D(X)$, and (a) follows with

$$\alpha_1 = \beta_1 v'(f) \quad \text{and} \quad \alpha_2 = \beta_2 v'(f)$$

If X is complex write $\Omega_0 = \sqrt{2}\,\mathrm{co}\{1, -1, i, -i\}$ and note that $\mathrm{co}(\Omega_0 D(X))$ is w^*-closed and contains $|\mathrm{co}|D(X)$. We can write

$$\frac{f}{v'(f)} = \sqrt{2}\,(\beta_1 f_1 - \beta_2 f_2 + i\beta_3 f_3 - i\beta_4 f_4)$$

with $\beta_1, \beta_2, \beta_3, \beta_4 \geq 0$, $\beta_1 + \beta_2 + \beta_3 + \beta_4 = 1$ and $f_1, f_2, f_3, f_4 \in D(X)$. Now (b) follows with $\alpha_i = \beta_i \sqrt{2}\, v'(f)$ $(i = 1, 2, 3, 4)$.

It remains to prove that (ii) $\Rightarrow$ (i). From (ii) we deduce that v is a norm on X and that v and $\|\cdot\|$ give the same continuous linear functionals. The identity mapping from X' with the dual norm of $\|\cdot\|$ onto X' with the norm v' has closed graph so $\|\cdot\|$ and v' are equivalent norms on X'. Hence v and $\|\cdot\|$ are equivalent norms on X; that is, (i) holds.

3.3 *Remark.* An alternative proof of (i) $\Rightarrow$ (ii) can be obtained from [2, Theorem 1].

Let A be a unital complete normed complex nonassociative algebra, denote by $\mathrm{BL}(A)$ the unital Banach algebra of bounded linear operators on A and for a in A define $L_a(b) = ab$ $(b \in A)$. Then $a \to L_a$ is a unit-preserving isometric linear mapping from A into $\mathrm{BL}(A)$ so

$$n(A) \geq n(\mathrm{BL}(A)) \geq 1/e.$$

3.4 COROLLARY. *Let A be a unital complete normed complex nonassociative algebra. Then* $\mathrm{Lin}\, D(A) = A'$ *and every element in A' can be written as in Theorem* 3.2(b).

3.5 COROLLARY. *Let X be a numerical range space. The following statements are equivalent.*

(i) $\mathrm{Lin}\, D(X)$ *is norm-closed in* X'.
(ii) $\mathrm{Lin}\, D(X)$ *is* w^**-closed.*
(iii) $\mathrm{Lin}\, D(X) = \mathrm{Rad}(X)^\circ$ *in the duality* (X', X).
(iv) $n(X/\mathrm{Rad}(X)) > 0$.

Proof. (i) $\Leftrightarrow$ (ii). Follows from [9, Corollary V.9.5].

(ii) $\Leftrightarrow$ (iii). Since $\mathrm{Lin}\, D(X)^\circ = \mathrm{Rad}(X)$ the bipolar theorem gives

$$w^*\text{-cl}(\mathrm{Lin}\, D(X)) = \mathrm{Rad}(X)^\circ.$$

(iii) $\Leftrightarrow$ (iv). Taking into account (3.4) and the usual identification $(X/\mathrm{Rad}(X))' \equiv \mathrm{Rad}(X)^{\circ}$, (iii) can be reformulated as

$$\mathrm{Lin}\, D(X/\mathrm{Rad}(X)) = (X/\mathrm{Rad}(X))'$$

So it is enough to apply Theorem 3.2 to the numerical range space $X/\mathrm{Rad}(X)$.

The above corollary will be specially relevant when applied to the real space underlying a complex numerical range space. Let (X, u) be a complex numerical range space and denote by X_r the real Banach space underlying X. For the numerical range space (X_r, u) clearly we have

$$D(X_r) = \{\mathrm{Re} f\colon f \in D(X)\} \tag{3.10}$$

so

$$V(X_r, x) = \mathrm{Re}\, V(X, x) \quad (x \in X) \tag{3.11}$$

and

$$\mathrm{Rad}(X_r) = iH(X) \tag{3.12}$$

In particular we have $iu \in \mathrm{Rad}(X_r)$ and $n(X_r) = 0$. We define the *real numerical index* $n_r(X)$ of X by

$$n_r(X) = n(X_r/\mathrm{Rad}(X_r)). \tag{3.13}$$

Note that in view of (3.6) we can have $n_r(X) > 0$ and this assertion becomes an additional desirable property of the complex numerical range space X. The reader can easily deduce from (3.11), (3.12) and (3.5) applied to X_r that the assertion $n_r(X) > 0$ is equivalent to the existence of a positive real number α such that for all x in X

$$d(x, H(X)) \leq \alpha \,\mathrm{Max}\{|\mathrm{Im}\,\lambda| : \lambda \in V(X, x)\} \tag{3.14}$$

In fact the least constant α which can appear in (3.14) is $1/n_r(X)$. It is shown in [17] that if A is a unital complex Banach algebra with $n_r(A) > 0$, then $H(A'') = w^*\text{-cl}(H(A))$.

3.6 THEOREM. *Let X be a complex numerical range space and let* $\mathrm{Lin}_r D(X)$ *denote the real linear span of $D(X)$. The following statements are equivalent.*

(i) $\mathrm{Lin}_r D(X)$ *is norm-closed in X'.*
(ii) $\mathrm{Lin}_r D(X)$ *is w^*-closed in X'.*
(iii) $\mathrm{Lin}_r D(X) = \{f \in X'\colon f(H(X)) \subset \mathbf{R}\}$.
(iv) $n_r(X) > 0$.

Proof. The mapping $f \to \operatorname{Re} f$ from $(X')_r$ onto $(X_r)'$ is a linear homeomorphism for both the norm and w^*-topologies and by (3.10) it maps $D(X)$ onto $D(X_r)$. With this in mind the proof reduces to applying Corollary 3.5 to the real numerical range space X_r.

3.7 *Remark.* Even in the particular case of unital complex Banach algebras the above theorem improves the results in [6, Theorem 31.12]. The improvement comes from the fact that the assertion $n_r(X) > 0$ is equivalent to the others. Note that if A is a unital complex Banach algebra, $A_r/\operatorname{Rad}(A_r)$ needs not be a real Banach algebra but it is a real numerical range space. It was only the unnecessary restriction of numerical ranges to the context of Banach algebras which prevented the consideration of the numerical index of $A_r/\operatorname{Rad}(A_r)$ namely $n_r(A)$ and the application of the Moore-Sinclair theorem to $A_r/\operatorname{Rad}(A_r)$ as we have done. The above theorem will play a central role in the proof of the main result in this section.

3.8 LEMMA. *Let X be a complex numerical range space.*
(a) *In the duality (X', X),*

$$\left[\operatorname{co}(D(X) \cup -D(X)) + i \operatorname{Lin}_r D(X)\right]^\circ = \{x \in X: V(x) \subset [-1,1]\}$$

(b) *In the duality (X'', X'),*

$$\left[\operatorname{co}(D(X) \cup -D(X)) + i \operatorname{Lin}_r D(X)\right]^\circ = \left\{F \in X'': V_l(F) \subset [-1,1]\right\}$$

Proof. Straightforward.

3.9 THEOREM. *Let X be a complex numerical range space such that $n_r(X) > 0$. Let Ω be a closed convex subset of $\mathbf{K}$ and suppose that Ω is not reduced to a point. Then*

$$w^*\text{-cl}\{x \in X: V(x) \subset \Omega\} = \left\{F \in X'': V_l(F) \subset \Omega\right\}.$$

In particular $H(X'') \subset w^*\text{-cl}(H(X))$.

Proof. If Ω has nonempty interior the result follows from Theorem 2.3 and the assumption $n_r(X) > 0$ is superfluous in this case.

Since $\{F \in X'': V_l(F) \subset \Omega\}$ is w^*-closed in X'' we have

$$w^*\text{-cl}\{x \in X: V(x) \subset \Omega\} \subset \left\{F \in X'': V_l(F) \subset \Omega\right\}.$$

We turn to the reverse inclusion and we can assume that Ω has empty interior so that Ω is contained in a straight line. Since Ω is not a point we can find $\alpha, \beta \in \mathbf{C}, \alpha \neq \beta$, such that the segment $[\alpha, \beta]$ is contained in Ω. Let $G \in X''$

be such that $V_l(G) \subset \Omega$. There are $\gamma, \lambda \in \mathbf{C}$, $\gamma \neq \lambda$, such that

$$\mathrm{co}\left(\overline{V_l(G)} \cup [\alpha, \beta]\right) = [\gamma, \lambda] \subset \Omega.$$

If we prove that

$$(3.15) \quad w^*\text{-cl}\{x \in X \colon V(x) \subset [\gamma, \lambda]\} = \left\{F \in X'' \colon V_l(F) \subset [\gamma, \lambda]\right\}$$

we will have $G \in w^*\text{-cl}\{x \in X \colon V(x) \subset \Omega\}$ and the proof will be finished.

To prove (3.15) note that in view of (2.4) we can suppose $\gamma = -1$, $\lambda = 1$. From the assumption $n_r(X) > 0$ we have by Theorem 3.6 that $\mathrm{Lin}_r D(X)$ is w^*-closed hence so is $\mathrm{co}(D(X) \cup -D(X)) + i\,\mathrm{Lin}_r D(X)$. Now by Lemma 3.8(a) we have that in the duality (X', X),

$$\{x \in X \colon V(x) \subset [-1, 1]\}^\circ = \mathrm{co}(D(X) \cup -D(X)) + i\,\mathrm{Lin}_r D(X)$$

and by Lemma 3.8(b), in the duality (X'', X'),

$$\begin{aligned} w^*\text{-cl}\{x \in X \colon V(x) \subset [-1, 1]\} &= \{x \in X \colon V(x) \subset [-1, 1]\}^{\circ\circ} \\ &= \left\{F \in X'' \colon V_l(F) \subset [-1, 1]\right\}. \end{aligned}$$

The following corollary improves Theorem 2.4 of [17].

3.10 COROLLARY. *Let A be a unital complex Banach algebra with $n_r(A) > 0$. For all F in A'' there is a net $\{a_\lambda\}_{\lambda \in \Lambda}$ of elements in A satisfying:*
(i) *$V(a_\lambda) \subset V(F)$ for all λ in Λ;*
(ii) *$\{a_\lambda\}_{\lambda \in \Lambda}$ converges to F in the w^*-topology of A''.*

3.11 *Remark.* The assumption $n_r(A) > 0$ can not be dropped in the above corollary as shown in [17, Example 2.3]. Note also that if $V(F)$ is a single point, then F is a scalar multiple of the unit and the result is obvious in this case. For a general numerical range space the case $V(F) = \{\alpha\}$ also follows easily from our results.

3.12 THEOREM. *Let X be a numerical range space with $n(X/\mathrm{Rad}(X)) > 0$. For all $\alpha \in \mathbf{K}$,*

$$w^*\text{-cl}\{x \in X \colon V(x) = \{\alpha\}\} = \left\{F \in X'' \colon V_l(F) = \{\alpha\}\right\}.$$

In particular $\mathrm{Rad}(X'') \subset w^*\text{-cl}(\mathrm{Rad}(X))$.

Proof. By (2.4) we can suppose $\alpha = 0$. Then by Corollary 3.5 we have that in the duality (X'', X'),

$$w^*\text{-cl}(\mathrm{Rad}(X)) = \mathrm{Rad}(X)^{\circ\circ} = \mathrm{Lin}\,D(X)^\circ = \left\{F \in X'' \colon V_l(F) = \{0\}\right\}.$$

Theorems 2.3, 3.9 and 3.12 can be summarized as follows.

3.13 COROLLARY. *Let X be a numerical range space such that*

$$n(X/\mathrm{Rad}(X)) > 0.$$

If X is complex assume further that $n_r(X) > 0$. Let Ω be a closed convex subset of $\mathbf{K}$. Then

$$w^*\text{-cl}\{x \in X\colon V(x) \subset \Omega\} = \{F \in X''\colon V_l(F) \subset \Omega\}.$$

In particular, for all F in X'' there is a net $\{x_\lambda\}_{\lambda \in \Lambda}$ of elements in X such that:
(i) *$V(x_\lambda) \subset V(F)$ for all λ in Λ;*
(ii) *$\{x_\lambda\}_{\lambda \in \Lambda}$ converges to F in the w^*-topology of X''.*

4. Applications

(A) *Dual characterization of noncommutative JB^*-algebras.*

The following geometric characterizations of unital C^*-algebras are well known. Let A be a unital complex Banach algebra. The Vidav-Palmer theorem (see [6, Chapter 2]) states that A is a C^*-algebra if and only if

$$A = H(A) + iH(A). \tag{4.1}$$

Moore's theorem [13] uses this result to show that A is a C^*-algebra if and only if

$$\mathrm{Lin}_r D(A) \cap i\,\mathrm{Lin}_r D(A) = \{0\}. \tag{4.2}$$

Next we show that the statements (4.1) and (4.2) for a complex numerical range space with positive numerical index are equivalent. As a consequence of this and the main result in [16], it is shown that the only unital complete normed complex nonassociative algebras satisfying (4.2) are the unital noncommutative JB^*-algebras.

4.1 THEOREM. *Let X be a complex numerical range space such that $n(X) > 0$. The following statements are equivalent.*
(i) $\mathrm{Lin}_r D(X) \cap i\,\mathrm{Lin}_r D(X) = \{0\}$.
(ii) $X = H(X) + iH(X)$.

Proof. (i) $\Rightarrow$ (ii). From the assumption $n(X) > 0$ it is easy to obtain that $H(X) + iH(X)$ is a closed subspace of X. In view of Theorem 3.2 we have

$$X' = \mathrm{Lin}_r D(X) + i\,\mathrm{Lin}_r D(X),$$

and by (i) the sum is algebraic-direct. But from Theorem 3.2(b) the real linear projection P onto $i\,\mathrm{Lin}_r D(X)$ is norm-continuous,

$$\|P(f)\| \leq \frac{\sqrt{2}}{n(X)}\|f\| \quad \text{for all } f \text{ in } X',$$

so $\mathrm{Lin}_r D(X)$ is norm-closed. Now we can use Theorem 3.6 ((i) $\Leftrightarrow$ (iii)) to show that if $f \in X'$ and $f(H(X) + iH(X)) = \{0\}$ then f and if belong to $\mathrm{Lin}_r D(X)$ so $f = 0$ by (i).

(ii) $\Rightarrow$ (i). If $f \in \mathrm{Lin}_r D(X) \cap i\,\mathrm{Lin}_r D(X)$ we have $f(H(X)) = \{0\}$ so $f = 0$ by (ii).

4.2 *Remark.* An alternative proof of the above result can be obtained from Theorem 4.0.2 of [1].

We recall (see [14]) that a noncommutative JB^*-algebra is a complete normed complex noncommutative Jordan algebra A on which an algebra involution $*$ is defined satisfying

$$\|U_a(a^*)\| = \|a\|^3 \quad \text{for all } a \text{ in } A,$$

where U_a is the bounded linear operator on A defined by

$$U_a(b) = a(ba) + (ba)a - ba^2 \quad (b \in A)$$

4.3 Corollary. *Let A be a unital complete normed complex nonassociative algebra. The following statements are equivalent.*

(i) $\mathrm{Lin}_r D(A) \cap i\,\mathrm{Lin}_r D(A) = \{0\}$.

(ii) $A = H(A) + iH(A)$.

(iii) *A is a noncommutative JB^*-algebra.*

Proof. As noticed in the comments preceding Corollary 3.4 we have $n(A) > 0$ so the equivalence between (i) and (ii) follows from Theorem 4.1.

By the main result in [16] if A satisfies (ii), then the mapping

$$h + ik \xrightarrow{*} h - ik \quad (h, k \in H(A))$$

is an algebra involution on A. Then by [12, Theorem 8], A is a noncommutative Jordan algebra. Now the equivalence between (ii) and (iii) follows from the nonassociative Vidav-Palmer theorem [12, Theorem 11].

(B) *Affine functions on compact convex sets.*

Let K be a compact convex subset of a Haussdorf locally convex space. Denote by $C(K)$ the Banach space of complex-valued continuous functions on

K with the uniform norm and by $A(K)$ the closed subspace of $C(K)$ whose elements are affine functions. Let us consider the complex numerical range space $(A(K), I)$ where $I(k) = 1$ for all k in K. Then by (2.5) and Vidav's Lemma [6, Theorem 5.14] we have

$$(4.3) \qquad V(A(K), f) = V(C(K), f) = f(K) \quad (f \in A(K)).$$

As a consequence we have $v(f) = \|f\|$ $(f \in A(K))$; that is, $n(A(K)) = 1$. Also from (4.3) we deduce that hermitian elements in $A(K)$ are the real-valued continuous affine functions on K. From this it is not difficult to obtain that $n_r(A(K)) = 1$.

It is known that the bidual space $A(K)''$ can be identified with the Banach space $A^b(K)$ of complex-valued bounded affine functions on K. In this identification the w^*-topology of $A(K)''$ is translated into the topology of pointwise convergence of functions in $A^b(K)$.

From (4.3) and the Hahn-Banach Theorem on separation of convex sets (see for example [3, Theorem 34.7 and Exercise 34.12]), $D(A(K))$ is the set of evaluation functionals on the points of K. With this in mind it is easy to see that if $F \in A^b(K) = A(K)''$ then $V_l(F) = F(K)$.

The above considerations allow us to state the following Corollary which improves Theorem 2.2 of [18]. To prove it one must simply apply Corollary 3.13 to the complex numerical range space $(A(K), I)$.

4.4 COROLLARY. *Let K be a compact convex subset of a Haussdorf locally convex space and let Ω be a closed convex subset of* **C**. *Then*

$$\{F \in A^b(K) \colon F(K) \subset \Omega\} = \tau\text{-cl}\{f \in A(K) \colon f(K) \subset \Omega\}$$

where τ is the topology of pointwise convergence on K. In particular if F is a complex-valued bounded affine function on K there is a net $\{f_\lambda\}$ of complex-valued continuous affine functions on K such that $f_\lambda(K) \subset \overline{F(K)}$ and $\{f_\lambda(k)\} \to F(k)$ for all k in K.

(C) *Numerical indexes of bidual spaces.*

Following [10], if X is a real or complex Banach space we define the *duality mapping* for X to be the set-valued mapping on $S(X)$ given by

$$x \to D(X, x) = \{f \in S(X') \colon f(x) = 1\}.$$

Denote by n (resp. w) the norm (resp. weak) topology on any Banach space. The duality mapping of X will be said to be (n, n)-*upper semicontinuous* (resp. (n, w)-*upper semicontinuous*) at a point x in $S(X)$ if for every n-neighbourhood (resp. w-neighbourhood) of zero U in X' there is an $\epsilon > 0$ such that if $y \in S(X)$ and $\|x - y\| < \epsilon$, then

$$D(X, y) \subset D(X, x) + U.$$

We are interested in numerical range spaces (X, u) whose duality mapping is (n, w)-upper semicontinuous at u. The following result provides a wide class of such spaces.

4.5 PROPOSITION. *Let A be a unital complete normed nonassociative real or complex algebra and let I be the unit of A. The duality mapping of the numerical range space (A, I) is (n, n) (so also (n, w)) upper semicontinuous at I.*

Proof. For a in $S(A)$ and f in $D(A, a)$ define f_a: $A \to \mathbf{K}$ by $f_a(b) = f(ab)$ $(b \in A)$. Clearly we have $f_a \in D(A, I)$ and $\|f_a - f\| \leq \|a - I\|$. So if $\varepsilon > 0$ and B denotes the open ball in X' with center at zero and radius ε we have

$$D(A, a) \subset D(A, I) + B \quad \text{for all } a \text{ in } S(A) \text{ with } \|a - I\| < \varepsilon.$$

The following result characterizes numerical range spaces whose duality maping is (n, w)-upper semicontinuous.

4.6 THEOREM. [10, Theorem 3.1]. *Let (X, u) be a real or complex numerical range space. The following statements are equivalent.*

(i) *The duality mapping of X is (n, w)-upper semicontinuous at u.*

(ii) $D(X'', u) = w^*\text{-cl}(D(X, u))$.

(iii) $V(F) = \overline{V_l(F)}$ *for all F in X''.*

Proof. The equivalence between (i) and (ii) appears in [10, Theorem 3.1] for real X. The extension to the complex case is a consequence of the fact that the mapping $f \to \operatorname{Re} f$ is a linear homeomorphism for both the norm and weak topologies from $(X')_r$ onto $(X_r)'$ which maps $D(X, x)$ onto $D(X_r, x)$ for all x in $S(X)$.

The equivalence between (ii) and (iii) for real or complex X is a straightforward consequence of the Hahn-Banach Theorem on separation of convex sets (for example, see [3, Theorem 34.7 and Exercise 34.12]).

4.7 *Remark.* In view of the above theorem, if the duality mapping of the numerical range space (X, u) is (n, w)-upper semicontinuous we can change $V_l(F)$ to $V(F)$ in Theorems 2.3, 3.9 and 3.12. In particular we would have

$$w^*\text{-cl}(H(X)) = H(X'')$$

in Theorem 3.9 and

$$w^*\text{-cl}(\operatorname{Rad}(X)) = \operatorname{Rad}(X'')$$

in Theorem 3.12. This applies in particular to unital complete normed nonassociative algebras in view of Proposition 4.5.

The next lemma seems to be new even for unital Banach algebras. It will be used to improve Corollary 3.3 of [18].

4.8 LEMMA. *Let X be a numerical range space. Then $n(X'') = n(X)$.*

Proof. Since X is a closed subspace of X'', clearly we have $n(X) \geq n(X'')$. For the converse inequality we can suppose $n(X) > 0$. From $n(X)\|x\| \leq v(x)$ for all x in X we deduce $n(X)\|F\| \leq v''(F)$ for all F in X'' where v'' denotes the bidual norm of v. But in view of Theorem 3.1 we have, for all F in X'',

$$\begin{aligned} v''(F) &= \operatorname{Sup}\{|F(f)|: v'(f) \leq 1\} \\ &= \operatorname{Sup}\{|F(f)|: f \in D(X)\} \\ &= \operatorname{Sup}\{|\lambda|: \lambda \in V_l(F)\} \\ &\leq \operatorname{Sup}\{|\lambda|: \lambda \in V(F)\} \\ &= v(F). \end{aligned}$$

4.9 THEOREM. *Let (X, u) be a complex numerical range space. Suppose that the duality mapping of X is (n, w)-upper semicontinuous at u and that $n_r(X) > 0$. Then $n_r(X) = n_r(X'')$.*

Proof. We have $\operatorname{Rad}(X_r'') = w^*\text{-cl}(\operatorname{Rad}(X_r)) = \operatorname{Rad}(X_r)^{\circ\circ}$ (see Remark 4.7). Then

$$X_r''/\operatorname{Rad}(X_r'') = X_r''/\operatorname{Rad}(X_r)^{\circ\circ} \cong (X_r/\operatorname{Rad}(X_r))''$$

Now apply Lemma 4.8 to $X_r/\operatorname{Rad}(X_r)$.

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