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BIVARIATE CARDINAL INTERPOLATION BY SPLINES ON A THREE-DIRECTION MESH

BY

CARL DE BOOR[1], KLAUS HÖLLIG[1,2] AND SHERMAN RIEMENSCHNEIDER[3]

**Dedicated to I.J. Schoenberg
to whose insight and sense of beauty we are all indebted**

1. Introduction

In this paper, we carry Schoenberg's beautiful cardinal spline theory [S_2], [S_3] over to a two-dimensional context which is not just the tensor product of the univariate situation. We find that we must work harder, yet must be satisfied with less precise results.

We are after a bounded cardinal interpolant to bounded data. This means that we are looking for a function of the form

$$If = \sum_{j \in \mathbf{Z}^2} a_j M(\cdot - j)$$

with $a \in l_\infty(\mathbf{Z}^2)$ which agrees with a given bounded function f on $\mathbf{Z}^2$. Here, M is a fixed function of compact support. In Section 2, we follow Schoenberg [S_1] in describing necessary and sufficient conditions on the Fourier transform of M to insure the correctness of the interpolation problem, i.e., the existence and uniqueness of solutions.

We are particularly interested in using for M a box spline, i.e., the two-dimensional "shadow" of an m-dimensional cube, as given explicitly in (1) below. Let Z be a set of vectors in $\mathbf{R}^2$. We find it convenient to change the definition [BH_1]

$$M\phi := \int_{[0,1]^Z} \phi\left(\sum_{\zeta \in Z} \lambda(\zeta)\zeta \right) d\lambda$$

of the box spline $M = M_Z$ to include an appropriate shift which makes the

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origin the center of the support of M. This means that we use the definition

$$M\phi := \int_{[-1/2,1/2]^Z} \phi\Big(\sum_{\zeta\in Z} \lambda(\zeta)\zeta\Big)\, d\lambda. \tag{1.1}$$

This gives the Fourier transform $M^\wedge$ of M the symmetric form

$$M^\wedge(x) = \prod_{\zeta\in Z} S(\zeta^* x) \tag{1.2}$$

with

$$S(t) := \frac{\sin t/2}{t/2}. \tag{1.3}$$

It is obvious from this formula that $M = M_Z$ is unchanged if one or more of the $\zeta \in Z$ are replaced by their negative; i.e.,

$$M_{AZ} = M_Z \tag{1.4}$$

if $A = \mathrm{diag}(\pm 1, \dots, \pm 1)$. Further, if A is any matrix, then

$$M_{AZ}^\wedge(x) = M_Z^\wedge(A^* x) \quad \text{and} \quad M_{AZ}(Ax) = M_Z(x)/\det A. \tag{1.5}$$

This allows one to deduce symmetries in $M^\wedge$ in case AZ equals Z after, possibly, some elements of AZ have been multiplied by -1.

The set Z of directions can, of course, be chosen arbitrarily. But since we are interested in having

$$\mathbf{S} := \mathrm{span}(M(\cdot\, - j))_{j\in\mathbf{Z}^2}$$

be a simple piecewise polynomial space, we choose Z from $\mathbf{Z}^2$. It is shown in [BH_1] that the integer translates $M(\cdot\, - j)$, $j \in \mathbf{Z}^2$, of the box spline are linearly dependent (when allowing for infinite linear combinations) in case the direction set Z contains two vectors which span a proper sublattice of $\mathbf{Z}^2$. Linear independence is an obvious necessary condition for the cardinal interpolation problem to be correct. Thus, up to obvious symmetries, this leaves the three vectors $(1,0)$, $(0,1)$ and $(1,1)$ as the only candidates for the directions ζ in Z.

With this restriction, $\mathbf{S}$ is a space of piecewise polynomial functions, of polynomial degree $|Z| - 2$ or less, and with possible discontinuities only across the three types

$$x(1) = k, \qquad x(2) = k, \qquad x(1) - x(2) = k, \quad k \in \mathbf{Z}$$

of mesh lines. The overall smoothness of the elements of $\mathbf{S}$ depends on the multiplicities of the directions in Z. Such details, as well as the relationship of $\mathbf{S}$ to the space of all piecewise polynomial functions on such a three-direction mesh, of degree $|Z| - 2$ and of specified smoothness, are all discussed in [BH_2].

In Section 3, we supply certain details concerning symmetries of such a three-direction box spline and its Fourier transform. We prove the correctness of cardinal interpolation with such a box spline in Section 4. We spend the major effort of this paper in Section 5 where we prove that, under reasonable conditions, the cardinal interpolant If of any suitably smooth function f converges to f as $|Z| \to \infty$. Specifically, we prove such convergence under the condition that f is the Fourier transform of some compactly supported measure, following entirely the path established by Schoenberg [S] who showed in the univariate case that such convergence could be had whenever $\operatorname{supp} f \subseteq (-\pi, \pi)$. We find, though, that, in our bivariate setup, there are many different sets playing the role of this interval, and which of these sets is relevant depends on the manner in which $|Z|$ goes to infinity.

The final section is devoted to the many detailed estimates on which the arguments in Section 5 are based.

2. Cardinal interpolation

Let M: $\mathbf{R}^2 \to \mathbf{R}$ be a continuous function with compact support, and denote by

$$\mathbf{S} := \mathbf{S}_M := \operatorname{span}\{ M(\cdot\ - j) \colon j \in \mathbf{Z}^2 \}$$

the space generated by its integer translates. Cardinal interpolation with M concerns inversion of the linear map

$$\mathbf{S} \cap \mathbf{L}_\infty \to l_\infty \colon f \mapsto f|_{\mathbf{Z}^2}. \tag{2.1}$$

We say that cardinal interpolation with M is correct if this map is 1-1 and onto, hence boundedly invertible, and denote its inverse by I_M or I. In other words, cardinal interpolation with M is correct iff there exists, for every bounded sequence $f \in l_\infty(\mathbf{Z}^2)$, a bounded function $If \in \mathbf{S}$ which agrees with f on $\mathbf{Z}^2$. The interpolation problem, i.e., the determination of If, is equivalent to the algebraic problem of determining the coefficient sequence a for $If = \Sigma a_j M(\cdot\ - j)$ so that $a \in l_\infty$ and $\Sigma a_j M(\cdot\ - j) = f$ on $\mathbf{Z}^2$. Hence the correctness of cardinal interpolation is equivalent to the invertibility of the matrix

$$A := (M(j-k))_{j,k \in \mathbf{Z}^2} \tag{2.2}$$

as a map on l_∞. Since A is a banded (bivariate) Toeplitz matrix, we have the following necessary and sufficient condition for the correctness of cardinal interpolation.

THEOREM 2. *Cardinal interpolation with M is correct iff*

$$P(x) := P_M(x) := \sum M(j) e^{ijx} \tag{2.3}$$

does not vanish.

Proof. If $P(x) = 0$, then $(e^{-ijx})_{j \in \mathbf{Z}^2} \in (\ker A) \cap l_\infty$, and this contradicts the assumption that A is 1-1. On the other hand, if P does not vanish, then the inverse of A can be expressed as a Toeplitz matrix,

$$(A^{-1})_{jk} := \int_{(-\pi,\pi)^2} \frac{e^{-i(j-k)x}}{P(x)} \, dx/2\pi. \tag{2.4}$$

In view of the geometric decay of the Fourier series for $1/P$, we have

$$|(A^{-1})_{jk}| \leq \text{const}\, \lambda^{|j-k|} \tag{2.5}$$

for some $\lambda = \lambda(P) \in (0,1)$. Therefore, A^{-1} is bounded on $l_p(\mathbf{Z}^2)$ for any $p \in [1, \infty]$. ■

It is convenient to write the cardinal interpolant in Lagrange form:

$$If = \sum f_j L(\cdot\ - j)$$

with

$$L := L_M := I\delta = \sum (A^{-1})_{0j} M(\cdot\ - j) \tag{2.6}$$

the *fundamental function* of the interpolation process. The Fourier transform $L\hat{}$ of L is particularly simple. Combining (4) with (6), we obtain

$$L\hat{} = M\hat{}/P. \tag{2.7}$$

We will also make use of the identity

$$P(x) = \sum M\hat{}\,(2\pi j - x) \tag{2.8}$$

which follows from applying the Poisson summation formula $\Sigma f(j) = \Sigma f\hat{}(2\pi j)$ to (3).

3. Cardinal interpolation with a box spline

In this section, we develop in some detail facts about cardinal interpolation with the box spline M_Z. Recall from Section 1 that $(M(\cdot\ - j))_{j \in \mathbf{Z}^2}$ is linearly dependent if Z contains two vectors which span a proper sublattice of $\mathbf{Z}^2$. Linear independence of $(M(\cdot - j))$ is an obvious necessary condition for cardinal interpolation with M_Z to be correct. Thus, up to obvious symmetries, the only relevant case to consider is the case when the only directions in Z are

$$d_1 := (1,0), \qquad d_2 := (0,1), \quad \text{and} \quad d_3 := (1,1).$$

We show in Section 4 that, with this restriction, cardinal interpolation with M_Z is always correct.

Assume from now on that $Z = (d_1 : r, d_2 : s, d_3 : t)$. In this case, Z is characterized by the vector

$$n := (n_1, n_2, n_3) := (r, s, t)$$

of direction multiplicities, and we will freely write n instead of Z whenever it is necessary to indicate by subscript the dependence on Z of some quantity. Further, the general formulae given in Section 1 simplify. For example,

$$M_n^{\hat{}}(u,v) = S(u)^r S(v)^s S(u+v)^t, \tag{3.1}$$

with

$$S(t) := \frac{\sin(t/2)}{t/2}.$$

Further, the characteristic polynomial $P = P_n$ and the Fourier transform $L^{\hat{}}$ of the fundamental spline $L = L_n$ have the representations

$$P(2\pi u, 2\pi v) = \pi^{-|n|}(\sin(\pi u))^r(\sin(\pi v))^s(\sin(\pi(u+v)))^t \times \sum_{k,l} \frac{(-)^{rk+sl+t(k+l)}}{(u+k)^r(v+l)^s(u+v+k+l)^t} \tag{3.2}$$

and

$$1/L^{\hat{}}(2\pi u, 2\pi v) = \sum_{k,l\in\mathbf{Z}} (-)^{rk+sl+t(k+l)} \left(\frac{u}{u+k}\right)^r \left(\frac{v}{v+l}\right)^s \left(\frac{u+v}{u+v+k+l}\right)^t. \tag{3.3}$$

Let A^* denote the transpose of A. The relation

$$M_{AZ}^{\hat{}}(x) = M^{\hat{}}_Z(A^*x) \tag{1.5}$$

valid for any matrix A together with the fact that

$$M_{AZ} = M_Z \tag{1.4}$$

in case $A = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_{|n|})$ with $\varepsilon_i \in \{-1, 1\}$, all i, implies certain symmetries of M and $\hat{M}$ if the matrix A leaves the set $Z_{\pm} := \{d_1, d_2, d_3, -d_1, -d_2, -d_3\}$ invariant. Denote by $\mathbf{A}$ the group of all such *invertible* matrices A. Each $A \in \mathbf{A}$ is associated with a permutation $\sigma_A \in \$_3$ ($:=$ symmetric group on 3 elements) by the condition

$$Ad_i \in \left\{d_{\sigma_A(i)}, -d_{\sigma_A(i)}\right\}, \quad i = 1, 2, 3.$$

From the two matrices corresponding to a given $\sigma \in \$_3$, we choose one, A_σ, in such a way that the six matrices form a group and we call this group $\mathbf{A}_+$. Thus,

$$A_\sigma d_i \in \left\{d_{\sigma(i)}, -d_{\sigma(i)}\right\} \quad \text{for all } \sigma \in \$_3, \tag{3.4}$$

and one choice for the group generators are the three matrices

$$A_{(12)} := \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad A_{(13)} := \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \qquad A_{(23)} := \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

corresponding to the transpositions (12), (13), and (23). With the definition

$$\sigma(n) := \left(n_{\sigma(1)}, n_{\sigma(2)}, n_{\sigma(3)}\right),$$

it follows from (1.5) and from (4) that

$$M_n(x) = M_{\sigma(n)}(\pm A_\sigma x), \quad \text{and} \quad \hat{M}_{\sigma(n)}(y) = \hat{M}_N(\pm A_\sigma^* y). \tag{3.5}$$

This implies

$$\hat{P}_n(\pm A_\sigma^* x) = \hat{P}_{\sigma(n)}(x) \tag{3.6}$$

and

$$L_n(x) = L_{\sigma(n)}(\pm A_\sigma x), \qquad \hat{L}_{\sigma(n)}(y) = \hat{L}_n(\pm A_\sigma^* y). \tag{3.7}$$

Of particular interest is the case $r = s = t$, i.e., when the direction multiplicities are all equal. In this case, $\sigma(n) = n$, all σ; i.e., (5)–(7) hold with $\sigma(n)$ replaced by n. For example, writing out in detail the relations (6) for $P = P_{(s,s,s)}$, we get

$$P(u, v) = P(-u, -v) = P(v, u) = P(u + v, -v) = P(-u, u + v). \tag{3.6$'$}$$

The relations for $\hat{M}$, $\hat{P}$ and $\hat{L}$ will be used frequently in the sequel. Since they are given in terms of the transposes of the matrices in $\mathbf{A}$, we now consider

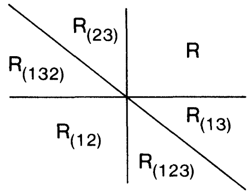

FIG. 3.1

$\mathbf{A}^* := \{A^*: A \in \mathbf{A}\}$ in more detail. Set

$$d' := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} d,$$

$$Z'_{\pm} := \{d': d \in Z_{\pm}\} = \{(0,1),(-1,0),(-1,1),(0,-1),(1,0),(1,-1)\}.$$

Since $d'^*d = 0$, we see from (4) that $\mathbf{A}^*$ leaves $Z'_{\pm}$ invariant. To further illustrate the action of the group $\mathbf{A}^*$, we divide $\mathbf{R}^2$ into the six cones R_σ, $\sigma \in \mathcal{S}_3$, as indicated in Figure 3.1. It is easily checked that

$$A^*_\sigma R = R_\sigma \quad \text{for all } \sigma \in \mathcal{S}_3. \tag{3.8}$$

4. The correctness of cardinal interpolation with M_n

In this section, we show that cardinal interpolation with M_n is correct for all choices of $n \in \mathbf{Z}^3_+$.

THEOREM 4. *For all $n \in \mathbf{Z}^3_+$, P_n is strictly positive.*

Since P is 2π-periodic, this amounts to the claim that $P_n(x) > 0$ for all $x \in [-\pi, \pi]^2$. This is the bivariate analogue of Schoenberg's well known result for univariate cardinal spline interpolation. To recall this result, denote by N_r the univariate cardinal B-spline of degree r, and by Q_r the corresponding characteristic polynomial given by

$$Q_r(x) := \sum N_r(j) e^{ijx}.$$

Schoenberg showed in [S$_1$] that

$$\min_x Q_r(x) = Q_r(\pi) = 2\left(\frac{2}{\pi}\right)^{r+1} \sum_{\nu=0}^{\infty} \frac{(-)^{\nu(r+1)}}{(2\nu+1)^{r+1}}. \tag{4.1}$$

The fact that, for any r, the minimum is attained at $x = \pi$ is a consequence of the total positivity of the matrix $(N_r(j-k))_{j,k\in\mathbf{Z}}$. In view of this result, one might think that, in the above theorem, $\min_{u,v} P(u,v) = P(\pi,\pi)$. This is trivially true in the tensor product case, i.e., when $n=(r,s,0)$. However, in general, the point at which P_n attains its minimum depends on n. It would be interesting to determine its location for special choices of n. The nicest conjecture in this context (cf. Section 5) is that

$$\min P(u,v) = P(2\pi/3, 2\pi/3) \quad \text{if } n=(r,r,r). \tag{4.2}$$

In the proof of the theorem, we make use of (3.6). This allows us to assume without loss of generality that $r\ge s\ge t$.

We first consider two cases which reduce to Schoenberg's result.

The tensor product case $n=(r,s,0)$. Here, we have $M_n(u,v)=N_r(u)N_s(v)$, and this implies that $P_n(u,v)=Q_r(u)Q_s(v)$.

The case $n=(r,1,1)$. Since the open support of M_n intersects exactly one mesh line of the form $(\cdot,l)$, viz. the meshline $(\cdot,0)$, it follows that, in this case,

$$M(k,l)=\begin{cases} N_r(k), & l=0,\\ 0, & l\in\mathbf{Z}\backslash 0\end{cases}.$$

This means that cardinal interpolation with M_n reduces to univariate interpolation with N_r on each of the lines $(\cdot,l)$, $l\in\mathbf{Z}$. In particular, $P(u,v)=Q_r(u)$.

For the proof of Theorem 4, it remains to consider the cases where the multiplicities are all at least 1, with equality for at most one. We make this assumption for the remainder of this section.

To prove the positivity of P, we use the representation (2.10) in the form

$$P(2\pi x)=\sum M^{\tilde{}}(x+j), \tag{4.3}$$

with

$$M^{\tilde{}}(x) := M^{\hat{}}(2\pi x).$$

Recall from (3.1) that, for $x=(u,v)$ and $j=(k,l)$,

$$M^{\tilde{}}(x+j)=\pi^{-|n|}(\sin\pi u)^r(\sin\pi v)^s(\sin\pi(u+v))^t \times\frac{(-)^{rk+sl+t(k+l)}}{(u+k)^r(v+l)^s(u+v+k+l)^t}. \tag{4.4}$$

It is sufficient to show the positivity of $P(2\pi\,\cdot\,)$ on $[0,1/2]^2$ for arbitrary n. This follows from (3.6) since, by (3.8), $[-1/2,1/2]^2\subseteq \cup_{A\in\mathbf{A}}A^*[0,1/2]^2$. For $x\in[0,1/2]^2$, we now show that the three positive terms

$$M^{\tilde{}}(x),\quad M^{\tilde{}}(x-d_1),\quad\text{and}\quad M^{\tilde{}}(x-d_2) \tag{4.5}$$

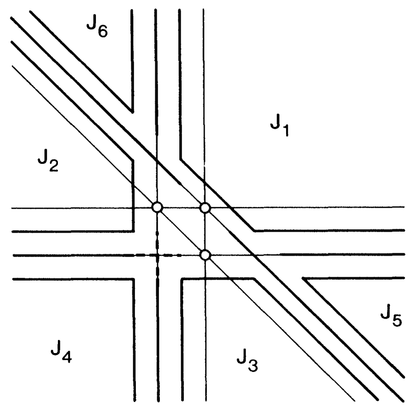

FIG. 4.1

dominate the sum in (3). To this end, we associate each of the other terms with one of these (even to the point of splitting one of the other terms between two of these) and show that each of the resulting three sums, when divided by their respective dominant term, is less than 1. For ease of argument, we actually split the sum into altogether ten parts, as indicated in part by Fig. 4.1.

To simplify notation, we set

$$b_\nu(j) := b_{\nu,n}(j,x) := \frac{|M^{\sim}(x+j)|}{M^{\sim}(x+j_\nu)}, \quad \nu = 1,2,3, \tag{4.6}$$

with

$$j_1 := 0, \qquad j_2 := -d_1, \qquad j_3 := -d_2.$$

We now prove that

$$\sum_{j \in J_1 \cup J_4} b_1(j) + \sum_{l \neq 0} b_1(l,-l) < 1, \tag{4.7}$$

$$\sum_{j \in J_2 \cup J_5} b_2(j) + \sum_{|l|>1} b_2(-1,l) + b_2(-1,-1)/2 < 1, \tag{4.8}$$

$$\sum_{j \in J_3 \cup J_6} b_3(j) + \sum_{|l|>1} b_3(l,-1) + b_3(-1,-1)/2 < 1. \tag{4.9}$$

Since each of the summands (divided by its appropriate dominant term) other than the three dominant terms (5) occurs in (7)–(9) exactly once, we conclude from (7)–(9) the positivity of P.

The estimation of the various sums in (7)–(9) is straightforward. In each case, we find a majorant which is independent of $x \in [0,1/2]^2$ and n. For this, recall that we are assuming that $r,s,t \geq 1$ with at most one equality.

We begin with the sum $\sum_{J_1} b_1(j)$. By (3.1) and (6), for $x = (u, v)$ and $j = (k, l)$ we have

$$b_1(j) = \left|\frac{u}{u+k}\right|^r \left|\frac{v}{v+l}\right|^s \left|\frac{u+v}{u+v+k+l}\right|^t.$$

Since $k, l \geq 0$ for $j = (k, l) \in J_1$ and we are assuming that $u, v \in [0, 1/2]$, this quotient is largest when $u = v = 1/2$; i.e.,

$$b_1(j) \leq \left(\frac{1/2}{1/2+k}\right)^r \left(\frac{1/2}{1/2+l}\right)^s \left(\frac{1}{1+k+l}\right)^t.$$

This bound is largest when the exponents r, s, t are as small as possible, i.e., when $n = (1, 2, 2)$, $(2, 1, 2)$ or $(2, 2, 1)$. Since

$$\sum_{\substack{k,l \geq 0 \\ (k,l) \neq 0}} \left(\frac{1/2}{1/2+k}\right)^r \left(\frac{1/2}{1/2+l}\right)^s \left(\frac{1}{1+k+l}\right)^t = .1723\ldots$$

for these values of n, we conclude that

$$\sum_{J_1} b_1(j) < .18. \tag{4.10}$$

Similarly, one verifies that, for $j = (-k, -l) \in J_4$ and $x \in [0, 1/2]^2$,

$$b_1(j) \leq \left(\frac{1/2}{k-1/2}\right)^r \left(\frac{1/2}{l-1/2}\right)^s \left(\frac{1}{k+l-1}\right)^t$$

and so obtains

$$\sum_{J_4} b_1(j) < .02, \tag{4.11}$$

since

$$\sum_{k,l>1} (2k-1)^{-r}(2l-1)^{-s}(k+l-1)^{-t} = .0101\ldots$$

for $(r, s, t) = (1, 2, 2)$, $(2, 1, 2)$, or $(2, 2, 1)$.

Finally, for $j = (-l, l)$ and $l \neq 0$, we have

$$b_1(-l, l) \leq \begin{cases} \left(\dfrac{1/2}{l+1/2}\right)^r \left(\dfrac{1/2}{l-1/2}\right)^s, & l > 0, \\ \left(\dfrac{1/2}{-l-1/2}\right)^r \left(\dfrac{1/2}{-l+1/2}\right)^s, & l < 0, \end{cases}$$

and so obtain

$$\sum_{l\neq 0} b_1(l,-l) \leq \sum_{1}^{\infty}\left[\left(\frac{1/2}{l+1/2}\right)\left(\frac{1/2}{l-1/2}\right)^2+\left(\frac{1/2}{l+1/2}\right)^2\left(\frac{1/2}{l-1/2}\right)\right]=.5. \tag{4.12}$$

Combining (10)–(12) establishes (7).

The other inequalities are proved in a similar fashion and we only list the estimates involved.

Proof of (8).

$$b_2(j)=\left|\frac{u-1}{u+k}\right|^r\left|\frac{v}{v+l}\right|^s\left|\frac{u+v-1}{u+v+k+l}\right|^t.$$

The case $j=(-k,l)\in J_2$.

$$b_2(j)\leq\left(\frac{1}{k}\right)^r\left(\frac{1/2}{l+1/2}\right)^s\left(\frac{1}{k-l}\right)^t$$

and

$$\sum_{k=2}^{\infty}\sum_{l=0}^{k-1}\left(\frac{1}{k}\right)^r\left(\frac{1/2}{l+1/2}\right)^s\left(\frac{1}{k-l}\right)^t=\begin{cases}.25\ldots\\.23\ldots\\.30\ldots\end{cases}\quad\text{for } (r,s,t)=\begin{cases}(2,2,1)\\(2,1,2)\\(1,2,2)\end{cases}.$$

Therefore $\Sigma_{J_2} b_2(j)<.35$.

The case $j=(k,-l)\in J_5$.

$$b_2(j)\leq\left(\frac{1}{k}\right)^r\left(\frac{1/2}{l-1/2}\right)^s\left(\frac{1}{k-l}\right)^t$$

and

$$\sum_{k=3}^{\infty}\sum_{l=2}^{k-1}\left(\frac{1}{k}\right)^r\left(\frac{1/2}{l-1/2}\right)^s\left(\frac{1}{k-l}\right)^t=\begin{cases}.027\ldots\\.081\ldots\\.010\ldots\end{cases}\quad\text{for } (r,s,t)=\begin{cases}(2,2,1)\\(2,1,2)\\(1,2,2)\end{cases}.$$

Therefore $\Sigma_{J_5} b_2(j)<.1$.

The case $j = (-1, l)$, $|l| > 1$.

$$b_2(j) \le \begin{cases} \left(\dfrac{1/2}{l+1/2}\right)^s \left(\dfrac{1}{l-1}\right)^t, & l > 1, \\ \left(\dfrac{1/2}{-l-1/2}\right)^s \left(\dfrac{1}{-l+1}\right)^t, & l < -1, \end{cases}$$

$$\sum_{|l|>1} b_2(j) \le \sum_{l>1} \left[\left(\frac{1/2}{l+1/2}\right)\left(\frac{1}{l-1}\right)^2 + \left(\frac{1/2}{l-1/2}\right)\left(\frac{1}{l+1}\right)^2\right] = .329\ldots.$$

The case $j = (-1, -1)$.

$$b_2(-1,-1) \le \frac{v}{1-v}\,\frac{1-u-v}{2-u-v} \le 1/3 \quad (\text{at } (u,v) = (0, 1/2)).$$

Proof of (9). Since $\hat{M}_{(r,s,t)}(u,v) = \hat{M}_{(s,r,t)}(v,u)$ and therefore

$$b_{2,(r,s,t)}((u,v),(k,l)) = b_{3,(s,r,t)}((v,u),(l,k)),$$

the inequality (9) follows from (8) by interchanging the roles of u and v as well as those of k and l.

This completes the proof of Theorem 4. ■

5. Convergence of cardinal interpolation

This section is devoted to the main goal of our paper, a study of the convergence of the cardinal interpolant to smooth functions as the degree tends to infinity. We prove the analogue of I.J. Schoenberg's basic result:

THEOREM [S_2]. *If f is the Fourier transform of a measure with support in $(-\pi, \pi)$, then its cardinal spline interpolant $I_r f$ of degree r converges to f as the degree tends to infinity, i.e.,* $\|f - I_r f\|_\infty \to 0$ *as* $r \to \infty$.

This theorem is a consequence of the fact that the Fourier transform $\hat{L}_r$ of the fundamental spline converges to the characteristic function of the interval $(-\pi, \pi)$.

The bivariate situation is more complicated. Here, the limit of $\hat{L}_n$ depends on just how n goes to infinity. Recall from (3.3) that

$$1/\hat{L}(2\pi x) = 1 + \sum_{j \in \mathbf{Z}^2 \setminus 0} \varepsilon_j(x) a_{n,j}(x), \tag{5.1}$$

with $\varepsilon_j(x) \in \{-1, 1\}$ and

$$a_{n,(k,l)}(u,v) := \left|\frac{u}{u+k}\right|^r \left|\frac{v}{v+l}\right|^s \left|\frac{u+v}{u+v+k+l}\right|^t. \tag{5.2}$$

Define the "middle component" $\mu(n)$ of n by the requirement that it equal the middle or second number in any ordering of the three numbers r, s, t, and set

$$n' := (r', s', t') := n/\mu(n).$$

Then, the typical summand in the right hand side of (1) is, up to sign,

$$a_{n,j}(x) = \left(a_{n',j}(x)\right)^{\mu(n)}.$$

This shows that $\hat{L}(2\pi x)$ is close to 1 for large $\mu(n)$ provided $a_{n',j}(x) < 1$ for all $j \neq 0$.

The set

$$\left\{x\colon a_{n',j}(x) < 1 \text{ for all } j \in \mathbf{Z}^2 \setminus 0\right\}$$

depends on n'. In particular, we cannot expect it to converge as $|n| \to \infty$ unless $n' = n/\mu(n)$ converges, to some 3-vector m, say. Here, we are willing to allow m to have infinite components. For example, if $n = (1, s, s^2)$, then $\mu(n) = s$ and $n' = (1/s, 1, s) \to (0, 1, \infty)$ as $s \to \infty$. But not every $m \in [0, \infty]^3$ is a possible limit. By construction of $\mu(n)$, $n' = n/\mu(n)$ has exactly one component equal to 1 and, among the other two, one must be ≤ 1 and the other must be ≥ 1. Thus the set

$$N := \left\{n \in [0, \infty]^3\colon n_{\sigma(1)} \leq 1 = n_{\sigma(2)} \leq n_{\sigma(3)} \text{ for some } \sigma \in \$_3\right\}$$

makes up the collection of all possible limits. On this set, we set up a topology of sorts by defining the open ball of radius r around $m \in N$ by

$$B_r(m) := \left\{n \in N\colon \max_i \begin{Bmatrix} |n_i - m_i|, & m_i < \infty \\ 1/n_i, & m_i = \infty \end{Bmatrix} < r\right\}.$$

We extend the definition (2) of $a_{n,j}$ to all $n \in N$, by pointwise limit if necessary.

In what is to follow, the sets

$$\Omega_m := \left\{x\colon a_{m,j}(x) < 1 \text{ for } j \in J\right\},$$

with

$$J := Z'_{\pm} = \{(1,0), (0,1), (-1,1), (-1,0), (0,-1), (1,-1)\}$$

play a major role. Note that $\Omega_n = \Omega_{n'}$. A qualitatively correct picture of Ω_n is given in Figure 5.1 which shows the roughly hexagonal shape of Ω_n and also

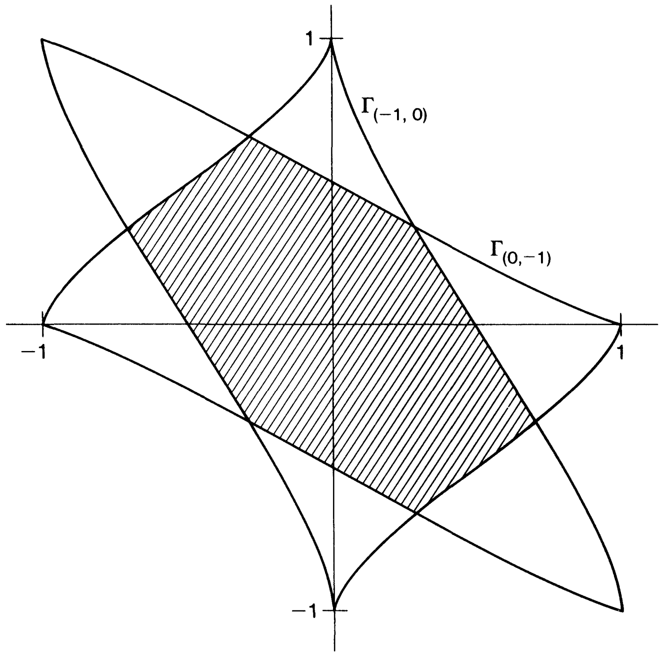

FIG. 5.1

shows the six curves

$$\Gamma_{n,j} := \left\{ x \in C_{-j} \colon a_{n,j}(x) = 1 \right\}, \quad j \in J$$

which contribute to the boundary, $\partial\Omega_n$. Here, C_{-j} is the union of the two cones R_σ which contain $-j$.

THEOREM 5.1. *For $m \in N$, let χ_m be the characteristic function of Ω_m. Then, for any $d > 0$, there exists $\varepsilon > 0$ so that*

$$|L_n^\wedge(2\pi x) - \chi_m(x)| \leq C_1(1 + C\operatorname{dist}(x, \partial\Omega_m))^{-\mu(n)} \tag{5.3}$$

for all x with $\operatorname{dist}(x, \partial\Omega_m) \geq d$ and all $n \in (0,\infty)^3$ with $n' := n/\mu(n) \in B_\varepsilon(m)$, and with the positive constants C_1 and C independent of m, n, d, or x.

Proof. The proof is based on a series of propositions which we merely state as needed and prove at leisure later. We begin with the following:

PROPOSITION 5.1. *Ω_n depends continuously on n in the Hausdorff topology.*

This is part of the corollary to Lemma 6.4 below. It implies, given $d > 0$, the existence of $\varepsilon > 0$ so that $\operatorname{dist}(\Omega_{n'}, \Omega_m) \leq d/2$ for all $n' \in B_\varepsilon(m)$. Consequently,

$$\operatorname{dist}(x, \partial\Omega_m) \leq 2\operatorname{dist}(x, \partial\Omega_{n'})$$

for all x with $\operatorname{dist}(x, \partial\Omega_m) \geq d$ and for all $n' \in B_\varepsilon(m)$. It is therefore sufficient to prove (3) with m replaced by n'.

For its proof, we use (1) and we consider two cases.

(i) $x \in \Omega_{n'}$. We need:

PROPOSITION 5.2. *Let*

$$J' := \mathbf{A}^*(1,1) = \{\pm(1,1), \pm(2,-1), \pm(-1,2)\}.$$

For $n \in N$ and $x \in \Omega_{n'} = \Omega_n$,

$$a_{n,j}(x) \leq \begin{cases} (1 + C\operatorname{dist}(x, \partial\Omega_n))^{-1}, & j \in J \cup J' \\ (1 + C|j|)^{-1}, & j \in \mathbf{Z}^2 \backslash (0 \cup J \cup J') \end{cases}, \tag{5.4}$$

with the positive constant C independent of n, j, or x.

This, together with (1), implies that

$$\begin{aligned} |1/\hat{L_n}(2\pi x) - 1| &\leq \sum_{j \neq 0} (a_{n',j}(x))^{\mu(n)} \\ &\leq 12(1 + C\operatorname{dist}(x, \partial\Omega_{n'}))^{-\mu(n)} + \sum_{j \notin 0 \cup J \cup J'} (1 + C|j|)^{-\mu(n)} \\ &\leq C_1(1 + C\operatorname{dist}(x, \partial\Omega_{n'}))^{-\mu(n)} \end{aligned} \tag{5.5}$$

and so proves (3) for this case.

(ii) $x \notin \Omega_{n'}$. For this case, we need:

PROPOSITION 5.3. *The integer translates of Ω_n form, up to a set of measure zero, a partition of $\mathbf{R}^2$; i.e.,*

$$\mathbf{R}^2 = \bigcup_{j \in \mathbf{Z}^2} j + \Omega_n^-, \qquad \Omega_n \cap (j + \Omega_n) = \emptyset \quad \textit{for } j \neq 0.$$

We conclude that there is $j \neq 0$ so that $x = x' + j$ with $x' \in \Omega_{n'}^{-}$. With this, we use the periodicity of the characteristic polynomial P to write for such x,

$$\begin{aligned} L^{\hat{}}(2\pi x) &= L^{\hat{}}(2\pi(x'+j)) \\ &= \frac{M^{\tilde{}}(x'+j)}{P(2\pi(x'+j))}\frac{M^{\tilde{}}(x')}{M^{\tilde{}}(x')} \\ &= L^{\hat{}}(2\pi x')\frac{M^{\tilde{}}(x'+j)}{M^{\tilde{}}(x')}. \end{aligned}$$

Therefore

$$|L_n^{\hat{}}(2\pi x)| = |L_n^{\hat{}}(2\pi x')| a_{n,j}(x') = |L_n^{\hat{}}(2\pi x')| \left(a_{n',j}(x')\right)^{\mu(n)}.$$

By (5), $|L_n^{\hat{}}(2\pi x')| \leq C$ since $x' \in \Omega_{n'}^{-}$. Thus (3) is proved for this case once we show:

Proposition 5.4. *Let $x = x' + j$ with $j \in \mathbf{Z}^2 \setminus 0$ and $x' \in \Omega_{n'}^{-}$, and with $n \in \mathbf{R}_+^3$. Then*

$$a_{n',j}(x') \leq \left(1 + C\operatorname{dist}(x, \partial\Omega_{n'})\right)^{-1} \tag{5.6}$$

for some positive constant C independent of n and x.

This finishes the proof of Theorem 5.1. ■

Theorem 5.2. *Let f be the Fourier transform of a measure with support strictly inside $2\pi\Omega_m$ for some $m \in N$; i.e.,*

$$d := \operatorname{dist}(\operatorname{supp} f^{\hat{}}, \partial(2\pi\Omega_m)) > 0.$$

Then there exists $\varepsilon > 0$ so that, for all $n \in B_\varepsilon(m)$,

$$\|f - I_n f\|_\infty \leq C_1(1 + Cd)^{-\mu(n)} \|f^{\hat{}}\|_1,$$

with $\|f^{\hat{}}\|_1$ the total variation of $f^{\hat{}}$. The positive constants C, C_1 do not depend on m, d, or n.

Proof. Fix $d > 0$ and choose $\varepsilon > 0$ so that

$$\operatorname{supp} f^{\hat{}} \subseteq (2\pi\Omega_{n'}) \quad \text{and} \quad \operatorname{dist}(\operatorname{supp} f^{\hat{}}, \partial(2\pi\Omega_{n'})) \geq d/2$$

for all $n' \in B_\varepsilon(m)$. We have to estimate

$$f(x) - \sum f(j) L_n(x-j) = f(x) - \sum (2\pi)^{-2} \int_{\mathbf{R}^2} f(j) e^{-ijy} L_n^{\wedge}(y) e^{ixy}\, dy.$$

Since $2\pi\Omega_{n'}$ is a fundamental domain, i.e., translates by its $2\pi j$, $j \in \mathbf{Z}^2$, form a partition of unity (by Proposition 5.3), and $\operatorname{supp} f^{\wedge} \subseteq 2\pi\Omega_{n'}$, $(f(j))_{j\in\mathbf{Z}^2}$ are the Fourier coefficients of the periodic extension $f_p^{\wedge}$ of the measure $f^{\wedge}$. Using the weak convergence of the Fourier series of a measure, we obtain

$$f(x) - (I_n f)(x) = (2\pi)^{-2} \int_{\mathbf{R}^2} \left[f^{\wedge}(y) - f_p^{\wedge}(y) L_n^{\wedge}(y) \right] e^{ixy}\, dy.$$

Applying Theorem 5.1 yields, for $n' \in B_\varepsilon(m)$,

$$\begin{aligned} \|f - I_n f\|_\infty &\le (2\pi)^{-2} C_1 (1 + Cd)^{-\mu(n)} \|f^{\wedge}\|_1 + \sum_{j \ne 0} \|L_n^{\wedge}(\cdot + 2\pi j)\|_{\infty, \operatorname{supp} f^{\wedge}} \|f^{\wedge}\|_1 \\ &\le \left[C_1 (1 + Cd)^{-\mu(n)} + \sum_{j \notin 0 \cup J \cup J'} (1 + C|j|)^{-\mu(n)} \right] \|f^{\wedge}\|_1 \\ &\le C_1 (1 + Cd)^{-\mu(n)} \|f^{\wedge}\|_1. \quad \blacksquare \end{aligned}$$

We now discuss briefly the particularly symmetric and special case $n = (r, r, r)$. Figure 5.2 shows the level lines for $P_r := P_{(r,r,r)}$ for $r = 3$. Note that its minimum seems to occur at $(2\pi/3, 2\pi/3)$, and this can be verified analytically for $r < 4$. We conjecture that this is no accident, but is the case for all r.

Figure 5.3 shows $L_r := L_{(r,r,r)}$ for $r = 3$. The fast decay is quite striking, making plain that cardinal interpolation with this L would be strongly essentially local.

If we assume that $f^{\wedge} \in \mathbf{L}_2$ with $\operatorname{supp} f^{\wedge} \subseteq 2\pi\Omega_m$, then the convergence of $I_n f$ can be stated in a particularly nice way. We define a bivariate "Whittaker" operator

$$W\colon l_2 \to \mathbf{L}_2\colon f \mapsto \sum_{j \in \mathbf{Z}^2} f(j) \chi^{\wedge}(\cdot - j),$$

with

$$\chi^{\wedge}(u,v) := \frac{9}{2\pi^2} \left[\frac{\cos\frac{2\pi}{3}(u+v)}{(u-2v)(v-2u)} + \frac{\cos\frac{2\pi}{3}(2u-v)}{(u+v)(u-2v)} + \frac{\cos\frac{2\pi}{3}(2v-u)}{(u+v)(v-2u)} \right]$$

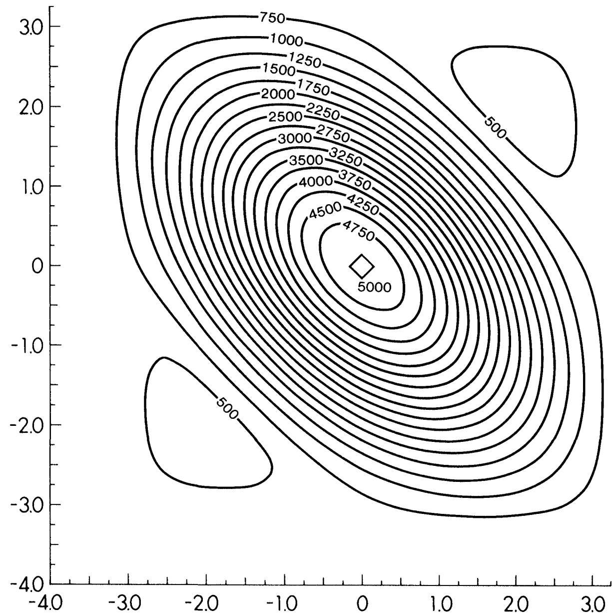
750
1000
1250
1500
1750
2000
2250
2500
2750
3000
3250
3500
3750
4000
4250
4500
4750
5000
500
500
3.0
2.0
1.0
0
-1.0
-2.0
-3.0
-4.0
-4.0
-3.0
-2.0
-1.0
0
1.0
2.0
3.0

FIG. 5.2

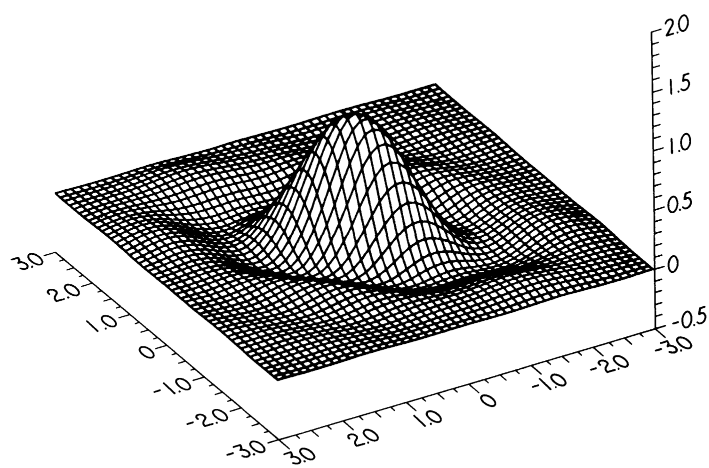
2.0
1.5
1.0
0.5
0
-0.5
3.0
2.0
1.0
0
-1.0
-2.0
-3.0
-3.0
-2.0
-1.0
0
1.0
2.0
3.0

FIG. 5.3

the Fourier transform of $\chi := \chi_\Omega$, $\Omega := 2\pi\Omega_{(1,1,1)}$. Note that the translates of $\chi\hat{}$ are orthogonal in $\mathbf{L}_2$. As in the univariate case [S₂], the "Whittaker" series provides the limiting operator for cardinal interpolation $I_{(r,r,r)}$ as $r \to \infty$. More precisely, we have

$$\| W - I_r \colon l_2 \to \mathbf{L}_2 \| \to 0 \quad \text{as } r \to \infty. \tag{5.7}$$

If $f \in \mathbf{L}_2$ and $\operatorname{supp} f\hat{} \subseteq 2\pi\Omega_{(1,1,1)}$, we have $W((f(j))_{\mathbf{Z}^2}) = f$ and hence (7) is an $\mathbf{L}_2$-version of Theorem 5.2.

To prove (7), we first show that the cardinal interpolation maps $I_r := I_{(r,r,r)}$ are bounded as maps from l_2 to $\mathbf{L}_2$ uniformly in r: For $f \in l_2$, we have

$$\|I_r f\|_2 = \left\| \sum f(j) e^{-ij\cdot} L_r\hat{} \right\|_2 \le \|f\|_2 \left(\sum \| L_r\hat{} (\cdot + 2\pi j) \|_{\infty,\Omega}^2 \right) \le C \|f\|_2 .$$

By the uniform boundedness principle, it is therefore sufficient to check the convergence (7) for the unit vector $e_j \in l_2$. But this is an obvious consequence of (3). ■

In the univariate setting, [MRR] proved the convergence in $\mathbf{L}_p$ for all $p \in (1, \infty)$, but we leave the corresponding bivariate problem to a later paper.

[R] extended Schoenberg's univariate result to include the possibility that $f\hat{}$ has support at $\pm\pi$. This requires the realization that $L_r\hat{}(\pm\pi) \to 1/2$. Theorem 5.1 says nothing about the limit of $L_r\hat{}(x)$ in case $x \in \partial(2\pi\Omega_m)$. For the special choice $n = (r, r, r)$, such a statement is relatively easy to make.

COROLLARY. *For the special choice* $n = (r, r, r)$,

$$\lim_{r \to \infty} L_n\hat{}(x) = \begin{cases} 1/3, & x \in \mathbf{A}z, \\ 1/2, & x \in \partial(2\pi\Omega) \setminus \mathbf{A}z, \end{cases} \tag{5.8}$$

with $z := (2\pi/3, 2\pi/3)$.

Clearly, our result concerning the convergence of I_n still holds if $f\hat{}$ is a measure, absolutely continuous in a neighborhood of $\partial 2\pi\Omega_m$ and supported in $2\pi\Omega_m$. Our result is best possible in the following sense. If $\operatorname{supp} f\hat{} \cap \partial 2\pi\Omega \neq \varnothing$, then, in general, $I_n f$ does not converge to f. For example, if $f(x) = \cos(z^* x)$, then

$$\begin{aligned} I_r f(x) &\underset{r\to\infty}{\to} \sum_{A \in \mathbf{A}} e^{i(Az)^* x} \\ &= \left[\cos\frac{2\pi}{3}(u + v) + \cos\frac{2\pi}{3}(2u - v) + \cos\frac{2\pi}{3}(2v - u) \right]. \end{aligned}$$

This follows from the corollary. However, it requires slightly more precise information about the convergence asserted in (8). The heuristic argument is that

$$(I_n f)\hat{} = \hat{f_p} \hat{L_n} = \frac{1}{2}(2\pi)^2 \sum \delta_{z+2\pi j} \hat{L_n}(z+2\pi j)$$

and therefore

$$I_n f \to \sum_{A \in \mathbf{A}^*} (\delta_{Az} \cdot 1/3) \quad \text{as } n \to \infty.$$

Here, δ_ξ denotes the Dirac measure at ξ.

6. Detailed estimates

In this section, we prove Propositions 5.1–5.4 and various lemmas needed in the proofs. This amounts to a detailed study of the functions $a_{n,j}$ and the set Ω_n and how they depend on $n \in N$. In particular, we need to study the boundary of Ω_n. This boundary is made up of pieces of curves given implicitly by the equation $a_{n,j}(x) = 1$ for some $j \in J$.

We use the symmetries of the given situation. Recall the notation

$$\sigma(n) := (n_{\sigma(1)}, n_{\sigma(2)}, n_{\sigma(3)}).$$

We conclude from (3.5) or directly from (5.2) that

$$a_{n,j}(x) = a_{\sigma(n), A_\sigma^* j}(A_\sigma^* x). \tag{6.1}$$

This implies that

$$\pm A_\sigma^* \Omega_n = \Omega_{\sigma(n)} \tag{6.2}$$

and therefore

$$A_\sigma^*(\Omega_n \cap R) = \Omega_{\sigma(n)} \cap R_\sigma, \tag{6.3}$$

where, to recall from Figure 3.1, $R = R_{(1)} = \mathbf{R}_+^2$.

Next, we consider the boundary of Ω_n (cf. Figure 5.1). Each $-j \in J$ lies in two cones R_σ. With C_{-j} their union, we define the curve

$$\Gamma_{n,j} := \{x \in C_{-j} : a_{n,j}(x) = 1\}. \tag{6.4}$$

The boundary of Ω_n is made up of segments of these curves. It follows from (1) that

$$A_\sigma^* \Gamma_{n,j} = \Gamma_{\sigma(n), A_\sigma^* j}. \tag{6.5}$$

LEMMA 6.1. *For $j \in J$, denote by j_0, j_1 the vectors spanning the union C_j of the two cones R_σ containing j. For $n \in (0,\infty)^3$, the curve $\Gamma_{n,j}$ passes through the points $-j_0$, $-j/2$, $-j_1$ and is symmetric with respect to the point $-j/2$. Moreover, it is monotone (in an appropriate coordinate system) and is Lipschitz continuous, uniformly in n.*

Proof. The symmetry with respect to the point $-j/2$ follows from the relations

$$a_{n,j}(x-j) = 1/a_{n,-j}(x) = 1/a_{n,j}(-x)$$

which are immediate consequences of the definition (5.2) of $a_{n,j}$. For the rest, it is, in view of (5), sufficient to consider $j = (-1,0)$. In this case, the curve $\Gamma_{n,j}$ is given by the equation

$$\left|\frac{u}{u-1}\right|^r \left|\frac{u+v}{u+v-1}\right|^t = 1. \tag{6.6}$$

Using the fact that $u \geq 0$, $u+v \geq 0$, and solving for v, we obtain

$$v = -u + 1\Big/\left[1 + \left(\frac{u}{1-u}\right)^\alpha\right], \quad 0 \leq u \leq 1, \tag{6.7}$$

where $\alpha := r/t$. This shows that, for any $\alpha \in (0,\infty)$, the points $-j_0 = (1,-1)$, $-j/2 = (1/2, 0)$, and $-j_1 = (0,1)$ lie on the curve. Moreover we have

$$dv/du = -1 - \left[1 + \left(\frac{u}{1-u}\right)^\alpha\right]^{-2} \alpha \left(\frac{u}{1-u}\right)^{\alpha-1} (1-u)^{-2} \tag{6.8}$$

which shows that

$$dv/du \leq -1, \tag{6.9}$$

with equality only if $u = 0$. This proves the remaining assertions of the Lemma. ■

LEMMA 6.2. *Let j_0, j_1 be the two vectors which span the cone R_σ. For $n \in (0,\infty)^3$, the curves $\Gamma_{n,-j_0}$ and $\Gamma_{n,-j_1}$ intersect at a unique point $z_{n,\sigma} \in R_\sigma$. The boundary of Ω_n consists of the segments of the curves $\Gamma_{n,j}$ connecting the points in $J/2$ and $z_{n,\sigma}$, $\sigma \in \$_3$.*

Proof. In view of (3) and (5), it is sufficient to consider the case $\sigma = (1)$, $R_\sigma = R$ and $j_0 = (1,0)$, $j_1 = (0,1)$. By (9), $\Gamma_{n,(-1,0)}$ has slope ≤ -1, with equality possible only at the point $(0,1)$. Similarly, a direct computation shows that $\Gamma_{n,(0,-1)}$ has slope between -1 and 0. Since $\Gamma_{n,(-1,0)}$ and $\Gamma_{n,(0,-1)}$ pass through the points $(0,1),(1/2,0)$ and $(0,1/2),(1,0)$ respectively, they intersect at a unique point $z_{n,(1)} \in (0,1/2)^2$. To show that the boundary of Ω_n intersected with R consists of the segments connecting $(0,1/2)$ with $z_{n,(1)}$, and $z_{n,(1)}$ with $(1/2,0)$, we prove that, for $x = (u,v) \in R$,

$$a_{n,(-1,0)}(x), a_{n,(0,-1)}(x) < 1$$

implies that

$$a_{n,j}(x) < 1 \quad \text{for all } j \in J.$$

Indeed,

$$a_{n,(-1,0)}(x) = \left|\frac{u}{u-1}\right|^r \left|\frac{u+v}{u+v-1}\right|^t < 1$$

implies $u < 1/2$, and, from $a_{n,(0,-1)}(x) < 1$, it follows that $v < 1/2$. This implies

$$a_{n,(-1,1)}(x) = \left|\frac{u}{u-1}\right|^r \left|\frac{v}{v+1}\right|^t < 1,$$

and the other cases can be checked just as easily. ■

LEMMA 6.3. *For all* $m \in N$, $\Gamma_{m,j} = \lim_{n \to m}\{\Gamma_{n,j}\colon\ n \in (0,\infty)^3 \cap N\}$. *Hence Lemma* 1 *is valid for all* $n \in N$.

Proof. Without loss of generality, we consider only the case $j = (-1,0)$. We claim that

$$\Gamma_{m,j} = \begin{cases} BL((0,1),(0,1/2),(1/2,0),(1,-1/2),(1,-1)) \\ \quad \text{if } m_1 = 0 \text{ and/or } m_3 = \infty \\ BL((0,1),(1/2,1/2),(1/2,0),(1/2,-1/2),(1,-1)) \\ \quad \text{if } m_1 = \infty \text{ and/or } m_3 = 0 \end{cases},$$

where $BL(x_1,\ldots,x_m)$ denotes the broken line with vertices $x_1,\ldots,x_m$. For example, consider the first case. By (9) and the symmetry of the curves $\Gamma_{n,j}$ and $\Gamma_{m,j}$, for $n \in (0,\infty)^3$ we have

$$\operatorname{dist}(\Gamma_{m,j}, \Gamma_{n,j}) \le u_n,$$

with u_n such that $a_{n,j}(u_n, 1/2) = 1$. From (6), we obtain

$$\gamma := t/r = \frac{\ln(1-u_n) - \ln u_n}{\ln(u_n + 1/2) - \ln(1/2 - u_n)}.$$

If $n \to m$ with $m_1 = 0$ and/or $m_3 = \infty$, we must have $\gamma \to \infty$ and, by the above equation, this implies that $u_n \to 0$. The second case can be handled similarly. ■

Lemma 3 is a particular case of the next lemma which states that $\Gamma_{n,j}$ depends continuously on n.

LEMMA 6.4. *For $n, m \in N$, $\operatorname{dist}(\Gamma_{n,j}, \Gamma_{m,j}) \to 0$ as $n \to m$.*

Proof. In view of Lemma 3, we may assume that $m \in (0, \infty)^3$. Moreover, it is sufficient to consider $j = (-1, 0)$. In this case, it follows from (7) that $\Gamma_{n,j} \to \Gamma_{m,j}$ pointwise, both curves being viewed as functions of u. By the uniform Lipschitz continuity of the curves, this implies the assertion of the lemma. ■

Lemmas 1–4 give a qualitative description of the boundary of Ω_n. We summarize the main features in the following result.

COROLLARY. (i) *$\partial\Omega_n$ consists of segments of the curves $\Gamma_{n,j}$ connecting the points in $J/2$ with the intersection points $z_{n,\sigma}$, $\sigma \in \$_3$.*

(ii) *$\partial\Omega_n$ is piecewise monotone and is Lipschitz continuous, uniformly in n.*

(iii) *Ω_n depends continuously on n in the Hausdorff topology.*

Note that this provides the proof of Proposition 5.1.

To give a few examples, we list in Figs. 6.1–6.3 all cases for which Ω_n has a piecewise linear boundary. Moreover, we have

$$\begin{aligned} \Omega_{(1,1,0)} &= \Omega_{(\infty,1,0)} = \Omega_{(1,\infty,0)} \\ \Omega_{(1,0,1)} &= \Omega_{(\infty,0,1)} = \Omega_{(1,0,\infty)} \\ \Omega_{(0,1,1)} &= \Omega_{(0,\infty,1)} = \Omega_{(0,1,\infty)}. \end{aligned} \tag{6.10}$$

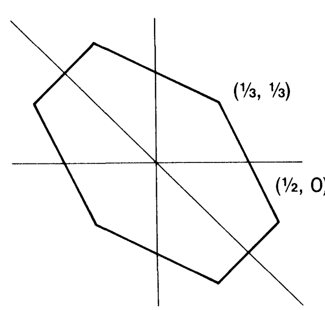

FIG. 6.1 $n = (1,1,1)$

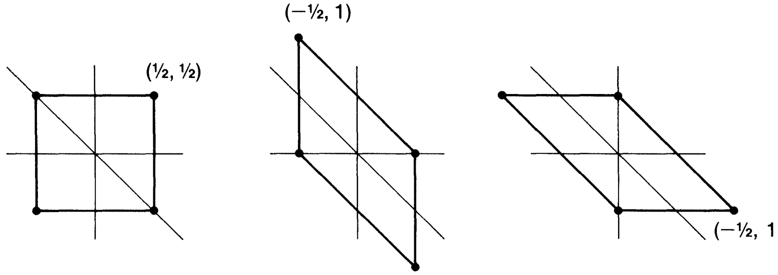

FIG. 6.2 $n = (1,1,0), (1,0,1), (0,1,1)$

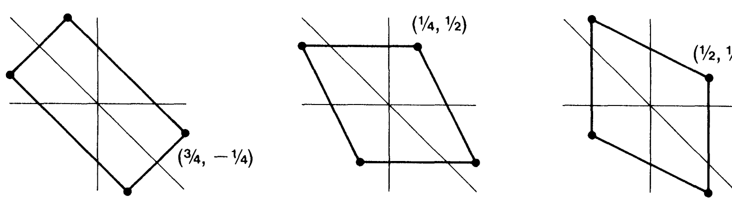

FIG. 6.3 $n = (1,1,\infty), (1,\infty,1), (\infty,1,1)$

We take the occasion to prove the following observation which stresses the underlying hexagonal structure.

PROPOSITION 6. $\Omega_0 := \cap_n \Omega_n = \operatorname{int}\operatorname{conv} J/2$, $\Omega_1 := \cup_n \Omega_n = \cup_{A \in \mathbf{A}_+} A^*[0,1/2)^2$.

Proof. We claim that, for any n,

$$\{(u,v)\colon u,v \geq 0,\ u+v < 1/2\} \subseteq R \cap \Omega_n \subseteq [0,1/2)^2. \tag{6.11}$$

This follows from Lemma 1, in particular from the fact that the curves $\Gamma_{n,(-1,0)}$ and $\Gamma_{n(0,-1)}$ pass through the points $(1/2,0)$ and $(0,1/2)$ and, as functions of u, have slopes ≤ -1 and ≥ -1, respectively. To complete the proof, note that $n = (0,1,1)$ gives equality in the first inclusion of (11) while $n = (1,1,0)$ gives equality in the second. ■

We are now also ready for:

Proof of Proposition 5.3. Because of the continuity of Ω_n as a function of n, it is sufficient to consider $n \in (0,\infty)^3$. In this situation, Figure 5.1 gives a qualitatively correct description of Ω_n. Because of the geometry of Ω_n and the

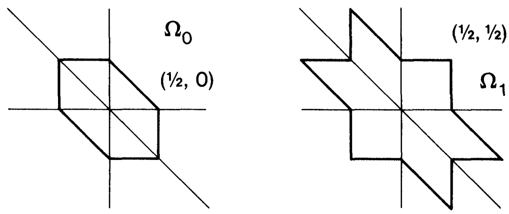

FIG. 6.4

symmetry relations (2), it is sufficient to establish the following claims:

(i) $j + \Gamma_{n,j} = \Gamma_{n,-j}$ for all $j \in J$.

(ii) The curve $(1,0) + \Gamma_{n,(1,-1)}$ passes through the point $z_{n,(1)}$.

The first assertion follows from the relation $a_{n,j}(x - j) = 1/a_{n,-j}(x)$ alluded to earlier and directly derivable from the definition (5.2) of $a_{n,j}$. As to (ii), note that

$$a_{n,(-1,0)}(z_{n,(1)}) = 1 = a_{n,(0,-1)}(z_{n,(1)})$$

implies that

$$1 = a_{n,(0,-1)}/a_{n,(-1,0)}(z_{n,(1)}) = a_{n,(1,-1)}(z_{n,(1)} - (1,0));$$

i.e., $z_{n,(1)} \in (1,0) + \Gamma_{n,(1,-1)}$. ■

The next three lemmas state various estimates for the functions $a_{n,j}$ needed for the proof of Proposition 5.2.

LEMMA 6.5. *For $n \in [0,\infty)^3$ with at most one component less than* 1, *we have*

$$a_{n,j}(x) \leq \left[1 + c\,\mathrm{dist}(x, \Gamma_{n,j} \cap \Omega_n^-)\right]^{-1}, \quad x \in \Omega_n^-, \quad j \in J, \quad (6.12)$$

with c a positive constant which does not depend on x, n, or j.

Proof. We may assume that $x := (u,v) \in R$, in particular that $u, v \in [0, 1/2]$. We consider each $j \in J$ separately and suppress all references to n.

(i) $j = (1,0)$. We have

$$a_j(x) = \left|\frac{u}{1+u}\right|^r \left|\frac{u+v}{1+u+v}\right|^t \leq 1/2.$$

This proves (12) since $\mathrm{dist}(x, \Gamma_{n,j} \cap \Omega_n^-) \leq 1$. For the estimate, we have used the fact that $\min\{r, t\} \geq 1$.

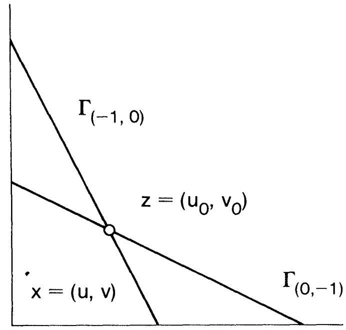

FIG. 6.5

The case $j = (0,1)$ is similar.

(ii) $j = (-1,0)$. Fig. 6.5 may be of help in following the argument. Let $i := (0,-1)$ and let $z =: (u_0, v_0) := z_{(1)}$ be the intersection of the two curves Γ_j and Γ_i. We consider two cases.

For $v \le v_0$, there exists

$$\varepsilon \ge \operatorname{dist}_\infty(x, \Gamma_j \cap \Omega^-)$$

such that $(u+\varepsilon, v) \in \Gamma_j$, i.e., $a_j(u+\varepsilon, v) = 1$. It follows that

$$\begin{aligned} a_j(u,v) &= a_j(u,v)/a_j(u+\varepsilon,v) \\ &= \left(\frac{u}{u+\varepsilon}\right)^r \left(\frac{1-u-\varepsilon}{1-u}\right)^r \left(\frac{u+v}{u+v+\varepsilon}\right)^t \left(\frac{1-u-v-\varepsilon}{1-u-v}\right)^t \\ &\le \left(\frac{u+v}{u+v+\varepsilon}\right)^{\max\{r,t\}} \\ &\le (1+\varepsilon)^{-1}. \end{aligned}$$

For $v \ge v_0$, we have $\operatorname{dist}_\infty(x, \Gamma_j \cap \Omega^-) = u_0 - u$. After possibly increasing v and therefore also increasing

$$a_j(x) = \left(\frac{u}{1-u}\right)^r \left(\frac{u+v}{1-u-v}\right)^t,$$

we may assume that $x \in \Gamma_i$, i.e., $a_i(x) = 1$. Using the fact that $a_j(z) = 1 = a_i(z)$, we obtain

$$a_j(x) = \frac{a_j(x)a_i(z)}{a_j(z)a_i(x)} = \left(\frac{u/(1-u)}{u_0/(1-u_0)}\right)^r \Big/ \left(\frac{v/(1-v)}{v_0/(1-v_0)}\right)^s.$$

For $0 \le p < q \le 1/2$, we have

$$\frac{q(1-p)}{p(1-q)} = 1 + \frac{q(1-p) - p(1-q)}{p(1-q)} = 1 + \frac{q-p}{p(1-q)} \ge 1 + 4(q-p).$$

This inequality implies that

$$1/a_j(x) \ge (1 + 4r|u - u_0|)\left(1 + 4s\left|\frac{v - v_0}{u - u_0}\right||u - u_0|\right). \tag{6.13}$$

This proves the desired estimate for $a_j(x)$ in case $r \ge 1$. If $r < 1$, we use the second factor of the product (13) together with the fact that $s|(v - v_0)/(u - u_0)|$ can be bounded below, uniformly in n and x. Since both (u, v) and (u_0, v_0) lie on the curve

$$\Gamma_i\colon \left(\frac{v}{1-v}\right)^s \left(\frac{u+v}{1-u-v}\right)^t = 1,$$

this last fact is established once we show that, on that curve,

$$\min_{v_0 \le v \le 1/2} s|dv/du| \ge c \tag{6.14}$$

for some positive c independent of n. For this, a direct calculation yields

$$1/|s\,dv/du| = \frac{1}{s} + \frac{1}{t}\frac{(u+v)(1-u-v)}{v(1-v)} \le 1 + \frac{1}{tv} \le 1 + \frac{1}{tv_0}.$$

Since, for $(u, v) \in \Gamma_i$,

$$s \ln\left|\frac{v}{1-v}\right| + t \ln\left|\frac{u+v}{1-u-v}\right| = 0,$$

and $u \in [0, 1/2]$, we conclude that, for $v_0 \le 1/4$,

$$tv_0 = -\frac{sv_0 \ln|v_0/(1-v_0)|}{\ln|(u_0+v_0)/(1-u_0-v_0)|} \ge v_0/\ln\left|\frac{1/2+v_0}{1/2-v_0}\right|,$$

and this shows (14).

The case $j = (0, -1)$ is treated similarly.

(iii) $j = (1, -1)$. We have to show that

$$\left|\frac{u+1}{u}\right|^r \left|\frac{1-v}{v}\right|^s \ge 1 + c\,\mathrm{dist}\big((u,v), \Gamma_j \cap \Omega^-\big), \tag{6.15}$$

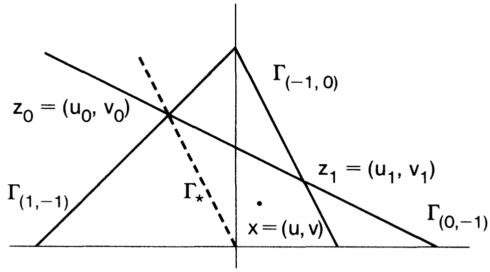

FIG. 6.6

and this is obvious for $r \geq 1$ in view of our assumption $(u, v) \in R \cap \Omega^-$. To prove (15) for $r < 1$, let $i := (-1, 0)$ and consider the situation as depicted in Figure 6.6. We may assume that $x = (u, v) \in \partial\Omega \cap R$, since increasing u decreases $|(u+1)/u|$ and increases $\operatorname{dist}(x, \Gamma_j \cap \Omega^-)$. For $x \in (\Gamma_{(-1,0)} \cup \Gamma_{(0,-1)}) \cap \Omega^-$, we have

$$\operatorname{dist}(x, \Gamma_j \cap \Omega^-) = \operatorname{dist}(x, z_0) = u - u_0. \tag{6.16}$$

This follows because Γ_j has nonnegative slope as a function of u and passes through the point $(-1/2, 1/2)$, while $(\Gamma_{(-1,0)} \cup \Gamma_{(0,-1)}) \cap \Omega^-$ is contained in the triangle spanned by $(1/2, 0), (1/2, 1/2), (0, 1/2)$. Also note that (cf. Figure 6.6)

$$u_0 = -u_1, \qquad v_0 - 1/2 = 1/2 - v_1. \tag{6.17}$$

This is a consequence of the radial symmetry of the curves Γ_i with respect to the point $-i/2$. In Figure 6.6, the curve Γ_* which passes through z_0 is meant to be the curve $\Gamma_{(-1,0)} - j$. For the proof of (15), we consider the two cases.

(a) $x \in \Gamma_{(-1,0)} \cap \Omega^-$; i.e., $u_1 \leq u$. In this case, from $a_{(-1,0)}(x) = 1$ we obtain the estimate

$$\begin{aligned} 1/a_j(x) - a_{(-1,0)}(x)/a_j(x) &= \left|\frac{u+1}{1-u}\right|^r \left|\frac{1-v}{v}\right|^s \left|\frac{u+v}{1-u-v}\right|^t \\ &\geq \left|\frac{1-v}{v}\right| \left|\frac{u+v}{1-u-v}\right| \\ &\geq 1 + u, \end{aligned}$$

where we have used the fact that $s, t \geq 1$, and the last inequality is easily checked. Since

$$|u - u_0| \leq |u| + |u_0| \leq 2|u|,$$

this proves (15) for this case.

(b) $x \in \Gamma_{(0,-1)} \cap \Omega^-$, i.e., $0 \leq u \leq u_1 = |u_0|$. First we assume that

$$v_0 - 1/2 \geq |u_0|/3. \tag{6.18}$$

Since $z_0 \in \Gamma_j$, we have

$$\begin{aligned} a_j(x) &= a_j(z_0)/a_j(x) \\ &= \left|\frac{(u+1)/u}{(u_0+1)/u_0}\right|^r \left|\frac{(1-v)/v}{(1-v_0)/v_0}\right|^s \\ &\geq \left|1 + \frac{(v_0 - v)/(vv_0)}{(1-v_0)/v_0}\right| \\ &\geq 1 + (v_0 - v). \end{aligned}$$

In view of $|u - u_0| \leq 2|u_0| \leq 6(v_0 - 1/2) \leq 6(v_0 - v)$, this proves (15) under the assumption (18).

Next, suppose that

$$v_0 - 1/2 \leq |u_0|/3. \tag{6.19}$$

We claim that

$$|(u_0 + 1)/u_0|^r \geq 1 + c|u_0|, \tag{6.20}$$

and this finishes the proof of (15) for this case, in view of (16) and the inequalities

$$1/a_j(x) \geq \left|\frac{u+1}{u}\right|^r \geq \left|\frac{u_0+1}{u_0}\right|^r \geq 1 + c|u_0| \geq 1 + (c/2)|u - u_0|.$$

To prove (20), we use the fact that $z_0 \in \Gamma_* = \Gamma_{(-1,0)} - j$ and therefore

$$|(u_0 + 1)/u_0|^r |(u_0 + v_0)/(1 - u_0 - v_0)|^t = 1.$$

Solving for r, we obtain

$$\begin{aligned} r &= t\frac{\ln|(u_0 + v_0)/(1 - u_0 - v_0)|}{\ln|(u_0 + 1)/u_0|} \\ &\geq -\ln\left|\frac{1/2 - (2/3)|u_0|}{1/2 + (2/3)|u_0|}\right| \Big/ \ln|(u_0 + 1)/u_0|. \end{aligned}$$

Here we used the assumption (19) and the fact that $t \geq 1$. Therefore, we have

$$r \ln |(u_0 + 1)/u_0| \geq \ln \left| \frac{1 + (4/3)|u_0|}{1 - (4/3)|u_0|} \right| \geq \ln |1 + (4/3)|u_0||$$

which establishes (20). This completes the case $j = (1, -1)$.

The case $j = (-1, 1)$ is treated similarly. ■

From the statement of Proposition 5.2, recall the definition

$$J' := \mathbf{A}^*(1,1) = \{\pm(1,1), \pm(2,-1), \pm(-1,2)\}.$$

LEMMA 6.6. *For $n \in [0, \infty)^3 \cap N$, $x \in \Omega_n^-$, and $j \in J'$*

$$a_{n,j}(x) \leq [1 + C \operatorname{dist}(x, -j/2)]^{-1},$$

with C a positive constant which does not depend on n, x, or j.

Proof. Assume without loss that $x = (u, v) \in R$.

First we consider $j = (-1, -1)$. For example, assume that

$$1/2 - u = \operatorname{dist}_\infty(x, (1,1)/2) =: \varepsilon.$$

Then we have

$$\begin{aligned} a_{(-1,-1)}(x) &= \left| \frac{u}{1-u} \right|^r \left| \frac{v}{1-v} \right|^s \left| \frac{u+v}{2-u-v} \right|^t \\ &\leq \left| \frac{1/2 - \varepsilon}{1/2 + \varepsilon} \right|^r \left| \frac{1-\varepsilon}{1+\varepsilon} \right|^t \\ &\leq 1/(1+\varepsilon). \end{aligned}$$

In the remaining cases $j \in J' \setminus (-1, -1)$, we have $\operatorname{dist}_\infty(-j/2, R) \leq 2$ and therefore it is sufficient to bound $a_j(x)$ by an absolute constant less than 1. This is straightforward, using the fact that $a_{(-1,0)}(x), a_{(0,-1)}(x) \leq 1$. We list only the estimates:

For $j = (1, 1)$,

$$\left| \frac{u}{u+1} \right|^r \left| \frac{v}{v+1} \right|^s \left| \frac{u+v}{u+v+2} \right|^t \leq 3^{-r} 3^{-s} 2^{-t}.$$

For $j = (2, -1)$,

$$\begin{aligned} a_{(2,-1)}(x) &\leq a_{(2,-1)}(x)/a_{(0,-1)}(x) \\ &= \left| \frac{u}{2+u} \right|^r \left| \frac{1-u-v}{2-u-v} \right|^t \\ &\leq 5^{-r} 2^{-t}. \end{aligned}$$

For $j = (-2,1)$,

$$\begin{aligned} a_{(-1,2)}(x) &\le a_{(-1,2)}(x)/a_{(-1,0)}(x) \\ &= \left|\frac{v}{2+v}\right|^s \left|\frac{1-u-v}{2-u-v}\right|^t \\ &\le 5^{-s}2^{-t}. \end{aligned}$$

The remaining two cases, $j = (-2,1), (1,-2)$ are similar. ∎

LEMMA 6.7. *For $n \in N$, $x \in \Omega_n^-$, and $j \in \mathbf{Z}^2 \setminus (0 \cup J \cup J')$,*

$$a_{n,j}(x) \le [1 + C|j|]^{-1},$$

with C a positive constant which does not depend on n, x, or j.

Proof. Let $j = (k,l)$ and $x = (u,v)$ and, without loss of generality, let $x \in R$; hence $u, v \in [0, 1/2]$. We consider several cases.

(i) $k, l \ne 0, -1$, and $k + l \ne 0, -1, -2$. Then

$$\left|\frac{u}{u+k}\right| \le (1+|k|)^{-1}, \qquad \left|\frac{v}{v+l}\right| \le (1+|l|)^{-1},$$
$$\left|\frac{u+v}{u+v+k+l}\right| \le (1+|k+l|/3)^{-1}.$$

Using the facts that

$$(1+p)^{-1}(1+q)^{-1} \le (1+p+q)^{-1} \quad \text{for } p, q \ge 0$$

and

$$|p| + |p+q| \ge \max\{|p|, |q|\},$$

we see that the product of any two of the above lefthand terms is bounded by

$$(1 + |j|_\infty/3)^{-1}.$$

Since at most one of r, s, t is less than 1, this yields

$$a_{n,j}(x) \le (1+|k|)^{-r}(1+|l|)^{-s}(1+|k+l|/3)^{-t} \le (1+|j|_\infty/3)^{-1}.$$

(ii) $j = (0,-2)$. Since $x \in \Omega_n$, we have

$$a_{n,(0,-1)}(x) = \left|\frac{v}{1-v}\right|^s \left|\frac{u+v}{1-u-v}\right|^t < 1$$

and therefore

$$\begin{aligned} a_{n,j}(x) &< a_{n,j}(x)/a_{n,(0,-1)}(x) \\ &= \left|\frac{1-v}{2-v}\right|^s \left|\frac{1-u-v}{2-u-v}\right|^t \\ &\le 2^{-s}2^{-t} \\ &\le 1/2 \\ &= (1+|j|_\infty/2)^{-1}. \end{aligned}$$

(iii) $k=0,\ l \neq -2,-1,0$. Here

$$\begin{aligned} a_{n,j}(x) &= \left|\frac{v}{v+l}\right|^s \left|\frac{u+v}{u+v+l}\right|^t \\ &\le (1+|l|)^{-s}(1+|l|/3)^{-t} \\ &\le (1+|j|_\infty/3)^{-1}. \end{aligned}$$

(iv) $k=-1,\ l \neq -1,0,1,2$. Here

$$\begin{aligned} \left|\frac{u}{1-u}\right|^r \left|\frac{v}{v+l}\right|^s \left|\frac{u+v}{u+v+l-1}\right|^t &\le 1\cdot(1+|l|)^{-s}(1+|l-1|/3)^{-t} \\ &\le (1+|j|_\infty/5)^{-1}. \end{aligned}$$

(v) $k=0,-1$ and $l=0,-1$. Treated in analogy to cases (ii)–(iv).
(vi) $l=-k,\ k\neq -1,0,1$. Here

$$\begin{aligned} a_{n,j}(x) &= \left|\frac{u}{u+k}\right|^r \left|\frac{v}{v-k}\right|^s \\ &\le (1+|k|)^{-r}(1+|k|)^{-s} \\ &\le (1+|j|_\infty)^{-1}. \end{aligned}$$

(vii) $l=-k-1,\ k\neq -2,-1,0,1$. Here

$$\begin{aligned} a_{n,j}(x) &< a_{n,j}(x)/a_{n,(-1,0)}(x) \\ &\le \left|\frac{1-u}{u+k}\right|^r \left|\frac{v}{v-k-1}\right|^s \\ &\le |k|^{-r}(1+|k-1|)^{-s} \\ &\le (1+|k|/2)^{-1} \\ &\le (1+|j|_\infty/3)^{-1}. \end{aligned}$$

(viii) $l = -k-2,\ k \neq -2, -1, 0$. Here

$$\begin{aligned} a_{n,j}(x) &\leq a_{n,j}(x)/a_{n,(-1,0)}(x) \\ &= \left|\frac{u-1}{u+k}\right|^r \left|\frac{v}{v-k-2}\right|^s \left|\frac{u+v-1}{u+v-2}\right|^t \\ &\leq (1+|k|/3)^{-1} \\ &\leq (1+|j|_\infty/9)^{-1}. \end{aligned}$$

This covers all $j \in \mathbf{Z}^2 \setminus (0 \cup J \cup J')$. ■

Proof of Proposition 5.2. Lemmas 5–7 prove (5.4) for $n \in N \cap [0,\infty)^3$, with the constants independent of n. Since Ω_n depends continuously on $n \in N$ (by Proposition 5.1), this proves (5.4) for all $n \in N$. ■

Proof of Proposition 5.4. To prove (5.6), we consider three cases.

(i) $j \in J$. From the geometry of the set $\Omega_n = \Omega_{n'}$ (cf. Figure 5.1 which gives a qualitatively accurate description of the general situation) and, in particular, from the estimates of the slopes of the curves $\Gamma_{n,j}$, we can see that in this case

$$\operatorname{dist}(x, \partial\Omega_n) = \operatorname{dist}\left(x', \Gamma_{n,j} \cap \Omega_n^-\right).$$

Therefore, (5.6) is a consequence of Lemma 6.5 in this case.

(ii) $j \in J'$. For example, assume that $x' \in R \cap \Omega_n$. Then, for $j \in J' \setminus (-1,-1)$, we have

$$\operatorname{dist}_\infty(x', -j/2) \geq 1/2$$

and (5.6) follows from Lemma 6.6 since $\operatorname{dist}_\infty(x, \partial\Omega_n) \leq C$.

It remains to consider the case $j = (-1,-1)$. From the bounds on the slopes of the curves $\Gamma_{n,(1,0)}, \Gamma_{n,(0,1)}$, we see that Ω_n lies in the half space

$$\left\{y\colon y^*(1,1) \geq -(u_0, v_0)^*(1,1)\right\}$$

where $(u_0, v_0) := z_{(1)}$ is the point of intersection of the curves $\Gamma_{n,(-1,0)}$ and $\Gamma_{n,(0,-1)}$. Since $\Omega_n \cap R \subseteq [0,1/2]^2$, it follows that, for $x' \in \Omega_n \cap R$,

$$\operatorname{dist}_1(x, \partial\Omega_n) = \operatorname{dist}_1\left(x', -z_{(1)} + (1,1)\right). \tag{6.21}$$

Here, dist_1 denotes the l_1-distance. Moreover, in view of

$$\left[-z_{(1)} + (1,1)\right] - (1/2,1/2) = (1/2,1/2) - z_{(1)},$$

we have

$$\operatorname{dist}_1\bigl(x', -z_{(1)} + (1,1)\bigr) = \operatorname{dist}_1\bigl(x', z_{(1)}\bigr) + 2\operatorname{dist}_1\bigl(z_{(1)}, (1/2, 1/2)\bigr)$$
$$\leq 2\operatorname{dist}_1\bigl(x', -j/2\bigr). \tag{6.22}$$

This, together with (21) and Lemma 6, proves (5.6) for this case.

(iii) $j \in \mathbf{Z}^2 \setminus (0 \cup J \cup J')$. In this case, (5.6) follows from Lemma 7 since, for any $j \neq 0$,

$$\operatorname{dist}(x, \partial\Omega_n) \leq C|j|.$$

This completes the proof of Proposition 5.4. ∎

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UNIVERSITY OF WISCONSIN
MADISON, WISCONSIN

UNIVERSITY OF ALBERTA
EDMONTON, CANADA