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VERDIER AND STRICT THOM STRATIFICATIONS IN O-MINIMAL STRUCTURES

TA LÊ LOI

0. Introduction

0.1 DEFINITION. An *o-minimal structure* on the real field $(\mathbf{R}, +, \cdot)$ is a family $\mathcal{D} = (\mathcal{D}_n)_{n \in \mathbf{N}}$ such that for each $n \in \mathbf{N}$:

(1) $\mathcal{D}_n$ is a boolean algebra of subsets of $\mathbf{R}^n$.
(2) If $A \in \mathcal{D}_n$, then $A \times \mathbf{R}$ and $\mathbf{R} \times A$ belong to $\mathcal{D}_{n+1}$.
(3) If $A \in \mathcal{D}_{n+1}$, then $\pi(A) \in \mathcal{D}_n$, where π: $\mathbf{R}^{n+1} \longrightarrow \mathbf{R}^n$ is the projection on the first n coordinates.
(4) $\mathcal{D}_n$ contains $\{x \in \mathbf{R}^n\colon P(x) = 0\}$ for every polynomial $P \in \mathbf{R}[X_1, \dots, X_n]$.
(5) Each set belonging to $\mathcal{D}_1$ is a finite union of intervals and points. (o-minimality)

A set belonging to $\mathcal{D}$ is called *definable* (in this structure). *Definable maps* are maps whose graphs belong to $\mathcal{D}$.

Many results in Semialgebraic Geometry and Subanalytic Geometry hold true for o-minimal structures on the real field. Recently, o-minimality of many interesting structures on $(\mathbf{R}, +, \cdot)$ has been established, for example, structures generated by the exponential function [W1](see also [LR] and [DM1]), real power functions [M2], Pfaffian functions [W2] or functions defined by multisummable powerseries [DS]. For more details on o-minimal structures we refer the readers to [D] and [DM2] (compare with [S]).

We now outline the main results of this paper. Let $\mathcal{D}$ be an o-minimal structure on $(\mathbf{R}, +, \cdot)$. In Section 1, we prove that the definable sets of $\mathcal{D}$ admit Verdier Stratification. We also show that the Verdier condition (w) implies the Whitney condition (b) in $\mathcal{D}$. Note that the theorems were proved for subanalytic sets in [V] and [LSW] (see also [DW]), the former based on Hironaka's Desingularization, and the latter on Puiseux's Theorem. But, in general, these tools cannot be applied to sets belonging to o-minimal structures (e.g., to the set $\{(x, y) \in \mathbf{R}^2\colon y = \exp(-1/x), x > 0\}$ in the structure generated by the exponential function). Section 2 is devoted to the study of stratifications of definable functions. In general, definable functions cannot be stratified to satisfy the strict Thom condition (w_f). However, if $\mathcal{D}$ is polynomially

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bounded, then its definable functions admit (w_f)-stratification. Our proof of this assertion is based on piecewise uniform asymptotics for definable functions from [M2], instead of Pawlucki's version of Puiseux's theorem with parameters, which is used in [KP] to prove the assertion for subanalytic functions.

Notations and conventions. Throughout this paper, let $\mathcal{D}$ denote some fixed, but arbitrary, o-minimal structure on $(\mathbf{R}, +, \cdot)$. *Definable* means definable in $\mathcal{D}$. If $\mathbf{R}^k \times \mathbf{R} \ni (y, t) \mapsto f(y, t) \in \mathbf{R}^m$ is a differentiable function, then $D_1 f$ denotes the derivative of f with respect to the first variables y. As usual, $d(\cdot, \cdot)$, $\| \cdot \|$ denote the Euclidean distance and norm respectively. We will often use Cell Decomposition [DM2, Th. 4.2], and Definable Choice [DM2, Th. 4.5] in our arguments without citations. Submanifolds will always be embedded submanifolds.

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1. Verdier stratifications

1.1. *Verdier condition.* Let Γ, Γ' be C^1 submanifolds of $\mathbf{R}^n$ such that $\Gamma \subset \overline{\Gamma'} \setminus \Gamma'$. Let y_0 be a point of Γ. We say that the pair (Γ, Γ') satisfies the Verdier condition at y_0 if the following holds:

(w) There exist a constant $C > 0$ and a neighborhood U of y_0 in $\mathbf{R}^n$ such that

$$\delta(T_y\Gamma, T_x\Gamma') \leq C\|x - y\| \quad \text{for all } x \in \Gamma' \cap U, y \in \Gamma \cap U,$$

where $T_y\Gamma$ denotes the tangent space of Γ at y, and

$$\delta(T, T') = \sup_{v \in T, \|v\|=1} d(v, T')$$

is the distance of vector subspaces of $\mathbf{R}^n$.

Note that (w) is invariant under C^2-diffeomorphisms.

1.2 DEFINITION. Let p be a positive integer. A *definable C^p stratification* of $\mathbf{R}^n$ is a partition $\mathcal{S}$ of $\mathbf{R}^n$ into finitely many subsets, called strata, such that:

(S1) Each stratum is a connected C^p submanifold of $\mathbf{R}^n$ and also a definable set.
(S2) For every $\Gamma \in \mathcal{S}$, $\overline{\Gamma} \setminus \Gamma$ is a union of strata.

We say that $\mathcal{S}$ is *compatible with* a class $\mathcal{A}$ of subsets of $\mathbf{R}^n$ if each $A \in \mathcal{A}$ is a finite union of some strata in $\mathcal{S}$.

A *definable* C^p *Verdier stratification* is a definable C^p stratification $\mathcal{S}$ such that for all $\Gamma, \Gamma' \in \mathcal{S}$, if $\Gamma \subset \overline{\Gamma'} \setminus \Gamma'$, then (Γ, Γ') satisfies the condition (w) at each point of Γ.

1.3 THEOREM (VERDIER STRATIFICATION). *Let p be a positive integer. Then given definable sets $A_1, \dots, A_k$ in $\mathbf{R}^n$, there exists a definable C^p Verdier stratification of $\mathbf{R}^n$ compatible with $\{A_1, \dots, A_k\}$.*

We first make an observation similar to that of [LSW]. Let (P) be a local property of pairs (Γ, Γ') at points y in Γ, where Γ, Γ' are subsets of $\mathbf{R}^n$, and where "local" means that if U is an open neighborhood of y, then (Γ, Γ') has property (P) at y if and only if $(\Gamma \cap U, \Gamma' \cap U)$ has property (P) at y. Let $P(\Gamma, \Gamma') = \{y \in \Gamma\colon (\Gamma, \Gamma') \text{ satisfies (P) at } y\}$.

1.4 PROPOSITION. *Suppose that for every pair (Γ, Γ') of definable C^p submanifolds of $\mathbf{R}^n$ with $\Gamma \subset \overline{\Gamma'} \setminus \Gamma'$ and $\Gamma \neq \emptyset$, the set $P(\Gamma, \Gamma')$ is definable and* $\dim(\Gamma \setminus P(\Gamma, \Gamma')) < \dim \Gamma$. *Then given definable sets $A_1, \dots, A_k$ contained in $\mathbf{R}^n$, there exists a definable C^p stratification $\mathcal{S}$ of $\mathbf{R}^n$ compatible with $\{A_1, \dots, A_k\}$ such that*

(*P*) $P(\Gamma, \Gamma') = \Gamma$ *for all* $\Gamma, \Gamma' \in \mathcal{S}$ *with* $\Gamma \subset \overline{\Gamma'} \setminus \Gamma'$.

Proof. Similar to the proof of [LSW, Prop. 2]. □

By the proposition, Theorem 1.3 is a consequence of the following.

1.5 PROPOSITION. *Let Γ, Γ' be definable C^p-submanifolds of $\mathbf{R}^n$. Suppose that $\Gamma \subset \overline{\Gamma'} \setminus \Gamma'$ and $\Gamma \neq \emptyset$. Then* $W = \{y \in \Gamma\colon (\Gamma, \Gamma') \text{ satisfies (w) at } y\}$ *is definable, and* $\dim(\Gamma \setminus W) < \dim \Gamma$.

To prove Proposition 1.5 we prepare some lemmas.

1.6 LEMMA. *Under the notation of Proposition* 1.5, *W is a definable set.*

Proof. Note that the Grassmannian $G_k(\mathbf{R}^n)$ of k-dimensional linear subspaces of $\mathbf{R}^n$ is semialgebraic, and hence definable; δ and the tangent map: $\Gamma \ni x \mapsto T_x\Gamma \in G_{\dim \Gamma}(\mathbf{R}^n)$ are also definable. Therefore,

$$W = \{y_0\colon \quad y_0 \in \Gamma, \exists C > 0, \exists t > 0, \forall x \in \Gamma', \forall y \in \Gamma \\ (\|x - y_0\| < t, \|y - y_0\| < t \Rightarrow \delta(T_y\Gamma, T_x\Gamma') \leq C\|x - y\|)\}$$

is a definable set. □

1.7 LEMMA (WING LEMMA). *Let $V \subset \mathbf{R}^k$ be a nonempty open definable set, and $S \subset \mathbf{R}^k \times \mathbf{R}^l$ be a definable set. Suppose $V \subset \overline{S} \setminus S$. Then there exist a nonempty open subset U of V, $\alpha > 0$, and a definable map $\bar{\rho}$: $U \times (0, \alpha) \longrightarrow S$, of class C^p, such that $\bar{\rho}(y, t) = (y, \rho(y, t))$ and $\|\rho(y, t)\| = t$, for all $y \in U, t \in (0, \alpha)$.*

Proof. Similar to the proof of [L1, Lemma 2.7] □

To control the tangent spaces we need the following lemma.

1.8 LEMMA. *Let $U \subset \mathbf{R}^k$ be a nonempty open definable set, and M: $U \times (0, \alpha) \longrightarrow \mathbf{R}^l$ be a C^1 definable map. Suppose there exists $K > 0$ such that $\|M(y, t)\| \leq K$, for all $y \in U$ and $t \in (0, \alpha)$. Then there exists a definable set F, closed in U with $\dim F < \dim U$, and continuous definable functions C, τ: $U \setminus F \longrightarrow \mathbf{R}_+$, such that*

$$\|D_1 M(y, t)\| \leq C(y)\ , \quad \text{for all } y \in U \setminus F \text{ and } t \in (0, \tau(y)).$$

Proof. It suffices to prove this for $l = 1$. Suppose the assertion of the lemma is false. Since $\{y \in U: \lim_{t \to 0^+} \|D_1 M(y, t)\| = +\infty\}$ is definable, there is an open subset B of U, such that

$$\lim_{t \to 0^+} \|D_1 M(y, t)\| = +\infty, \quad \text{for all } y \text{ in } B.$$

By monotonicity [DM2, Th. 4.1], for each $y \in B$, there is $s > 0$ such that $t \mapsto \|D_1 M(y, t)\|$ is strictly decreasing on $(0, s)$. Let

$$\tau(y) = \sup\{s: \|D_1 M(y, \cdot)\| \text{ is strictly decreasing on } (0, s)\}.$$

Note that τ is a definable function, and, by Cell Decomposition, τ is continuous on an open subset B' of B, and $\tau > \alpha'$ on B', for some $\alpha' > 0$. Let $\psi(t) = \inf\{\|D_1 M(y, t)\|: y \in B', 0 < t < \alpha'\}$. Shrinking B', we can assume that $\lim_{t \to 0^+} \psi(t) = +\infty$. Then, for each $y \in B'$, we have

$$\|D_1 M(y, t)\| > \psi(t)\ , \quad \text{for all } t \in (0, \alpha').$$

This implies $|M(y, t) - M(y', t)| > \psi(t)\|y - y'\|$, for all $y, y' \in B'$, and $t < \alpha'$. Therefore, $\psi(t) \leq \frac{2K}{\operatorname{diam} B'}$, for all $t \in (0, \alpha')$, a contradiction. □

1.9 *Proof of Proposition* 1.5. The first part of the proposition was proved in Lemma 1.6. To prove the second part we suppose, contrary to the assertion, that $\dim(\Gamma \setminus W) = \dim \Gamma = k$.

Since (w) is a local property and invariant under C^2 local diffeomorphisms, we can suppose Γ is an open subset of $\mathbf{R}^k \subset \mathbf{R}^k \times \mathbf{R}^{n-k}$. In this case $T_y\Gamma = \mathbf{R}^k$, for all $y \in \Gamma$. Then by the assumption, applying Lemma 1.7, we get an open subset U of

Γ, a C^p definable map $\bar{\rho}$: $U \times (0, \alpha) \longrightarrow \Gamma'$ such that $\bar{\rho}(y, t) = (y, \rho(y, t))$ and $\|\rho(y, t)\| = t$, and, moreover, for each $y \in U$,

$$\frac{\delta(\mathbf{R}^k, T_{(y,\rho(y,t))}\Gamma')}{\|\rho(y, t)\|} \to +\infty \quad \text{when } t \to 0^+.$$

On the other hand, applying Lemma 1.8 to $M(y, t) := \frac{\rho(y,t)}{t}$ and shrinking U and α, we have

$$\|D_1\rho(y, t)\| \le Ct, \quad \text{for all } y \in U, t \in (0, \alpha),$$

with some $C > 0$. Note that $T_{(y,\rho(y,t))}\Gamma' \supset \operatorname{graph} D_1\rho(y, t)$. Therefore,

$$\frac{\delta(\mathbf{R}^k, T_{(y,\rho(y,t))}\Gamma')}{\|\rho(y, t)\|} \le \frac{\|D_1\rho(y, t)\|}{\|\rho(y, t)\|} \le C \quad \text{for } y \in U, 0 < t < \alpha.$$

This is a contradiction. □

Note that Whitney's condition (b) (defined in [Wh]) does not imply condition (w), even for algebraic sets (see [BT]). And, in general, we do not have (w) $\Rightarrow$ (b) (e.g., $\Gamma = (0, 0)$, $\Gamma' = \{(x, y) \in \mathbf{R}^2\colon x = r\cos r, y = r\sin r, r > 0\}$, or $\Gamma' = \{(x, y) \in \mathbf{R}^2\colon y = x\sin(1/x), x > 0\}$). In o-minimal structures such spiral phenomena or oscillation cannot occur. The following is a version of Kuo-Verdier's Theorem (see [K] and [V]).

1.10 PROPOSITION. *Let $\Gamma, \Gamma' \subset \mathbf{R}^n$ be definable C^p-submanifolds* ($p \ge 2$), *with* $\Gamma \subset \overline{\Gamma'} \setminus \Gamma$. *If* (Γ, Γ') *satisfies the condition* (w) *at* $y \in \Gamma$, *then it satisfies the Whitney condition* (b) *at* y.

Proof. Our proof is an adaptation of [V, Theorem 1.5] and based on the following observation: If f: $(0, \alpha) \longrightarrow \mathbf{R}$ is definable with $f(t) \ne 0$, for all t, and $\lim_{t\to 0^+} f(t) = 0$, then, by Cell Decomposition and monotonicity [DM2. Th.4.1], there is $0 < \alpha' < \alpha$, such that f is of class C^1 and strictly monotone on $(0, \alpha')$. By the Mean Value Theorem and Definable Choice, there exists a definable function θ: $(0, \alpha') \to (0, \alpha')$ with $0 < \theta(t) < t$, such that $f(t) = f'(\theta(t))t$. Since $|f(t)| > |f(\theta(t))|$, by monotonicity, $\lim_{t\to 0^+} \frac{f(t)}{f'(t)} = 0$.

Now we prove the proposition. By a C^2 change of local coordinates, we can suppose that Γ is an open subset of $\mathbf{R}^k \subset \mathbf{R}^k \times \mathbf{R}^l$ ($l = n - k$), and $y = 0$. Let π: $\mathbf{R}^k \times \mathbf{R}^l \longrightarrow \mathbf{R}^l$ be the orthogonal projection. Since (Γ, Γ') satisfies (w) at 0, there exists $C > 0$ and a neighborhood U of 0 in $\mathbf{R}^n$, such that

$$(*) \qquad \delta(T_y\Gamma, T_x\Gamma') \le C\|x - y\|, \quad \text{for all } x \in \Gamma' \cap U, y \in \Gamma \cap U.$$

If the condition (b) is not satisfied at 0 for (Γ, Γ'), then there exists $\epsilon > 0$, such that $0 \in \overline{S} \setminus S$, where

$$S = \{x \in \Gamma'\colon \delta(\mathbf{R}\pi(x), T_x\Gamma') \ge 2\epsilon\}.$$

Since $S \cap \{x\colon \|x\| \leq t\} \neq \emptyset$, for all $t > 0$, by Curve selection [DM2, Th.4.6], there exists a definable curve $\varphi\colon (0,\alpha) \longrightarrow S$, such that $\|\varphi(t)\| \leq t$, for all t. By the above observation, we can assume φ is of class C^1. Write $\varphi(t) = (a(t), b(t)) \in \mathbf{R}^k \times \mathbf{R}^l$. Then $\|b'(t)\|$ is bounded. Since $\varphi((0,\alpha)) \subset \Gamma'$, $a \not\equiv 0$. Shrinking α, we can assume $a'(t) \neq 0$, for all t. Since $\lim_{t\to 0^+} a'(t)$ exists, we have $\delta(\mathbf{R}a'(t), \mathbf{R}a(t)) \to 0$, when $t \to 0$. Therefore

$$(**) \qquad \delta(\mathbf{R}a'(t), T_{\varphi(t)}\Gamma') \geq \epsilon\ , \text{ for all } t \text{ sufficiently small.}$$

On the other hand, we have

$$\begin{aligned}\delta(\mathbf{R}a'(t), T_{\varphi(t)}\Gamma') &= \frac{1}{\|a'(t)\|}\delta(a'(t), T_{\varphi(t)}\Gamma') = \frac{1}{\|a'(t)\|}\delta(b'(t), T_{\varphi(t)}\Gamma') \\ &\leq \frac{\|b'(t)\|}{\|a'(t)\|}\delta(\mathbf{R}b'(t), T_{\varphi(t)}\Gamma').\end{aligned}$$

From $(*)$ and $(**)$, we have $\epsilon \leq C\|a(t)\|\frac{\|b'(t)\|}{\|a'(t)\|}$. By the observation, the right-hand side of the inequality tends to 0 (when $t \to 0$), which is a contradiction. □

Note that Theorem 1.3 and Proposition 1.10 together yield an alternative proof of the Whitney Stratification Theorem for o-minimal structures on the real field in [DM2].

2. (w_f)-stratifications

Thoughout this section, let $X \subset \mathbf{R}^n$ be a definable set and $f\colon X \longrightarrow \mathbf{R}$ be a continuous definable function. Let p be a positive integer.

2.1 Definition. A *definable C^p stratification of f* is a definable C^p stratification $\mathcal{S}$ of $\mathbf{R}^n$ compatible with X, such that for every stratum $\Gamma \in \mathcal{S}$ with $\Gamma \subset X$, the restriction $f|_\Gamma$ is C^p and of constant rank.

For each $x \in \Gamma$, $T_{x,f}$ denotes the tangent space of the level of $f|_\Gamma$ at x, i.e. $T_{x,f} = \ker D(f|_\Gamma)(x)$.

Let $\Gamma, \Gamma' \in \mathcal{S}$ with $\Gamma \subset \overline{\Gamma'} \setminus \Gamma'$. We say that the pair (Γ, Γ') satisfies the *Thom condition* (a_f) at $y_0 \in \Gamma$ if and only if the following holds:

(a_f) For every sequence (x_k) in Γ', converging to y_0, we have

$$\delta(T_{y_0,f}, T_{x_k,f}) \longrightarrow 0.$$

We say that (Γ, Γ') satisfies the *strict Thom condition* (w_f) at y_0 if:

(w_f) There exist a constant $C > 0$ and a neighborhood U of y_0 in $\mathbf{R}^n$, such that

$$\delta(T_{y,f}, T_{x,f}) \le C\|x - y\| \quad \text{for all } x \in \Gamma' \cap U,\, y \in \Gamma \cap U.$$

Note that the conditions are C^2-invariant.

The existence of stratifications satisfying (w_f) (and hence (a_f)) for subanalytic functions was proved in [KP] (see also [B] and [KR]). For functions definable in o-minimal structures on the real field we have:

2.2 THEOREM. *There exists a definable C^p stratification of f satisfying the Thom condition* (a_f) *at every point of the strata.*

Proof. See [L2]. □

2.3 *Remark.* In general, definable functions cannot be stratified to satisfy the condition (w_f). The following example is given by Kurdyka.

Let $f\colon (a,b) \times [0,+\infty) \longrightarrow \mathbf{R}$ be defined by $f(x,y) = y^x$ $(0 < a < b)$. Let $\Gamma = (a,b) \times 0$, and $\Gamma' = (a,b) \times (0,+\infty)$. Then the fiber of $f|_{\Gamma'}$ over $c \in \mathbf{R}_+$ equals

$$\left\{\left(x, y(x) = \exp\left(-\frac{1}{tx}\right)\right) : x \in (a,b)\right\}, \quad t = -\frac{1}{\ln c}.$$

Then $\frac{y'(x)}{y(x)} = \frac{1}{tx^2} \to +\infty$, when $t \to 0^+$, for all $x \in (a,b)$, i.e., $\frac{\delta(T_{x,f}, T_{(x,y(x)),f})}{\|y(x)\|}$ cannot be locally bounded along Γ.

The remainder of this section is devoted to the proof of the existence of (w_f)-stratification of functions definable in polynomially bounded o-minimal structures.

2.4 DEFINITION. A structure $\mathcal{D}$ on the real field $(\mathbf{R}, +, \cdot)$ is *polynomially bounded* if for every function $f\colon \mathbf{R} \longrightarrow \mathbf{R}$ definable in $\mathcal{D}$, there exists $N \in \mathbf{N}$ such that

$$|f(t)| \le t^N \quad \text{for all sufficiently large } t.$$

For example, the structure of global subanalytic sets, the structure generated by real power functions [M2], or by functions given by multisummable powerseries [DS] are polynomially bounded.

2.5 THEOREM. *Suppose that $\mathcal{D}$ is polynomially bounded. Then there exists a definable C^p stratification of f satisfying the condition* (w_f) *at each point of the strata.*

Note. The converse of the theorem is also true: If $\mathcal{D}$ is not polynomially bounded, then it must contain the exponential function, by [M1]. So the function given in Remark 2.3 is definable in $\mathcal{D}$ and cannot be (w_f)-stratified.

2.6 PROPOSITION. *There exists a definable C^p stratification of f.*

Proof (cf. [DM2, Th. 4.8]). First note that if $f\colon \Gamma \longrightarrow \mathbf{R}^l$ is a C^1 definable map on a C^1-submanifold Γ of $\mathbf{R}^n$, then the set

$$P = \{y \in \Gamma\colon \exists t > 0, \forall x \in \Gamma(\|x - y\| < t \Rightarrow \operatorname{rank} f(x) = \operatorname{rank} f(y))\}$$

is definable and $\dim(\Gamma \setminus P) < \dim \Gamma$.

Therefore, applying Proposition 1.4, we have a C^p stratification of f. □

By the previous proposition and Proposition 1.4, Theorem 2.5 is implied by the following.

2.7 PROPOSITION. *Suppose that $\mathcal{D}$ is polynomially bounded. Let Γ, Γ' be definable C^p submanifolds of $\mathbf{R}^n$. Suppose $\Gamma \subset \overline{\Gamma'} \setminus \Gamma'$, $\Gamma \neq \emptyset$, and $f\colon \Gamma \cup \Gamma' \longrightarrow \mathbf{R}$ is a continuous definable function such that $f|_\Gamma$ and $f|_{\Gamma'}$ have constant rank. Then*

(i) *$W_f = \{x \in \Gamma\colon$ (w_f) is satisfied at $x\}$ is definable, and*

(ii) $\dim(\Gamma \setminus W_f) < \dim \Gamma$.

Proof. The proof is much the same as that for the condition (a_f) in [L2].

(i) Since $x \mapsto D(f|_\Gamma)$ is a definable map (see [DM2]), the kernel bundle of $f|_\Gamma$ is definable. Therefore,

$$\begin{aligned} W_f = \{&y_0\colon y_0 \in \Gamma, \exists C > 0, \exists t > 0, \forall x \in \Gamma', \forall y \in \Gamma \\ &\|x - y_0\| < t, \|y - y_0\| < t \Rightarrow \delta(\ker D(f|_\Gamma)(y), \ker D(f|_{\Gamma'})(x) \\ &\leq C\|x - y\|\} \end{aligned}$$

is definable.

(ii) To prove the second assertion there are three cases to consider.

Case 1. $\operatorname{rank} f|_\Gamma = \operatorname{rank} f|_{\Gamma'} = 0$. In this case

$$W_f = \{y \in \Gamma\colon (\Gamma, \Gamma') \text{ satisfies Verdier condition (w) at } y\}.$$

The assertion follows from Theorem 1.3.

Case 2. $\operatorname{rank} f|_\Gamma = 0$ and $\operatorname{rank} f|_{\Gamma'} = 1$. Suppose the contrary: $\dim(\Gamma \setminus W_f) < \dim \Gamma$. Since ($\mathrm{w}_f$) is C^2 invariant, by Cell Decomposition, we can assume that Γ is an open subset of $\mathbf{R}^k \subset \mathbf{R}^k \times \mathbf{R}^{n-k}$, and $f|_{\Gamma'} > 0$, $f|_\Gamma \equiv 0$. So $T_{y,f} = \mathbf{R}^k$, for all $y \in \Gamma$. Let

$$A = \{(y, s, t)\colon (y, s) \in \Gamma \cup \Gamma', t > 0, f(y, s) = t\}.$$

Then A is a definable set. By Definable Choice and the assumption, there exists an open subset U of Γ, $\alpha > 0$, and a definable map $\theta\colon U \times [0, \alpha) \longrightarrow \mathbf{R}^{n-k}$, such that θ is C^p on $U \times (0, \alpha)$, $\theta|_\Gamma \equiv 0$, and $f(y, \theta(y, t)) = t$, and, moreover, for all $y \in U$, we have

$$(*) \qquad \frac{\|D_1\theta(y,t)\|}{\|\theta(y,t)\|} \geq \frac{\delta(\mathbf{R}^k, T_{(y,\theta(y,t)),f})}{\|\theta(y,t)\|} \to +\infty, \quad \text{when } t \to 0^+.$$

On the other hand, by [M2, Prop. 5.2], there exist a nonempty open subset B of U and $r > 0$, such that

$$(***) \qquad \theta(y,t) = c(y)t^r + \varphi(y,t)t^{r_1}, \; y \in B, t > 0 \text{ sufficiently small,}$$

where c is C^p on B, $c \not\equiv 0$, $r_1 > r$, and φ is C^p with $\lim_{t\to 0^+} \varphi(y,t) = 0$, for all $y \in B$. Moreover, by Lemma 1.8, we can suppose that $D_1\varphi$ is bounded. Substituting $(**)$ to the left-hand side of $(*)$ we get a contradiction.

Case 3. rank $f|_\Gamma$ = rank $f|_{\Gamma'} = 1$. If $\dim(\Gamma \setminus W_f) = \dim \Gamma$, then the condition (w_f) is false for (Γ, Γ') over a nonempty open subset B of Γ. It is easy to see that there is $c \in \mathbf{R}$ such that (w_f) is false for the pair $(\Gamma \cap f^{-1}(c), \Gamma')$ over a nonempty open subset of $B \cap f^{-1}(c)$, and hence open in $\Gamma \cap f^{-1}(c)$. This contradicts Case 2. □

2.8 *Remark.* If the structure admits analytic cell decomposition, then the theorems hold true with "analytic" in place of "C^p". Our results can be translated to the setting of analytic-geometric categories in the sense of [DM2].

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Department of Mathematics, University of Dalat, Dalat, Vietnam