

BOUNDARY FUNCTIONS FOR IDEALS IN ANALYTIC LIMIT ALGEBRAS

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1. Introduction

Boundary functions form a useful tool in the study of ideals in various classes of nest algebras. In the simplest case, where the nest algebra is T_n , the algebra of $n \times n$ upper triangular matrices, it is a simple matter to associate to each ideal in T_n an appropriate boundary function. This was generalized to weakly closed ideals in general nest subalgebras of $\mathcal{B}(\mathcal{H})$ by Erdos and Power in [EP] and to Volterra nest subalgebras of C^* -algebras by Power in [P1]. Larson and Solel extended the Erdos-Power theory to the context of nest subalgebras of factor von Neumann algebras [LS]. Both theories apply to modules over the nest algebra, not just to ideals in the nest algebra. Davidson, Donsig and Hudson in [DDH] study support functions for norm closed bimodules of nest algebras; their support functions come in pairs which allow the determination of a maximal and sometimes a minimal bimodule for a given pair

Amongst algebras which are direct limit of T_n 's, the full nest algebras introduced in [HP] have the most in common with weakly closed nest algebras. It is not surprising, therefore, that it is possible to define boundary functions in this context. The definition of boundary functions for ideals given in this paper is based on the possibility of coordinatization for these algebras. In point of fact, since everything is based on the properties of the coordinate system, the theory is valid for a wider class of operator algebras than full nest algebras. This class will be the trivially analytic subalgebras of simple AF C^* -algebras with an injective 0-cocycle. Although the theory of boundary functions in this paper resembles the theory in the various papers cited in the first paragraph, one significant difference is that the boundary functions in the cited papers are all maps from the nest of invariant subspaces to itself while the boundary functions in this paper are maps from the spectrum of the diagonal of the algebra to itself.

We shall assume throughout this paper that B is a simple AF C^* -algebra and that A is an analytic subalgebra of B with a trivial cocycle c which is the coboundary of an injective 0-cocycle. These analytic subalgebras are necessarily maximal triangular. The diagonal, $D = A \cap A^*$, of A is a canonical masa in B .

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AF C^* -algebras are groupoid C^* -algebras; it is this groupoid and substructures of the groupoid that provide the necessary coordinatization for the existence of boundary functions for arbitrary ideals. We describe these coordinates very briefly; consult [MS] for a more detailed treatment. Since D is an abelian C^* -algebra, there is a compact Hausdorff topological space X such that $D \cong C(X)$. In the present context, the spectrum X will, in fact, be a Cantor space.

The groupoid for B will be a principal groupoid based on X ; i.e., an equivalence relation on X . One way to obtain this equivalence relation is as follows: write B as a direct limit of finite dimensional C^* -algebras and choose a system of matrix units for this system. Each matrix unit in the system acts on D by conjugation and consequently induces a partial homeomorphism of X , the spectrum of D . The groupoid, G , is the union of the graphs of all the matrix units of the system. Since the same result is obtained if one uses all normalizing partial isometries in B instead of a system of matrix units, G is independent of the choice of matrix unit system.

In this context, G is an equivalence relation on X and the groupoid operations are as follows:

- (i) (w, x) and (y, z) are composable if, and only if, $x = y$, in which case the product is (w, z) , and
- (ii) the inverse of (w, x) is (x, w) .

Furthermore, G is a topological groupoid. The topology is obtained by declaring that the graph of each matrix unit will be an open set. It turns out that each such graph is also a compact set. Note that the groupoid topology is not the relative product topology on G as a subset of $X \times X$.

The space, X , may be identified with the diagonal of $X \times X$ (which is an open, compact subset of G) via the homeomorphism $x \rightarrow (x, x)$. We make this identification hereafter. Also, note that the two coordinate projections, π_1 and π_2 of $X \times X$ onto X are, when restricted to G , local homeomorphisms with respect to the topology on G . An open set on which π_1 and π_2 are one-to-one is known as a G -set. Since any neighborhood of a point in G contains a smaller neighborhood which is a G -set, we will always assume in the sequel that neighborhoods are G -sets.

The subalgebra, A , is a canonical subalgebra, and as such is generated by the matrix units which it contains (after a matrix unit system for B has been selected). This makes it easy to describe the support set, P , for A (a subset of G whose existence is guaranteed by the spectral theorem for bimodules [MS]); P is the union of the graphs of those matrix units which lie in A . This support set has been called by a variety of terms in the literature; we shall refer to it as the spectrum of the algebra A (based on the fact that the relationship between A and P is strongly analogous to the relationship between D and X). Similarly, we will sometimes refer to G as the spectrum of B and, by extension, to the triple (X, P, G) as the spectral triple for (D, A, B) . By a theorem of Power [P2], the spectrum is a complete invariant for isometric isomorphism of triangular subalgebras of AF C^* -algebras.

The spectrum, P , satisfies several important properties in the analytic subalgebra case. For example, $P \cap P^{-1} = X$ and $P \cup P^{-1} = G$. Also, P induces a total order on each equivalence class from G . If y is an element of X , let orb_y denote the equivalence class of x ; i.e., $\text{orb}_y = \{x \mid (x, y) \in G\}$. The assumption that B is a simple C^* -algebra is equivalent to the property that each orbit, orb_y , is dense in X [R, p. 112].

Examples of spectral triples of the type under discussion are provided by the refinement algebras. Here, X will be the space of all sequences (x_n) , where each term x_n is an element of a set of positive integers of the form $\{1, \dots, k_n\}$. Now X is, in fact, the Cartesian product of countably many finite sets; the topology on X is the product topology for this Cartesian product. The equivalence relation, G , is the following: $(x, y) \in G$ if, and only if, there is an integer N such that $x_n = y_n$ for all $n \geq N$. (This is the “tails are the same” equivalence relation.) Each pair of finite sequences, (a_1, \dots, a_N) and (b_1, \dots, b_N) , determines a basic open set for the topology in G ; this set is $\{(x, y) \in G \mid x_i = a_i \text{ and } y_i = b_i \text{ for } i = 1, \dots, N\}$. Finally, the spectrum, P , is given as follows: $(x, y) \in P$ if, and only if, there is an integer, N , such that $(x_1, \dots, x_N) \leq (y_1, \dots, y_N)$ (in the lexicographic order) and $x_i = y_i$ for $i > N$. The spectral triple for any refinement algebra can be represented as described above.

We mention in passing that the spectral triple for any full nest algebra has a similar representation. The only change that is needed is to replace the “tails the same” equivalence relation by a possibly much more complicated equivalence relation which can be determined from a presentation for the algebra.

Just as the subalgebra, A , of B has a support set P contained in G , so does any ideal in A or, for that matter, any A -module, M , in B . Suppose that $\mathcal{I} \subseteq A$ is an ideal. Then \mathcal{I} is canonical and so is generated by the matrix units which it contains. Let σ be the union of the graphs of the matrix units in \mathcal{I} . The open set σ is the support set of \mathcal{I} . (Since \mathcal{I} is a D -bimodule, the spectral theorem for bimodules may be used to obtain the existence of σ .) The same discussion applies to an A -module, M . The support set for an ideal or an A -module satisfies the following definition:

DEFINITION. An open subset σ of P is an *ideal set* if $(w, x), (y, z) \in P$ and $(x, y) \in \sigma$ imply that $(w, z) \in \sigma$. If σ satisfies the same condition but is merely contained in G , then it is an *A-module set*.

There is a one-to-one correspondence between ideals in A and ideal sets in P and between A -modules in B and A -module sets in G . Consequently, in the rest of this paper, we shall discuss ideal sets (or, once, A -module sets) only.

A 1-cocycle, c , is a continuous groupoid homomorphism from G into the real numbers, \mathbb{R} , such that $c^{-1}(0) = X$. (Keep in mind that X is identified with the diagonal of G .) The cocycle property asserts that $c(x, z) = c(x, y) + c(y, z)$ for all $(x, y), (y, z) \in G$. The canonical subalgebra A is analytic in B if $P = c^{-1}[0, \infty)$. Finally, A is trivially analytic if c can be written in the form $c(x, y) = b(y) - b(x)$ for some continuous function $b: X \rightarrow \mathbb{R}$. As stated above, all the algebras under

consideration are analytic subalgebras of an AF C^* -algebra which possess a trivial cocycle.

For example, suppose that (X, P, G) is the spectral triple of a refinement algebra and has the form described above. Then let

$$\tilde{c}(x, y) = \sum_{n=1}^{\infty} \frac{y_n - x_n}{k_1 k_2 \cdots k_n}.$$

The 1-cocycle, \tilde{c} , is the coboundary of the 0-cocycle $\tilde{b}: X \rightarrow \mathbb{R}$ defined by

$$\tilde{b}(x) = \sum_{n=1}^{\infty} \frac{x_n - 1}{k_1 k_2 \cdots k_n}.$$

It is easy to see that \tilde{b} is a continuous function on X and that $\tilde{c}(x, y) = \tilde{b}(y) - \tilde{b}(x)$, for all $(x, y) \in P$.

We know that P induces a total order on each equivalence class; in fact, there is a total order on X which agrees with P on equivalence classes. This is the lexicographic order: $x \leq y$ if, and only if, $x = y$ or there is an integer N such that $(x_1, \dots, x_N) < (y_1, \dots, y_N)$ in the lexicographic order. [Note that if $(x_1, \dots, x_N) < (y_1, \dots, y_N)$ and $M > N$, then $(x_1, \dots, x_M) < (y_1, \dots, y_M)$.] The lexicographic order on X has countably many gaps; let $a^{(n)}$ be an enumeration of the points with an immediate successor and let $b^{(n)}$ be the immediate successor of $a^{(n)}$. These are precisely the points where \tilde{b} fails to be one-to-one. However, we can define a new and injective 0-cocycle, b , which induces a 1-cocycle, c , given by $c(x, y) = b(y) - b(x)$. If $x \in X$, let $S(x) = \{n \mid a^{(n)} < x\}$. Define $b: X \rightarrow \mathbb{R}$ by

$$b(x) = \tilde{b}(x) + \sum_{n \in S(x)} \frac{1}{2^n}.$$

Since the order topology induced by \leq on X is the same as the topology which X carries as the spectrum of D , b is continuous. If $(x, y) \in P$, then $S(x) \subseteq S(y)$ and $c^{-1}[0, \infty) = \tilde{c}^{-1}[0, \infty)$; thus c is a cocycle for the same analytic subalgebra as \tilde{c} is.

This same discussion can be carried out in the case in which (X, P, G) is the spectral triple for a full nest algebra. The only change is that the “equal tails” equivalence relation is replaced by a more complicated equivalence relation. In the full nest algebra case it is possible that $(a^{(n)}, b^{(n)}) \in P$, for certain values of n . When this occurs, the formula for \tilde{c} given above is modified so that $\tilde{c}(a^{(n)}, b^{(n)}) > 0$; consequently $\tilde{b}(b^{(n)}) > \tilde{b}(a^{(n)})$ and the corresponding term may be omitted in the formula for b .

The point of this discussion is that for the class of algebras of primary interest, the full nest algebras, there exists an injective 0-cocycle, b , on X whose coboundary, c , renders the algebra trivially analytic. Whenever a trivially analytic algebra is induced

from an injective 0-cocycle, it is possible to define boundary functions for ideals (or A -modules). Henceforth, we assume the existence of such an injective 0-cocycle.

The 0-cocycle, b , then induces a total order on X : $x \preceq y$ if, and only if $b(x) \leq b(y)$. The two principal properties of this order are that it agrees on each equivalence class with the total order induced by P and that the order topology induced by \preceq agrees with the original topology on X . This total order on X is the feature which makes it possible to define, for each ideal set, a boundary function ϕ . A simple list of properties characterizes those functions from X to itself which arise as boundary functions and there are natural partitions on the family of ideal sets and on the family of boundary functions so that the quotients are in bijective correspondence. (The mapping from ideal sets to boundary functions is surjective, but not injective.)

Before proceeding to the definition of boundary functions, we recapitulate the properties of the spectral triple which are critical to the notion of a boundary function. Keep in mind that G has a principal groupoid structure and that X is identified with the diagonal in G , i.e. with the units of the groupoid.

1. G is a topological equivalence relation based on the compact Hausdorff space X .
2. P is an open subset of G which satisfies the properties: $P \circ P \subseteq P$, $P \cap P^{-1} = X$, and $P \cup P^{-1} = G$.
3. The two projection maps, π_1 and π_2 of $X \times X$ onto X are, when restricted to G , local homeomorphisms with respect to the topology on G . In particular, they are continuous and open mappings.
4. Each equivalence class, orb_y , from G is countable and dense in X .
5. There is a total order \preceq on X which, on each equivalence class, orb_y , agrees with the order induced by P . Furthermore, the order topology on X is the same as the original topology on X .
6. Since X is compact, it has a minimal element and a maximal element with respect to the order \preceq . Denote these elements by p_{\min} and p_{\max} .

This list of properties will suffice for the definition and properties of boundary functions. While there are spectral triples which satisfy these properties which do not come from trivially analytic subalgebras of AF C^* -algebras with injective 0-cocycle, the one's that the author knows of are associated with algebras which lack tractable properties and which do not appear to be of any interest. If one adds one further property—the assumption that X has at most countably many gaps—then it is possible to prove the existence of a trivial 1-cocycle for G with injective 0-cocycle.

2. Boundary functions

With the preliminaries out of the way, we now turn attention to the definition and properties of boundary functions. Assume that σ is an ideal set contained in P . Let

$y \in X$. Divide orb_y into two disjoint subsets as follows:

$$\begin{aligned} A &= \{x \in \text{orb}_y \mid (x, y) \in \sigma\}, \\ B &= \{x \in \text{orb}_y \mid (x, y) \notin \sigma\}. \end{aligned}$$

It follows from the definition of ideal set that A is an initial segment of orb_y and B is a terminal segment of orb_y (in the order induced by P). Now view A and B as subsets of X and let $v = \sup A$ and $w = \inf B$, where \sup and \inf are interpreted with respect to the total order on X (which agrees with P on orb_y). The existence of the \sup and \inf is guaranteed by the fact that X is compact in the order topology. Observe that the open order interval with endpoints v and w is empty. Indeed, if this open interval were not empty, then it would have to contain points of orb_y (all equivalence classes are dense in X) and this would contradict the obvious fact that X is the union of A and B . This leaves two possibilities: either $v = w$ or v is the immediate predecessor of w in the order, \preceq , on X .

In general, when $x < y$ in X and the open order interval with endpoints x and y is empty, we will say that x has a gap above and that y has a gap below. We will also write $x = \text{pred } y$ and $y = \text{succ } x$.

DEFINITION. Let σ be an ideal set in P . The boundary function for σ is the function $\phi_\sigma: X \rightarrow X$ is given by the formula $\phi_\sigma(y) = \sup\{x \in \text{orb}_y \mid (x, y) \in \sigma\}$.

Note that $\phi_\sigma(y)$ satisfies the following:

- (i) if $(w, y) \in P$ and $w < \phi_\sigma(y)$, then $(w, y) \in \sigma$, and
- (ii) if $\phi_\sigma(y) < w$ then $(w, y) \notin \sigma$.

It is not possible to say anything about $(\phi_\sigma(y), y)$ itself; $(\phi_\sigma(y), y)$ may or may not be an element of P and, if it is an element of P , it may or may not be an element of σ .

PROPOSITION 1. If $\phi = \phi_\sigma$ is the boundary function for an ideal σ , then ϕ has the following properties.

1. $\phi(y) \preceq y$, for all $y \in X$.
2. When $\phi(y)$ has a gap below, the following hold:
 - a. $(\phi(y), y) \in P$.
 - b. y has a gap below.
 - c. $y < z \implies \phi(y) < \phi(z)$.
 - d. There is a neighborhood N (a G -set) of $(\phi(y), y)$ such that $(s, t) \in N \implies s \preceq \phi(t)$.

- 3. If $y < z$ then $\phi(y) \preceq \phi(z)$, for all $y, z \in X$.
- 4. If y does not have a gap below, then $\phi(y) = \sup\{\phi(t) \mid t < y\}$.

DEFINITION. A function $\phi: X \rightarrow X$ which satisfies properties 1) through 4) in Proposition 1 will be called a *boundary function*.

Remark. If σ is an A -module set rather than an ideal set, a boundary function for σ may be defined in precisely the same way as for ideal sets. Boundary functions for A -module sets satisfy the properties in Proposition 1 with two exceptions: condition 1) must be dropped and condition 2a) must be changed to $(\phi(y), y) \in G$. With these two changes, the general notion of a boundary function can again be defined and the theory described below remains valid, provided some obvious trivial changes are made. In the (slightly) modified theory, the sets associated with boundary functions are, of course, A -module sets rather than ideal sets. From here on, the exposition will be limited to ideal sets.

Proof of Proposition 1. Property 1) follows immediately from the fact that $x \preceq y$ for all $(x, y) \in \sigma$.

Assume that $\phi_\sigma(y)$ has a gap below. We must have $(\phi_\sigma(y), y) \in \sigma$, for the other possibility violates the definition of ϕ_σ . In particular, $(\phi_\sigma(y), y) \in P$ and 2a) is verified.

Since $(\phi_\sigma(y), y) \in \sigma$, there is a neighborhood N of $(\phi_\sigma(y), y)$ which is contained in σ and is a G -set. We may also assume that $\pi_1(N)$ is contained in the order interval $[\phi_\sigma(y), p_{\max}]$ (This order interval is open, since $\phi_\sigma(y)$ has a gap below; simply intersect the original neighborhood with $[\phi_\sigma(y), p_{\max}] \times X$.) Now suppose that y does not have a gap below. Since π_2 is an open map, $\pi_2(N)$ contains an open order interval whose upper endpoint is y . This implies that there is a point $(a, b) \in N$ such that $a, b \in \text{orb}_y$ and $b < y$. (This uses, once again, the fact that orb_y is dense in X .) By the assumptions on N , we also have $\phi_\sigma(y) < a$. The open order interval with endpoints $\phi_\sigma(y)$ and a is non-empty (since $\phi_\sigma(y)$ does not have a gap above) and therefore contains a point z from orb_y . Now observe that $(z, y) = (z, a) \circ (a, b) \circ (b, y)$ with (z, a) and (b, y) in P and (a, b) in σ . Thus $(z, y) \in \sigma$. But $\phi_\sigma(y) < z$, contradicting the definition of ϕ_σ . This proves that y must have a gap below and condition 2b) is verified.

In order to verify condition 2c), assume that $y < z$. Since both $\phi_\sigma(y)$ and y have gaps below and $(\phi_\sigma(y), y) \in \sigma$, there is a neighborhood N (a G -set) of $(\phi_\sigma(y), y)$ which is contained in $[\phi_\sigma(y), p_{\max}] \times [y, p_{\max}]$. In particular, we can find a point $(a, b) \in N$ such that $a, b \in \text{orb}_z$ and $\phi_\sigma(y) < a$ and $y < b < z$. (The possibility $\phi_\sigma(y) = a$ is eliminated by the assumption that N is a G -set.) Thus we have $(a, z) = (a, b) \circ (b, z)$ with $(a, b) \in \sigma$ and $(b, z) \in P$. This shows that $(a, z) \in \sigma$ and hence that $a \preceq \phi_\sigma(z)$. Since $\phi_\sigma(y) < a$, we have $\phi_\sigma(y) < \phi_\sigma(z)$, as desired.

Property 2d) follows from property 2a): since $(\phi_\sigma(y), y) \in \sigma$, there is a neighborhood N of $(\phi_\sigma(y), y)$ which is contained in σ . If $(s, t) \in N$, then $(s, t) \in \sigma$ and hence $s \preceq \phi_\sigma(t)$.

Assume that property 3) does not hold; i.e., assume that there are points $y, z \in X$ such that $y < z$ and $\phi_\sigma(z) < \phi_\sigma(y)$. By property 2c), $\phi_\sigma(y)$ does not have a gap below. Choose an element $a \in \text{orb}_y$ such that $\phi_\sigma(z) < a < \phi_\sigma(y)$. By the definition of ϕ_σ , $(a, y) \in \sigma$. So, there is a neighborhood N of (a, y) such that $N \subset \sigma$ and such that $(s, t) \in N$ implies $\phi_\sigma(z) < s < \phi_\sigma(y)$ and $t < z$. Choose an element $(s, t) \in N$ with $s, t \in \text{orb}_z$. We then have $(s, z) = (s, t) \circ (t, z)$ with $(s, t) \in \sigma$ and $(t, z) \in P$; hence $(s, z) \in \sigma$. But now we have both $\phi_\sigma(z) < s$ and $s < \phi_\sigma(z)$, a contradiction. So 3) holds.

Assume that y has no gap below (and hence that $\phi_\sigma(y)$ also has no gap below). Property 3) shows that $t < y \implies \phi_\sigma(t) \preceq \phi_\sigma(y)$; thus $\sup\{\phi_\sigma(t) \mid t < y\} \preceq \phi_\sigma(y)$. In order to prove equality, we need to assume that $w < \phi_\sigma(y)$ and show that there is $t \in X$ such that $t < y$ and $w < \phi_\sigma(t)$. Since $\phi_\sigma(y)$ has no gap below, there is $x \in \text{orb}_y$ such that $w < x < \phi_\sigma(y)$. By the definition of ϕ_σ , $(x, y) \in \sigma$. Let N be a neighborhood of (x, y) which is contained in σ . Since y has no gap below, there is a point $(s, t) \in N$ such that $t < y$ and $w < s$. Since $(s, t) \in \sigma$, we have also $s \preceq \phi_\sigma(t)$; thus $w < \phi_\sigma(t)$ and 4) is verified. \square

The mapping $\sigma \longrightarrow \phi_\sigma$ from ideal sets to boundary functions is surjective (as we shall see later) but is not injective. To find a simple example of two ideals with the same boundary function, assume that $(a, b) \in P$, that $a \neq b$ and that a does not have a gap below. Define

$$\begin{aligned} \sigma_{a,b} &= \{(x, y) \in P \mid \text{either } x < a \text{ or } b < y\}, \\ \tau_{a,b} &= \sigma_{a,b} \cup \{(a, b)\}. \end{aligned}$$

We need to make one other assumption: that $\tau_{a,b}$ is an open subset of P . In the case of refinement algebras, this assumption holds for all points $(a, b) \in P$. For general full nest algebras, there may be points in P for which it fails. If $\tau_{a,b}$ is open then it is an ideal set; $\sigma_{a,b}$ is always an ideal set.

Define a function $\psi: X \longrightarrow X$ as follows:

$$\psi(y) = \begin{cases} y, & \text{if } y \preceq a, \\ a, & \text{if } a < y \preceq b, \\ y, & \text{if } b < y. \end{cases}$$

It is a simple matter to check that ψ is the boundary function for both ideals, $\sigma_{a,b}$ and $\tau_{a,b}$.

Next, we consider how to associate ideals to boundary functions. So, assume that $\phi: X \longrightarrow X$ satisfies the four conditions in the definition of a boundary function. We define three subsets of P as follows:

$$\sigma(\phi) = \{(x, y) \in P \mid x < \phi(y)\},$$

$$\eta(\phi) = \{(x, y) \in P \mid x \preceq \phi(y)\},$$

$$\sigma[\phi] = \{(x, y) \in P \mid \text{there is a neighborhood } N \text{ of } (x, y) \text{ with } N \subseteq \eta(\phi)\}.$$

PROPOSITION 2. *The set $\sigma(\phi)$ is an ideal set in P .*

Proof. First, we show that $\sigma(\phi)$ is open. Let $(x, y) \in \sigma(\phi)$. We must find a neighborhood, N , of (x, y) such that $N \subseteq \sigma(\phi)$.

First, assume that y has a gap below. The two order intervals $[p_{\min}, \phi(y))$ and $[y, p_{\max}]$ are open subsets of X and $(x, y) \in [p_{\min}, \phi(y)) \times [y, p_{\max}]$. Consequently, there is a neighborhood, N , of (x, y) such that $\pi_1(N) \subseteq [p_{\min}, \phi(y))$ and $\pi_2(N) \subseteq [y, p_{\max}]$. If $(w, z) \in N$, we have $w \prec \phi(y)$ and $y \preceq z$. By property 3), $\phi(y) \preceq \phi(z)$. Thus, $w \prec \phi(z)$ and $(w, z) \in \sigma(\phi)$. This shows that $N \subseteq \sigma(\phi)$.

Now we consider the case when y has no gap below. Property 2b) implies that $\phi(y)$ has no gap below. Consequently, there is $s \in X$ such that $x \prec s \prec \phi(y)$. By property 4), there is $t \prec y$ such that $s \prec \phi(t)$. Now, the order intervals, $[p_{\min}, s)$ and $(t, p_{\max}]$ are open in X and $(x, y) \in [p_{\min}, s) \times (t, p_{\max}]$. Consequently, there is a neighborhood, N , of (x, y) such that $\pi_1(N) \subseteq [p_{\min}, s)$ and $\pi_2(N) \subseteq (t, p_{\max}]$. Let $(w, z) \in N$. Then $w \prec s$ and $t \prec z$. By property 3), $\phi(t) \preceq \phi(z)$. But $w \prec s \prec \phi(t)$, so $w \prec \phi(z)$. Thus $(w, z) \in \sigma(\phi)$ and $N \subseteq \sigma(\phi)$. This shows that $\sigma(\phi)$ is an open subset of P .

It remains to show that $\sigma(\phi)$ satisfies the ideal property. Assume $(a, x) \in P$, $(x, y) \in \sigma(\phi)$, and $(y, b) \in P$. Property 3) implies that $\phi(y) \preceq \phi(b)$. Thus, we have $a \preceq x \prec \phi(y) \preceq \phi(b)$; hence $(a, b) \in \sigma(\phi)$ and $\sigma(\phi)$ is an ideal set. \square

PROPOSITION 3. *The set $\sigma[\phi]$ is an ideal set in P .*

Proof. Let $(x, y) \in \sigma[\phi]$. Then there is a neighborhood, N , of (x, y) such that $N \subseteq \eta(\phi)$. Clearly, any point in N is in $\sigma[\phi]$; thus $N \subseteq \sigma[\phi]$ and $\sigma[\phi]$ is an open subset of P .

To see that $\sigma[\phi]$ satisfies the ideal property, let $(a, x) \in P$, $(x, y) \in \sigma[\phi]$, and $(y, b) \in P$. All neighborhoods in the following argument are to be open G -sets which are subsets of P . Let N_2 be a neighborhood of (x, y) such that $N_2 \subseteq \eta(\phi)$. Let N_1 be a neighborhood of (a, x) and N_3 , a neighborhood of (y, b) . Let $N = N_1 \circ N_2 \circ N_3$. Then N is a neighborhood of (a, b) . If $(s, t) \in N$, then there exist $s', t' \in X$ such that $(s, s') \in N_1$, $(s', t') \in N_2$, and $(t', t) \in N_3$. Then, using $N_1 \subseteq P$, $N_2 \subseteq \eta(\phi)$, $N_3 \subseteq P$, and property 3) applied to $t' \preceq t$, we have $s \preceq s' \preceq \phi(t') \preceq \phi(t)$. Thus $(s, t) \in \eta(\phi)$; since (s, t) is arbitrary in N , $N \subseteq \eta(\phi)$. This proves that $(a, b) \in \sigma[\phi]$, so $\sigma[\phi]$ satisfies the ideal property. \square

We shall see shortly that the boundary function for the ideal $\sigma[\phi]$ is ϕ , thus verifying that the mapping from ideals to boundary functions is surjective. The ideal $\sigma(\phi)$ need not have ϕ as its boundary function. An examination of some examples

indicates that the boundary function for $\sigma(\phi)$ is closely related to ϕ and suggests the following definition.

DEFINITION. Let ϕ be a boundary function. Define another function ϕ^- by the formula

$$\phi^-(y) = \begin{cases} \phi(y), & \text{if } \phi(y) \text{ has no gap below,} \\ \text{pred } \phi(y), & \text{if } \phi(y) \text{ has a gap below.} \end{cases}$$

Remark. When ϕ is a boundary function, so is ϕ^- . Property 1) is obvious, since $\phi^-(y) \leq \phi(y)$, for all y . There is nothing to prove for property 2), since $\phi^-(y)$ never has a gap below. For property 3), assume that $y < z$. If $\phi(z)$ does not have a gap below, then $\phi^-(y) \leq \phi(y) \leq \phi(z) = \phi^-(z)$. If $\phi(z)$ does have a gap below and $\phi(y) < \phi(z)$, then $\phi^-(y) \leq \phi(y) \leq \phi^-(z)$. Finally, if $\phi(z)$ has a gap below and $\phi(y) = \phi(z)$, then clearly $\phi^-(y) = \phi^-(z)$.

This leaves 4) to be verified. When y has no gap below, neither does $\phi(y)$, by property 2b). We then have

$$\begin{aligned} \phi^-(y) &= \phi(y) = \sup\{\phi(t) \mid t < y\} \\ &= \sup\{\phi(t) \mid t < y \text{ and } \phi(t) \text{ has no gap below}\} \\ &= \sup\{\phi^-(t) \mid t < y\}. \end{aligned}$$

Since $\phi^-(y)$ never has a gap below, we have $(\phi^-)^- = \phi^-$; thus there is never any need to iterate the “minus” operation.

PROPOSITION 4. Let ϕ be a boundary function. Let σ be any ideal set such that $\sigma(\phi) \subseteq \sigma \subseteq \sigma[\phi]$. Then the boundary function ψ for σ satisfies $\phi^- \leq \psi \leq \phi$.

Proof. Let $y \in X$. First, we show that $\psi(y) \leq \phi(y)$. We distinguish two cases. First, assume that $\psi(y)$ has a gap below. Then, by property 2a), $(\psi(y), y) \in \sigma$. Since $\sigma \subseteq \sigma[\phi]$, we have $(\psi(y), y) \in \sigma[\phi]$ and hence $\psi(y) \leq \phi(y)$.

Now assume that $\psi(y)$ has no gap below. Suppose that $\phi(y) < \psi(y)$. Since orbits are dense, there is $t \in \text{orb}_y$ such that $\phi(y) < t < \psi(y)$. From the definition of boundary functions for an ideal, we have $(t, y) \in \sigma \subseteq \sigma[\phi]$. But this implies that $t \leq \phi(y)$, a contradiction. Thus $\psi(y) \leq \phi(y)$ in this case also.

Next we prove that $\phi^-(y) \leq \psi(y)$. Assume, to the contrary, that $\psi(y) < \phi^-(y)$. Since orbits are dense and $\phi^-(y)$ has no gap below, there is $t \in \text{orb}_y$ such that $\psi(y) < t < \phi^-(y)$. In particular, $t < \phi(y)$, so $(t, y) \in \sigma(\phi)$. But $\sigma(\phi) \subset \sigma$, so $(t, y) \in \sigma$. The combination $\psi(y) < t$ and $(t, y) \in \sigma$ contradicts the fact that ψ is the boundary function for σ . Thus $\phi^-(y) \leq \psi(y)$. \square

PROPOSITION 5. Assume that σ is an ideal set and that ϕ is the boundary function for σ . Then $\sigma(\phi) \subseteq \sigma \subseteq \sigma[\phi]$.

Proof. Let $(x, y) \in \sigma(\phi)$. Then $(x, y) \in P$ and $x < \phi(y)$. From the definition of boundary function, $(x, y) \in \sigma$. Thus $\sigma(\phi) \subseteq \sigma$.

Now suppose that $(x, y) \in \sigma$. The definition of boundary function precludes the possibility that $\phi(y) < x$; thus $x \leq \phi(y)$. This shows that $\sigma \subseteq \eta(\phi)$. Since σ is open, there is a neighborhood, N , of (x, y) such that $N \subseteq \sigma$. In particular, $N \subseteq \eta(\phi)$ and so $(x, y) \in \sigma[\phi]$. Thus $\sigma \subseteq \sigma[\phi]$. \square

The next proposition shows that the mapping from ideals to the class of boundary functions is surjective.

PROPOSITION 6. *Let ϕ be a boundary function. Then the boundary function for the ideal $\sigma[\phi]$ is ϕ and the boundary function for the ideal $\sigma(\phi)$ is ϕ^- .*

Proof. Let ψ denote the boundary function for the ideal $\sigma[\phi]$. Proposition 4 implies that $\phi^- \preceq \psi \preceq \phi$. If $\phi(y)$ has no gap below, $\psi(y) = \phi(y)$, so we need only consider the case in which $\phi(y)$ has a gap below. Property 2d) in the definition of boundary functions implies that $(\phi(y), y) \in \sigma[\phi]$. Since ψ is the boundary function for $\sigma[\phi]$, we have $\phi(y) \leq \psi(y)$. Thus $\psi(y) = \phi(y)$. This shows that the boundary function for $\sigma[\phi]$ is ϕ .

Now, let ψ denote the boundary function for $\sigma(\phi)$. Again, we have that $\phi^- \preceq \psi \preceq \phi$. Since $\phi^-(y) = \phi(y)$ when $\phi(y)$ has no gap below, we need only show that $\psi(y) = \phi^-(y)$ whenever $\phi(y)$ has a gap below. We know from the properties of boundary functions that $(\phi(y), y) \in P$. From the definition of $\sigma(\phi)$ we also know that $(\phi(y), y) \notin \sigma(\phi)$. It now follows that $\psi(y) = \text{pred } \phi(y)$, i.e. that $\psi(y) = \phi^-(y)$. This shows that the boundary function of $\sigma(\phi)$ is ϕ^- . \square

While the mapping from ideal sets to boundary functions is not injective, it is possible to say something about the family of ideals whose boundary function is a particular function ϕ . Of course, all of these ideals must lie between $\sigma(\phi)$ and $\sigma[\phi]$. The next result says that σ will have ϕ for its boundary function provided that σ contains an appropriate subset of the graph of ϕ .

Let B_ϕ denote the portion of the graph of ϕ which is contained in $\sigma[\phi]$; i.e.,

$$B_\phi = \{(\phi(y), y) \mid y \in X\} \cap \sigma[\phi].$$

Let

$$L_\phi = \{(\phi(y), y) \in B_\phi \mid \phi(y) \text{ has a gap below}\}.$$

We then have:

PROPOSITION 7. *Let ϕ be a boundary function. An ideal set σ will have ϕ for its boundary function if, and only if, $\sigma(\phi) \cup L_\phi \subseteq \sigma \subseteq \sigma[\phi]$.*

Proof. If ϕ is the boundary function for σ , then the inclusion $L_\phi \subseteq \sigma$ follows immediately from the way in which boundary functions are associated with ideal sets. (This was pointed out in the proof of part 2a) in Proposition 1.) For the converse, assume that $\sigma(\phi) \cup L_\phi \subseteq \sigma \subseteq \sigma[\phi]$. Then we know that $\phi^- \leq \phi_\sigma \leq \phi$. We need only check those points y for which $\phi^-(y) \neq \phi(y)$. For such y , $(\phi(y), y) \in L_\phi$; hence $(\phi(y), y) \in \sigma$. Therefore, $\phi_\sigma(y) = \phi(y)$ and $\phi_\sigma = \phi$ as desired. \square

Remark. Note that the set $\sigma(\phi) \cup L_\phi$ need not be open in P . In particular, $\sigma(\phi) \cup L_\phi$ will not, in general, be an ideal set. As a consequence, there need not exist a minimal ideal set which has ϕ as its boundary function. It is not difficult to produce specific examples of this phenomenon.

We obtained ϕ^- from ϕ basically by replacing $\phi(y)$ by its immediate predecessor whenever $\phi(y)$ has a gap below. This suggests defining a function ϕ^+ in an analogous way, replacing $\phi(y)$ by its immediate successor whenever $\phi(y)$ has a gap above. This turns out to be too simplistic, however; doing so will not produce a boundary function. For example, if $\phi(y)$ has a gap above and y does not have a gap below, then any function ψ for which $\psi(y) = \text{succ } \phi(y)$ would fail property 2b) from the definition of boundary function. Similar obstacles are presented by properties 2a) and 2d). The following definition tells just where we should redefine $\phi(y)$ to obtain ϕ^+ .

DEFINITION. Let $y \in X$. Say that y is a *point of modification* for ϕ if the following hold:

- (i) y has a gap below;
- (ii) $\phi(y)$ has a gap above;
- (iii) there is a neighborhood, N , (a G -set) of $(\text{succ } \phi(y), y)$ such that $N \subseteq P$ and, for all $(s, t) \in N$, $s \leq \phi(t)$ when $\phi(t)$ has no gap above and $s \leq \text{succ } \phi(t)$ when $\phi(t)$ does have a gap above.

Remark. If y is a point of modification for ϕ , then the neighborhood N in condition (iii) may be selected so that it satisfies the additional property that if $(s, t) \in N$ then $\text{succ } \phi(y) \leq s$ and $y \leq t$. To do so, simply intersect a neighborhood satisfying condition (iii) with $[\text{succ } \phi(y), p_{\max}] \times [y, p_{\max}]$.

We now define a boundary function ϕ^+ which is larger than ϕ and is closely related to ϕ .

DEFINITION. If ϕ is a boundary function, define a function $\phi^+ : X \rightarrow X$ by

$$\phi^+(y) = \begin{cases} \text{succ } \phi(y), & \text{if } y \text{ is a point of modification for } \phi, \\ \phi(y), & \text{otherwise.} \end{cases}$$

LEMMA 8. *Let ϕ be a boundary function. Suppose that $\phi(y) < \phi^+(y)$ and $y < z$. Then $\phi^+(y) < \phi(z)$.*

Proof. Choose a neighborhood, N , of $(\phi^+(y), y)$ which satisfies both the conditions in the definition of point of modification and the remark immediately following the definition. Since y does not have a gap above (it has a gap below), there is $t \in X$ such that $t \in \pi_2(N)$, $y < t < z$, and t has no gap below. Let s be such that $(s, t) \in N$. Since t is not a point of modification, we have $\phi^+(y) \leq s \leq \phi(t)$. But $(\phi^+(y), y) \in N$, $y \neq t$, and N is a G -set; so $\phi^+(y) < \phi(t)$. Now $t < z$ implies that $\phi(t) \leq \phi(z)$; hence, $\phi^+(y) < \phi(z)$. \square

Proposition 9. *If ϕ is a boundary function then so is ϕ^+ .*

Proof. We must show that if ϕ satisfies the 4 properties in the definition of boundary function (see Proposition 1), then so does ϕ^+ .

Property 1) is automatic, except at points of modification. If y is a point of modification, then y has a gap below and $\phi(y)$ has a gap above. In particular, $\phi(y) \neq y$; so $\phi(y) < y$. Since $\phi^+(y) = \text{succ } \phi(y)$, we have $\phi^+(y) \leq y$.

To verify 2), assume that $\phi^+(y)$ has a gap below. If y is not a point of modification for ϕ , then the four conditions in property 2) hold trivially for ϕ^+ . So assume that y is a point of modification. The first condition, $(\phi^+(y), y) \in P$, follows from condition (iii) in the definition of point of modification. Condition 2b), that y has a gap below, is immediate. If $y < z$ then, by Lemma 8, $\phi^+(y) < \phi(z) \leq \phi^+(z)$; so condition 2c) holds. Condition 2d), like condition 2a), follows from property iii) in the definition of point of modification.

For the verification of property 3), when y is not a point of modification, then $\phi^+(y) = \phi(y) < \phi(z) \leq \phi^+(z)$. If y is a point of modification, Lemma 8 implies condition 3).

Condition 4) is vacuous at points of modification and trivial elsewhere, since $t < y$ implies $\phi(t) \leq \phi^+(t) \leq \phi^+(y)$. \square

LEMMA 10. *Let ϕ be a boundary function. Then $(\phi^+)^- = \phi^-$ and $(\phi^-)^+ = \phi^+$.*

Proof. If $\phi^+(y) = \phi(y)$, then $(\phi^+)^-(y) = \phi^-(y)$ is automatic. Otherwise, $\phi^+(y)$ is the immediate successor of $\phi(y)$, in which case both $(\phi^+)^-(y)$ and $\phi^-(y)$ are equal to $\phi(y)$.

We certainly have $(\phi^-)^+(y) = \phi^+(y)$ when $\phi^-(y) = \phi(y)$, so assume that $\phi^-(y)$ is unequal to $\phi(y)$ and hence is the immediate predecessor of $\phi(y)$. Since $\phi^+(y) = \phi(y)$, we only need to show that y is a point of modification for ϕ^- . We have that $\phi^-(y)$ has a gap above by assumption and that y has a gap below by property 2b) for the boundary function ϕ . The third condition in the definition of point of modification applied to ϕ^- follows from the fact that ϕ satisfies condition 2d) in the definition of boundary function. \square

PROPOSITION 11. *Let ϕ and ψ be two boundary functions. The the following are equivalent:*

- A. $\phi^- = \psi^-$,
- B. $\phi^+ = \psi^+$,
- C. $\phi^- \leq \psi \leq \phi^+$.

Proof. The equivalence of conditions A and B follows directly from Lemma 10. If A and B hold, then $\phi^- = \psi^- \leq \psi \leq \psi^+ = \phi^+$, so condition C holds. Now, assume that C is valid. For each $y \in X$, either $\phi^-(y) = \phi^+(y)$ or $\phi^-(y)$ is the immediate predecessor of $\phi^+(y)$. Consequently, either $\psi(y) = \phi^-(y)$ or $\psi(y)$ is the immediate successor of $\phi^-(y)$. In either case, $\psi^-(y) = \phi^-(y)$. Thus A holds. \square

Let \mathcal{B} denote the family of all boundary functions on X . Define an equivalence relation on \mathcal{B} as follows: $\phi \approx \psi$ if, and only if, $\phi^- = \psi^-$. Proposition 11 implies that the equivalence classes have the following form: $[\phi] = \{\psi \mid \phi^- \leq \psi \leq \phi^+\}$.

Let \mathcal{S} denote the family of all ideal sets in P . Define an equivalence relation on \mathcal{S} : $\sigma \approx \tau$ if, and only if $\phi_\sigma^- = \phi_\tau^-$. Equivalence classes can be identified easily: $[\sigma] = \{\tau \mid \sigma(\phi_\sigma^-) \subseteq \tau \subseteq \sigma(\phi_\sigma^+)\}$.

While the mapping $\mathcal{S} \rightarrow \mathcal{B}$ given by $\sigma \rightarrow \phi_\sigma$ is not surjective, it does induce a natural bijection of \mathcal{S}/\approx onto \mathcal{B}/\approx ; viz. $[\sigma] \rightarrow [\phi_\sigma]$.

3. Examples

Just prior to Proposition 2 we gave an example of two ideal sets $(\sigma_{a,b}$ and $\tau_{a,b})$ which have the same boundary function. We add here a few more very simple examples which illustrate the properties of boundary functions.

The boundary function for the trivial ideal set, \emptyset , (which corresponds to the trivial ideal $\mathcal{I} = (0)$) is the function $\phi(y) = p_{\min}$, for all $y \in X$. The boundary function for the ideal set $\sigma = P$, (which corresponds to the improper ideal $\mathcal{I} = A$) is the identity function on X .

Let $a \in X$ and let $\sigma = P \setminus \{(a, a)\}$. Then σ is an ideal set which corresponds to a maximal ideal in A . If a has an immediate predecessor, then the boundary function ϕ for σ is given by

$$\phi(y) = \begin{cases} \text{pred } a, & \text{if } y = a, \\ y, & \text{otherwise.} \end{cases}$$

If a has no immediate predecessor, then the boundary function for σ is the identity function. Thus, when a has no gap below, P and $P \setminus \{(a, a)\}$ are another pair of ideal sets with the same boundary function.

The final example is a variation on the example preceding Proposition 2. For this example we must assume that A is a refinement algebra and that (a, b) is a point in

P such that a (and hence b) does not have a gap below. Let

$$\begin{aligned}\sigma' &= \{(x, y) \in \sigma_{a,b} \mid x \neq y\} \\ \tau' &= \{(x, y) \in \tau_{a,b} \mid x \neq y\}\end{aligned}$$

Then σ' and τ' have the same boundary function, ϕ , given by

$$\phi(y) = \begin{cases} y, & \text{if } y \text{ has no gap below and either } y < a \text{ or } b < y, \\ \text{pred } y, & \text{if } y \text{ has a gap below and } y < a \text{ or } b < y, \\ a, & \text{if } a \leq y \leq b. \end{cases}$$

Observe that $\phi^- = \phi$ in this case. In fact, we have $\sigma[\phi] = \sigma(\phi) \cup \{(a, b)\}$.

4. Meet and join irreducible boundary functions

In [DHLS] the meet irreducible ideal sets are explicitly described for algebras with spectral triple (X, P, G) for which there is a total order on X compatible with P . The description runs as follows: for each pair of points $a, b \in X$, let

$$\begin{aligned}\sigma_{a,b} &= \{(x, y) \in P \mid x < a \text{ or } b < y\} \\ \tau_{a,b} &= \sigma_{a,b} \cup \{(a, b)\}.\end{aligned}$$

While $\sigma_{a,b}$ is always an ideal set in P , in order for $\tau_{a,b}$ to be an ideal set we must assume that $(a, b) \in P$ and that $\tau_{a,b}$ is an open subset of P . Whenever we use $\tau_{a,b}$, we will assume that these two conditions are satisfied. A complete list of all the meet irreducible ideal sets in P is then given as follows:

1. $\sigma_{a,b}$ if $(a, b) \in P$.
2. $\sigma_{a,b}$ if $(a, b) \notin P$ and there is either no gap above for a or no gap below for b .
3. $\tau_{a,b}$ if $(a, b) \in P$, there is either no gap above for a or no gap below for b , and $\tau_{a,b}$ is open.

Later, we will give a description of all the join irreducible ideal sets. We shall also see that the boundary function for an ideal set is meet irreducible or join irreducible (in an appropriate sense) whenever the ideal set is meet or join irreducible.

In order to talk about meet and join irreducibility for boundary functions, we need appropriate lattice operations. The choices are the obvious ones: $\phi \vee \psi = \max(\phi, \psi)$ and $\phi \wedge \psi = \min(\phi, \psi)$, both computed pointwise. We then have:

LEMMA 11. *If ϕ and ψ are boundary functions, then so are $\phi \vee \psi$ and $\phi \wedge \psi$.*

Proof. The verification that $\phi \vee \psi$ satisfies the conditions which define boundary function (given in Proposition 1) is completely routine. For $\phi \wedge \psi$, the only conditions

whose verification has some content are conditions 2c), 2d) and 4). We give arguments for these only. For convenience, let ν denote $\phi \wedge \psi$ and let $y \in X$. Without loss of generality, we may assume that $\nu(y) = \phi(y)$.

To verify condition 2c), we assume that $\nu(y)$ has a gap below and that z is a point in X which satisfies $y < z$. Since ϕ is a boundary function we know that $\phi(y) < \phi(z)$; i.e., $\nu(y) < \phi(z)$. From the assumption in the last sentence of the preceding paragraph, we also have $\phi(y) \leq \psi(y)$. If, in fact, $\phi(y) = \psi(y)$, then condition 2c) applied to ψ yields $\nu(y) = \psi(y) < \psi(z)$. Thus, in this case, $\nu(y) < \nu(z)$. So assume that $\phi(y) < \psi(y)$. Use property 3) for ψ to see that $\nu(y) = \phi(y) < \psi(y) \leq \psi(z)$. Thus, we have both $\nu(y) < \phi(z)$ and $\nu(y) < \psi(z)$, whence $\nu(y) < \nu(z)$.

We continue the assumption that $\nu(y)$ has a gap below for the verification of condition 2d). Let N_1 be a neighborhood of $(\phi(y), y)$ such that $(s, t) \in N_1$ implies $s \leq \phi(t)$. As before, we have $\phi(y) \leq \psi(y)$. Assume first that $\phi(y) = \psi(y)$. Then there is a neighborhood N_2 of $(\psi(y), y) = (\phi(y), y)$ such that $(s, t) \in N_2$ implies $s \leq \psi(t)$. Let $N = N_1 \cap N_2$. Then N is a neighborhood of $(\nu(y), y) = (\phi(y), y) = (\psi(y), y)$ such that $(s, t) \in N$ implies both $s \leq \phi(t)$ and $s \leq \psi(t)$; i.e. $(s, t) \in N$ implies $s \leq \nu(t)$. Now assume that $\phi(y) < \psi(y)$. With N_1 as above, let $N = N_1 \cap ([\phi(y), \psi(y)] \times [y, p_{\max}])$. Then N is also a neighborhood of $(\phi(y), y)$. If $(s, t) \in N$, then $s < \psi(y)$ and $y \leq t$. By property 3) for ψ , we have $\psi(y) \leq \psi(t)$; thus $s \leq \psi(t)$. Since $N \subseteq N_1$, we also have $s \leq \phi(t)$. This shows that $s \leq \nu(t)$ and completes the verification of 2d).

We now turn to the verification of condition 4) for ν and assume that y does not have a gap below. For any $t < y$, $\phi(t) \leq \phi(y)$. Hence $\nu(t) \leq \phi(y) = \nu(y)$ and we have $\sup\{\nu(y) \mid t < y\} \leq \nu(y)$. Let $x < \nu(y) = \phi(y)$. Since ϕ satisfies condition 4), there is t_1 such that $t_1 < y$ and $x < \phi(t_1)$. But $x < \psi(y)$ also (since $\phi(y) \leq \psi(y)$) and ψ satisfies condition 4); therefore there is t_2 such that $t_2 < y$ and $x < \psi(t_2)$. Let $t_3 = \max(t_1, t_2)$. Clearly $t_3 < y$. Also, $x < \phi(t_1) \leq \phi(t_3)$ and $x < \psi(t_2) \leq \psi(t_3)$. Thus $x < \nu(t_3)$ and $\sup\{\nu(t) \mid t < y\} = \nu(y)$. \square

As to be expected, the lattice operations on boundary functions are related to the lattice operations (set union and intersection) on ideal sets. Recall that the boundary function of an ideal set, σ , is given by $\phi_\sigma = \sup\{x \in \text{orb}_y \mid (x, y) \in \sigma\}$.

LEMMA 12. *Let σ and τ be ideal sets with boundary functions ϕ_σ and ϕ_τ . Then the boundary functions for the ideal sets $\sigma \cap \tau$ and $\sigma \cup \tau$ are given by*

$$\begin{aligned} \phi_{\sigma \cap \tau} &= \phi_\sigma \wedge \phi_\tau \\ \phi_{\sigma \cup \tau} &= \phi_\sigma \vee \phi_\tau \end{aligned}$$

Proof. For each $y \in X$ and for each ideal set σ , $\{x \in \text{orb}_y \mid (x, y) \in \sigma\}$ is an initial segment of orb_y . Therefore, for a fixed y , the initial segments for σ and for τ

are related by inclusion. Assume, without loss of generality, that

$$\{x \in \text{orb}_y \mid (x, y) \in \sigma\} \subseteq \{x \in \text{orb}_y \mid (x, y) \in \tau\};$$

in other words, assume that $\phi_\sigma(y) \leq \phi_\tau(y)$. We then have

$$\begin{aligned} \{x \in \text{orb}_y \mid (x, y) \in \sigma\} &= \{x \in \text{orb}_y \mid (x, y) \in \sigma \cap \tau\} \\ \{x \in \text{orb}_y \mid (x, y) \in \tau\} &= \{x \in \text{orb}_y \mid (x, y) \in \sigma \cup \tau\}. \end{aligned}$$

From this we conclude that $\phi_\sigma(y) = \phi_{\sigma \cap \tau}(y)$ and $\phi_\tau(y) = \phi_{\sigma \cup \tau}(y)$. But since $\phi_\sigma(y) \leq \phi_\tau(y)$, we have

$$\begin{aligned} (\phi_\sigma \wedge \phi_\tau)(y) &= \phi_{\sigma \cap \tau}(y) \\ (\phi_\sigma \vee \phi_\tau)(y) &= \phi_{\sigma \cup \tau}(y). \end{aligned} \quad \square$$

In view of Lemma 12, it is natural to expect that the boundary functions for meet and join irreducible ideal sets are themselves meet or join irreducible (as appropriate) with respect to the lattice operations on boundary functions. First, we consider the meet operation, for which the following function will be relevant. For all $a, b \in X$ with $a < b$, define a function $\phi_{a,b}: X \rightarrow X$ by

$$\phi_{a,b}(y) = \begin{cases} y, & \text{if } y < a, \\ a, & \text{if } a \leq y \leq b, \\ y, & \text{if } b < y. \end{cases}$$

Provided that a has no gap below, $\phi_{a,b}$ is a boundary function. (When a does have a gap below, $\phi_{a,b}$ is not a boundary function, by property 2b.) It is straightforward to check that whenever $\phi_{a,b}$ is a boundary function, it is a meet irreducible boundary function.

The ideal set $\sigma_{a,b}$ is a meet irreducible ideal set except when $(a, b) \notin P$, a has a gap above, and b has a gap below. First, assume that $\sigma_{a,b}$ is meet irreducible. If a has no gap below, then the boundary function for $\sigma_{a,b}$ is $\phi_{a,b}$. If a does have a gap below, and if $pa = \text{pred } a$, then the boundary function for $\sigma_{a,b}$ is $\phi_{pa,b}$. In this case, $\sigma_{a,b}$ and $\sigma_{pa,b}$ (which may fail to be a meet irreducible ideal set) have the same boundary function. In any event, when $\sigma_{a,b}$ is meet irreducible, so is its boundary function.

If $\sigma_{a,b}$ is not meet irreducible, i.e., if $(a, b) \notin P$, a has a gap above and b has a gap below, then $\phi_{a,b}$ (which is meet irreducible) is the boundary function for $\sigma_{a,b}$. But $\phi_{a,b}$ is also the boundary function of the meet irreducible ideal set $\sigma_{sa,b}$, where $sa = \text{succ } a$.

Thus, whenever $\phi_{a,b}$ is a boundary function (i.e., whenever a has no gap below), $\phi_{a,b}$ is meet irreducible and the boundary function of a meet irreducible ideal set.

Next, we consider meet irreducible ideals of the form $\tau_{a,b}$ and their boundary functions. If a has no gap below, the boundary function for $\tau_{a,b}$ is $\phi_{a,b}$. (In this case,

$\phi_{a,b}$ is the boundary function of two distinct meet irreducible ideal sets.) If a has a gap below, then the boundary function for $\tau_{a,b}$ is the function $\psi_{pa,a,b}$ defined by

$$\psi_{pa,a,b}(y) = \begin{cases} y, & \text{if } y < a, \\ pa, & \text{if } a \leq y < b, \\ a, & \text{if } y = b, \\ y, & \text{if } b < y. \end{cases}$$

It is straightforward to check that $\psi_{pa,a,b}$ is meet irreducible.

Note in passing that if a function of the form $\psi_{pa,a,b}$ is a boundary function, then property 2a) implies that $(a, b) \in P$. In this case it is also true that b must have a gap below. (This is required by property 2b) for boundary functions; it is also necessary in order that $\tau_{a,b}$ be an open set.)

In the case in which $\tau_{a,b}$ is not meet irreducible, i.e., when $(a, b) \in P$, a has a gap above, and b has a gap below, the boundary function for $\tau_{a,b}$ is $\phi_{a,b}$. This function is, of course, meet irreducible and is also the boundary function of a meet irreducible ideal set.

In the discussion above, we have assumed that $a < b$. When $a = b$, the ideal set $\sigma_{a,a}$ is a maximal ideal set and hence is meet irreducible. If a has no gap below, the boundary function for $\sigma_{a,a}$ is the identity function, which is trivially a meet irreducible boundary function. If a has a gap below, the boundary function for $\sigma_{a,a}$ is $\phi_{a,sa}$, a meet irreducible boundary function.

PROPOSITION 13. *Let $a, b \in P$ with $a < b$. If a has no gap below, the function $\phi_{a,b}$ defined above is meet irreducible. If a has a gap below, the function $\psi_{pa,a,b}$ defined above is meet irreducible. These functions, together with the identity function, are the only meet irreducible boundary functions. Every meet irreducible boundary function is the boundary function of a meet irreducible ideal set. Furthermore, if an ideal set is meet irreducible, then its boundary function is meet irreducible.*

Proof. We need to show that the boundary functions listed above are the only meet irreducible boundary functions. All the remaining assertions are either straightforward or have been dealt with in the discussion preceding the statement of the Proposition.

It is evident from the nature of the meet irreducible boundary functions, that for a given boundary function ϕ , we need to focus on the points $\phi(y)$ for which $\phi(y) < y$. Accordingly, define two sets:

$$ED_\phi = \{y \mid \phi(y) < y\} \quad \text{and} \quad RD_\phi = \{\phi(y) \mid y \in ED_\phi\}.$$

It is possible that $RD_\phi = \emptyset$. This happens when ϕ is the identity function. If RD_ϕ is a singleton, say $RD_\phi = \{a\}$, then ϕ has the form $\phi_{a,b}$, for some $b \in X$. This is evident from the general fact that when $a \in RD_\phi$, $\phi^{-1}(a)$ is an order interval from X . If RD_ϕ consists of two points which are the endpoints of a gap, i.e., if $RD_\phi = \{a, b\}$

where a is the immediate predecessor of b , then ϕ has the form $\psi_{pa,a,b}$. In all of these cases, ϕ is a meet irreducible boundary function.

For any other boundary function, ϕ , there will be two distinct points in RD_ϕ with a third point from X between the two. We must show that in this case, ϕ is not meet irreducible. So assume that $a < b < c$ in X and that $a, c \in RD_\phi$.

Define an auxiliary function η by

$$\eta(y) = \begin{cases} y, & \text{if } y \leq b, \\ b, & \text{if } b < y \end{cases}$$

and let $\psi_1 = \phi \vee \eta$.

It is evident that $\phi \leq \psi_1$; furthermore, $\phi \neq \psi_1$. Indeed, since $a \in RD_\phi$, there is z such that $a = \phi(z) < z$. Since $a < b$, we have $\phi(z) < b$. Now $\eta(z)$ is either b or z . In either case, $\phi(z) < \eta(z)$. This means that $\psi_1(z) = \eta(z) \neq \phi(z)$.

Now let $t = \sup\{y \in X \mid \phi(y) \leq b\}$. By properties 3) and 4) for boundary functions, $\phi(t) \leq b$. Define a boundary function ψ_2 by

$$\psi_2(y) = \begin{cases} \phi(y), & \text{if } y \leq t, \\ y, & \text{if } t < y. \end{cases}$$

It is easy to see that ψ_2 is a boundary function and that $\phi \leq \psi_2$. Furthermore, $\phi \neq \psi_2$: there is $s \in X$ such that $c = \phi(s) < s$. Since $b < c = \phi(s)$, we have $s \notin \{y \mid \phi(y) \leq b\}$. Since $\phi(t) \leq b < c$, we have $t < s$. Therefore, $\psi_2(s) = s$; in particular, $\psi_2(s) \neq \phi(s)$.

To prove that ϕ is not meet irreducible we need only show that $\phi = \psi_1 \wedge \psi_2$. Clearly, $\phi \leq \psi_1 \wedge \psi_2$. Let $y \in X$. If $t < y$, then $b < \phi(y)$; hence $\psi_1(y) = \max\{\phi(y), \eta(y)\} = \phi(y)$ (since $\eta(y) \leq b$). Thus, $\phi(y) = (\psi_1 \wedge \psi_2)(y)$. On the other hand, if $y \leq t$, then $\phi(y) = \psi_2(y)$ and $\phi(y) \leq \psi_1(y)$, so $\phi(y) = (\psi_1 \wedge \psi_2)(y)$. Thus $\phi = \psi_1 \wedge \psi_2$. \square

Next, we turn to a description of the join irreducible boundary functions. Whether or not a boundary function is join irreducible depends only on the range of the boundary function. For any boundary function ϕ , let $\text{ran } \phi = \{\phi(y) \mid y \in X\}$. Note that, since $\phi(p_{\min}) = p_{\min}$, we always have $p_{\min} \in \text{ran } \phi$.

PROPOSITION 14. *A boundary function ϕ is join irreducible if, and only if, the cardinality of $\text{ran } \phi$ is at most 2.*

Proof. If $\text{ran } \phi$ contains one element only (necessarily p_{\min}), then $\phi(y) = p_{\min}$ for all $y \in X$. Thus ϕ is the minimal boundary function and so is trivially join irreducible.

Assume that the cardinality of $\text{ran } \phi$ is 2. Then $\text{ran } \phi = \{p_{\min}, a\}$, where $p_{\min} < a$. Observe that $\phi^{-1}(p_{\min})$ and $\phi^{-1}(a)$ are intervals in X which satisfy the property that

if $y_1 \in \phi^{-1}(p_{\min})$ and $y_2 \in \phi^{-1}(y)$, then $y_1 < y_2$. Furthermore, the union of these two intervals is all of X . Consequently, there is an element $t \in X$ such that

$$\begin{aligned} y < t &\implies \phi(y) = p_{\min}, \\ t < y &\implies \phi(y) = a. \end{aligned}$$

If t does not have a gap below, then it follows from property 4) of boundary functions that $\phi(t) = p_{\min}$. If t does have a gap below, then either alternative, $\phi(t) = p_{\min}$ or $\phi(t) = a$ is possible. Note also that $a \leq t$, since $\phi(y) \leq y$, for all y .

Now suppose that $\phi = \psi_1 \vee \psi_2$, where both ψ_1 and ψ_2 are boundary function. It is evident that on the interval $\phi^{-1}(p_{\min})$ we have $\psi_1 = \psi_2 = \phi$.

First consider the case in which $\phi(t) = a$. Then either $\psi_1(t) = a$ or $\psi_2(t) = a$. Assume, without loss of generality, that $\psi_1(t) = a$. Then, for any y with $t < y$, we have $a = \psi_1(t) \leq \psi_1(y) \leq a$. This shows that $\psi_1(y) = a$ on $\phi^{-1}(a)$, and thus that $\phi = \psi_1$.

This leaves the case in which $\phi(t) = p_{\min}$. Suppose that both $\phi \neq \psi_1$ and $\phi \neq \psi_2$. Then there exist elements t_1 and t_2 such that $\psi_1(t_1) < a$, $\psi_2(t_2) < a$, $t < t_1$, and $t < t_2$. Let $t_3 = \min(t_1, t_2)$. Then $t < t_3$, $\psi_1(t_3) \leq \psi_1(t_1) < a$, and $\psi_2(t_3) \leq \psi_2(t_2) < a$. Thus, $(\psi_1 \vee \psi_2)(t_3) < a$ while $\phi(t_3) = a$, contradicting the assumption that $\phi = \psi_1 \vee \psi_2$.

We have shown that ϕ is join irreducible whenever the cardinality of $\text{ran } \phi$ is at most 2. We now assume that the cardinality of $\text{ran } \phi$ is greater than 2 and show that ϕ is not join irreducible.

Assume that $p_{\min} < a < b$ and that $a, b \in \text{ran } \phi$. We first consider the case in which there is an element $c \in X$ such that $a < c < b$. If there are any points at all between a and b , then there are infinitely many. In particular, there are points between a and b with no gap below; so we assume without loss of generality that c has no gap below.

Let

$$\begin{aligned} S &= \{y \mid \phi(y) \leq c\}, \\ T &= \{y \mid c < \phi(y)\}. \end{aligned}$$

Note that $X = S \cup T$ and that $s \in S, t \in T \implies s < t$.

Next, define

$$\begin{aligned} \psi_1(y) &= \begin{cases} \phi(y), & \text{if } y \in S, \\ c, & \text{if } y \in T, \end{cases} \\ \psi_2(y) &= \begin{cases} p_{\min}, & \text{if } y \in S, \\ \phi(y), & \text{if } y \in T. \end{cases} \end{aligned}$$

A routine, but tedious, argument (which we omit) shows that ψ_1 and ψ_2 are boundary functions. Since $b \in \text{ran } \phi$ and $b \notin \text{ran } \psi_1$, we have $\phi \neq \psi_1$. Similarly, since a is in $\text{ran } \phi$ but not in $\text{ran } \psi_2$, $\phi \neq \psi_2$. On S it is evident that $\phi = \psi_1 \vee \psi_2$; since $c < \phi(y)$

for all $y \in T$, the same equality is valid on T . Thus $\phi = \psi_1 \vee \psi_2$ and ϕ is not join irreducible.

This leaves the case in which $a, b \in \text{ran } \phi$ and b is the immediate successor of a . In particular, b has a gap below. Let t be such that $\phi(t) = b$. By condition 2) for boundary functions, t has a gap below. Furthermore, if $t < z$, then $b = \phi(t) < \phi(z)$. So, choose z such that $t < z$ (which can be done since $t \neq p_{\max}$), and let $d = \phi(z)$. We now have $b < d, b, d \in \text{ran } \phi$ and, since b has no gap above, there is c such that $b < c < d$. By the preceding argument, ϕ is not join irreducible. \square

Remark. If ϕ is a boundary function whose range is $\{p_{\min}, a\}$ with $p_{\min} \neq a$, then, by property 2), a cannot have a gap below. Note that it is also impossible to have $a = p_{\max}$.

Using Proposition 14, it is easy to describe the join irreducible boundary functions explicitly. For each pair of elements $a, t \in X$ such that $a \leq t < p_{\max}$, define $\phi^{a,t}$ by

$$\phi^{a,t}(y) = \begin{cases} p_{\min}, & \text{if } y \leq t, \\ a, & \text{if } t < y. \end{cases}$$

Then $\phi^{a,t}$ is a join irreducible boundary function. Furthermore, every join irreducible boundary function is of this form. (The main issue is the case in which t has a gap below and ϕ is the boundary function for which $\phi(y) = p_{\min}$ when $y < t$ and $\phi(y) = a$ when $t \leq y$. Let $pt = \text{pred } t$ and note that, since a has no gap below, $a < t$; in particular, $a \leq pt$. Then $\phi = \phi^{a,pt}$. The only other point to note is that $\phi^{p_{\min},t}$ is the minimal boundary function, whose range has cardinality 1.)

If $\phi = \phi^{p_{\min},t}$ is the minimal boundary function, then $\sigma(\phi) = \sigma[\phi] = \emptyset$, the ideal set for the trivial ideal (0). This is the only ideal set whose boundary function is the minimal boundary function and it is trivially a join irreducible ideal set.

For any pair $a, t \in X$ with $p_{\min} < a \leq t < p_{\max}$, define an ideal set $\sigma^{a,t}$ by

$$\sigma^{a,t} = \{(x, y) \in P \mid x < a \text{ and } t < y\}.$$

We do not need to assume that a has no gap below; $\sigma^{a,t}$ is always an ideal set. However, the boundary function for $\sigma^{a,t}$ is $\phi^{a,t}$ if, and only if, a has no gap below.

Generally speaking, $\sigma^{a,t}$ will be join irreducible. There is, in fact, only one circumstance when it is not join irreducible. This occurs when a has a gap below (let $pa = \text{pred } a$), t has a gap above (let $st = \text{succ } t$), and $(pa, st) \notin P$. In this case, $\sigma^{a,t} = \sigma^{a,st} \cup \sigma^{pa,t}$ while $\sigma^{a,t} \neq \sigma^{a,st}$ and $\sigma^{a,t} \neq \sigma^{pa,t}$.

If $(pa, st) \in P$, then $\sigma^{a,t}$ is join irreducible, as it is in all other cases when either a has no gap below or t has no gap above. The verification that $\sigma^{a,t}$ is join irreducible in all these cases is routine.

If a has a gap below, then the boundary function for $\sigma^{a,t}$ is $\phi^{pa,t}$ (and not $\phi^{a,t}$, which fails property 2) for boundary functions). As we shall see shortly, $\sigma^{a,t}$ is the maximal ideal set whose boundary function is $\phi^{pa,t}$.

Now assume that a has no gap below, so that $\phi^{a,t}$ is a boundary function. It is evident that $\sigma(\phi^{a,t}) = \sigma^{a,t}$; thus every join irreducible boundary function is the boundary function of a join irreducible ideal set. If a has no gap above, then the properties of boundary functions ensure that $\sigma[\phi^{a,t}] = \sigma^{a,t}$. In particular, when a has no gap above, there is only one ideal set whose boundary function is $\phi^{a,t}$. (Use Proposition 7.)

This leaves the case when a does have a gap above. We then have $\sigma[\phi^{a,t}] = \sigma^{sa,t}$. (Roughly speaking, because a has a gap above, we can adjoin all the “boundary points” (a, y) , $y < t$ to $\sigma^{a,t}$ to obtain a set which is open and satisfies the ideal property and therefore is an ideal set with the same boundary function.)

As noted earlier, the only time that $\sigma[\phi^{a,t}] = \sigma^{sa,t}$ will fail to be join irreducible is when a has a gap above, t has a gap above, and $(a, st) \notin P$. Thus, when a has a gap above, $\phi^{a,t}$ has distinct minimal and maximal ideal sets amongst the ideal sets whose boundary function is $\phi^{a,t}$. The minimal ideal set is always join irreducible and the maximal ideal set is also join irreducible outside of one exceptional case.

There are other ideal sets properly between $\sigma(\phi^{a,t})$ and $\sigma[\phi^{a,t}]$ when a has a gap above. While all of these have the same join irreducible boundary function, a routine argument shows that none of these ideal sets is join irreducible.

This completes the discussion of all ideal sets whose boundary function has cardinality at most 2. As for ideal sets whose boundary function has cardinality greater than 2, none are join irreducible. The routine argument is omitted; it is similar in spirit to the argument in Proposition 14.

We summarize this discussion as Proposition 15:

PROPOSITION 15. *Assume that $a, t \in X$ and $p_{\min} < a \leq t < p_{\max}$. Let*

$$\sigma^{a,t} = \{(x, y) \in P \mid x < a \text{ and } t < y\}.$$

Then $\sigma^{a,t}$ is a join irreducible ideal set except when a has a gap below, t has a gap above, and $(pa, st) \notin P$ (where $pa = \text{pred } a$ and $st = \text{succ } t$). Every non-empty join irreducible ideal set is of this form. Every join irreducible ideal set has a join irreducible boundary function. Every join irreducible boundary function is the boundary function of at least one join irreducible ideal set (and at most two join irreducible ideal sets).

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