ILLINOIS JOURNAL OF MATHEMATICS
Volume 42, Number 2, Summer 1998

# THE PRIME ELEMENT THEOREM IN ADDITIVE ARITHMETIC SEMIGROUPS, II

WEN-BIN ZHANG

## 1. Introduction

The main purpose of this paper is to give an estimate of the remainder of the prime element theorems on additive arithmetic semigroups proved in [6], Theorems 6.1 and 6.2, which are analogs of Beurling's classical generalization of the prime number theorem and have a remainder of $o(1)$ form. In applications, a better estimate of the remainder than $o(1)$ is required. Thus the new estimate can be used in the investigation of mean values of counterparts of classical arithmetic functions $\mu(n)$ and $\lambda(n)$ in additive arithmetic semigroups. The investigation reveals an interesting situation in which the counterparts of $\mu(n)$ and $\lambda(n)$ do not have a mean value. This phenomenon does not occur in the theory of the Beurling generalized integers [8]. The new estimate may also have other applications.

Let $f(n)$ and $g(n)$ be two arithmetic functions defined for all nonnegative integers $n$. The function $h(n)$ defined by setting

$$h(n) = \sum_{k=0}^{n} f(k)g(n-k), \qquad n = 0, 1, 2, \ldots$$

is called the additive convolution of $f$ and $g$ and denoted by $f * g$.

The prime element theorems proved in [6] are essentially a tauberian theorem about the solution $\bar{\lambda}(n)$ (which is not the same function $\lambda(n)$ in classical number theory!) to the convolution equation

$$\bar{\lambda} * \bar{g}(n) = n\bar{g}(n), \qquad n = 0, 1, 2, \ldots. \tag{1.1}$$

Suppose there are non-zero constants $A_1, \ldots, A_r$ with $A_r > 0$, constant $\gamma > 1$, and constants $\rho_1 < \cdots < \rho_r$ with $\rho_r > 0$ and $\rho_1 > 1 - \gamma$ such that

$$\bar{g}(n) = \sum_{j=1}^{r} A_j n^{\rho_j - 1} + O(n^{-\gamma}). \tag{1.2}$$

Let $R_1$, $R_2$ and $R_3$ denote the sets of $\rho_j$ which are positive integers, 0 or negative integers, and non-integers, respectively.

Received December 15, 1996.
1991 Mathematics Subject Classification. Primary 11N80; Secondary 11T55, 30B30.

Let $[a]_1$ denote the greatest integer less than $a$. Also, as usual, let $[a]$ denote the greatest integer less than or equal to $a$. Thus $[a]_1 = [a] - 1$ or $[a]_1 = [a]$ according as $a$ is or is not an integer.

Set

$$m_1 := \min\{[\rho_r - \rho_j] : \rho_j \in (R_1 \cup R_3) - \{\rho_r\}\} \tag{1.3}$$

if $(R_1 \cup R_3) - \{\rho_r\}$ is not empty. Also, set

$$m_2 := \begin{cases} [\rho_r]_1, & \text{if some } \rho_j = 0; \\ [\gamma]_1 - 1, & \text{if } R_2 = \emptyset \text{ and if } \rho_r \text{ is an integer}; \\ [\rho_r], & \text{else.} \end{cases} \tag{1.4}$$

We shall first prove the following tauberian theorem.

THEOREM 1.1. *Let $\bar{g}(n)$ and $\bar{\lambda}(n)$ be two nonnegative arithmetic functions satisfying $\bar{g}(0) = 1$ and equation* (1.1). *Assume* (1.2) *with $\gamma > \max\{2 + \rho_r, 3\}$. Then there exist k real numbers $0 < \theta_1 < \cdots < \theta_k \le \frac{1}{2}$, and k positive integers $n_1, \ldots, n_k$, where nonnegative integer $k \le (\rho_r + 1)/2$, and further there exits some constant $\sigma_0 > 0$ (see the following Remark* 1) *such that*

$$\bar{\lambda}(n) = \rho_r - 2\sum_{l=1}^{k-1} n_l \cos 2n\pi\theta_l - (-1)^n n_k + O(n^{-t-\sigma}) \tag{1.5}$$

*and*

$$n_k + 2\sum_{l=1}^{k-1} n_l \le \rho_r \tag{1.6}$$

*if $\theta_k = \frac{1}{2}$ or such that*

$$\bar{\lambda}(n) = \rho_r - 2\sum_{l=1}^{k} n_l \cos 2n\pi\theta_l + O(n^{-t-\sigma}) \tag{1.7}$$

*and*

$$2\sum_{l=1}^{k} n_l \le \rho_r \tag{1.8}$$

*if $\theta_k < \frac{1}{2}$ for every constant $\sigma$ with $0 < \sigma < \sigma_0$. Here*

$$t := \min\{m_1, m_2, m_3, [\gamma]_1 - 3\} \tag{1.9$_1$}$$

*with*

$$m_3 := \begin{cases} [\gamma]_1 - 1 - \max\{n_1, \ldots, n_k\}, & \text{if } k \ge 1; \\ [\gamma]_1 - 1, & \text{if } k = 0 \end{cases} \tag{1.10}$$

*if* $(R_1 \cup R_3) - \{\rho_r\}$ *is not empty and if* $\rho_r - \rho_j$ *are not all positive integers for* $\rho_j \in (R_1 \cup R_3) - \{\rho_r\}$ *and*

$$t := \min\{m_2, m_3, [\gamma]_1 - 3\} \tag{1.9$_2$}$$

*otherwise.*

*Remark* 1. For $\sigma_0$ the following values are suitable. (1) If $(R_1 \cup R_3) - \{\rho_r\} \neq \emptyset$ and if $\rho_r - \rho_j$ are not all positive integers for $\rho_j \in (R_1 \cup R_3) - \{\rho_r\}$, let

$$j_0 := \max\{j : \rho_j \in (R_1 \cup R_3) - \{\rho_r\}, [\rho_r - \rho_j] = m_1\}. \tag{1.11}$$

Then, as we shall show in the proof of Lemma 2.5, one may take

$$\sigma_0 := \begin{cases} \min\{\gamma - [\gamma]_1, \rho_r - [\rho_r]_1, \rho_r - \rho_{j_0} - [\rho_r - \rho_{j_0}]\}, \\ \min\{\rho_r - [\rho_r]_1, \rho_r - \rho_{j_0} - [\rho_r - \rho_{j_0}]\}, \\ \gamma - [\gamma]_1 \end{cases} \tag{1.12$_1$}$$

according as

$$\begin{cases} \min\{[\gamma]_1 - 3, m_1, m_2\} = \min\{[\gamma]_1 - 3, m_3\}, \\ \min\{[\gamma]_1 - 3, m_1, m_2\} < \min\{[\gamma]_1 - 3, m_3\}, \\ \min\{[\gamma]_1 - 3, m_1, m_2\} > \min\{[\gamma]_1 - 3, m_3\}. \end{cases}$$

(2) Otherwise, that is, if $(R_1 \cup R_3) - \{\rho_r\} = \emptyset$ or if $\rho_r - \rho_j$ are all positive integers for $\rho_j \in (R_1 \cup R_3) - \{\rho_r\}$, one may take

$$\sigma_0 := \begin{cases} \min\{\gamma - [\gamma]_1, \rho_r - [\rho_r]_1\}, & \text{if } \min\{[\gamma]_1 - 3, m_2\} = \min\{[\gamma]_1 - 3, m_3\}; \\ \rho_r - [\rho_r]_1, & \text{if } \min\{[\gamma]_1 - 3, m_2\} < \min\{[\gamma]_1 - 3, m_3\}; \\ \gamma - [\gamma]_1, & \text{if } \min\{[\gamma]_1 - 3, m_2\} > \min\{[\gamma]_1 - 3, m_3\}. \end{cases} \tag{1.12$_2$}$$

*Remark* 2. A more accurate estimate of the remainder in (1.5) and (1.7) can be obtained. For simplicity of statements, we shall not pursue this estimate in this paper and shall content ourselves with giving in Lemmas 2.3 and 2.4 necessary details for obtaining it.

The particular case of (1.2) with $r = 1$ and $\rho_r = 1$ is of most interest. It has been considered in [2] and [5] under conditions of remainder $O(q^{vn})$ with $0 < v < 1$. In this case, Theorem 1.1 says that if $\gamma > 3$ then

$$\bar{\lambda}(n) = 1 + O_\epsilon(n^{-\gamma+3+\epsilon}) \quad \text{or} \quad \bar{\lambda}(n) = 1 - (-1)^n + O_\epsilon(n^{-\gamma+3+\epsilon})$$

for any $\epsilon > 0$. This result can be sharpened significantly in Theorem 1.2. Also, a comparison of this result with Theorem 6.3 in [**6**] with $\tau = 1$ and $\gamma > 2$ indicates a gap in Theorem 1.1 for $2 < \gamma \leq 3$. This gap is filled with Theorem 1.2 too.

THEOREM 1.2. *If*

$$\bar{g}(n) = A + O(n^{-\gamma}), \qquad n \geq 1$$

*with* $A > 0, \gamma > 2$, *then either*

$$\bar{\lambda}(n) = \begin{cases} 1 + O(n^{-\gamma+2}), & \text{if } \gamma \text{ is not an integer;} \\ 1 + O(n^{-\gamma+2}\log n), & \text{if } \gamma \text{ is an integer} \end{cases} \tag{1.13}$$

*or*

$$\bar{\lambda}(n) = \begin{cases} 1 - (-1)^n + O(n^{-\gamma+2}\log n), & \text{if } \gamma \text{ is not an integer;} \\ 1 - (-1)^n + O(n^{-\gamma+2}\log^2 n), & \text{if } \gamma \text{ is an integer.} \end{cases} \tag{1.14}$$

A variant of Theorem 1.1 or Theorem 1.2 in terms of additive arithmetic semigroups is a prime element theorem with a new estimate of the remainder. We recall that an additive arithmetic semigroup $\mathcal{G}$ is, by definition [3], [5], a free commutative semigroup with identity element 1 such that $\mathcal{G}$ has a countable free generating set $\mathcal{P}$ of "primes" $p$ and such that $\mathcal{G}$ admits an integer-valued "degree" mapping $\partial$: $\mathcal{G} \to \mathbf{N} \cup \{0\}$ satisfying

(i) $\partial(1) = 0$ and $\partial(p) > 0$ for all $p \in \mathcal{P}$,
(ii) $\partial(ab) = \partial(a) + \partial(b)$ for all $a, b \in \mathcal{G}$, and
(iii) the total number $\bar{G}(n)$ of elements of degree $n$ in $\mathcal{G}$ is finite for each $n \geq 0$.

Additive arithmetic semigroups cover concrete cases [3] such as (i) the multiplicative semigroup $G_q$ of all monic polynomials in one indeterminate over a finite field $\mathbf{F}_q$, (ii) semigroups of ideals in principal orders within algebraic function fields over $\mathbf{F}_q$, (iii) semigroups formed under direct sum by the isomorphism classes of certain kinds of finite modules or algebras over such principal orders.

Let $\bar{P}(n)$ be the total number of primes of degree $n$ in $\mathcal{G}$. Also let

$$\bar{\Lambda}(n) = \sum_{r|n} r\bar{P}(r).$$

Then [6]

$$\bar{\Lambda} * \bar{G}(n) = n\bar{G}(n). \tag{1.15}$$

Under the condition

$$\bar{G}(n) = q^n \sum_{j=1}^{r} A_j n^{\rho_j - 1} + O(q^n n^{-\gamma}) \tag{1.16}$$

with $\gamma > \max\{1 + \rho_r, \frac{3}{2}\}$, the prime element theorems proved in [6] has remainder of the form $q^n o(1)$. Let now $\bar{g}(n) = \bar{G}(n)q^{-n}$ and $\bar{\lambda}(n) = \bar{\Lambda}(n)q^{-n}$. If $\bar{G}(n)$ and $\bar{\Lambda}(n)$ satisfy (1.15) and (1.16) with $\gamma > \max\{2 + \rho_r, 3\}$, then $\bar{g}(n)$ and $\bar{\lambda}(n)$ satisfy the conditions of Theorem 1.1. Hence Theorem 1.1 implies a better estimate of the remainder of the prime element theorem. The same can be said about Theorem 1.2.

THEOREM 1.3. *Assume* (1.16) *with* $q > 1$ *and* $\gamma > \max\{2 + \rho_r, 3\}$. *Then there exist* $k$ *real numbers* $0 < \theta_1 < \cdots < \theta_k \le \frac{1}{2}$ *and* $k$ *positive integers* $n_1, \ldots, n_k$, *where nonnegative integer* $k \le (\rho_r + 1)/2$, *such that*

$$\bar{\Lambda}(n) = q^n \left( \rho_r - 2 \sum_{l=1}^{k-1} n_l \cos 2n\pi\theta_l - (-1)^n n_k \right) + O(q^n n^{-t-\sigma}) \tag{1.17}$$

*and*

$$n_k + 2 \sum_{l=1}^{k-1} n_l \le \rho_r \tag{1.18}$$

*if* $\theta_k = \frac{1}{2}$ *or such that*

$$\bar{\Lambda}(n) = q^n \left( \rho_r - 2 \sum_{l=1}^{k} n_l \cos 2n\pi\theta_l \right) + O(q^n n^{-t-\sigma}) \tag{1.19}$$

*and*

$$2 \sum_{l=1}^{k} n_l \le \rho_r \tag{1.20}$$

*if* $\theta_k < \frac{1}{2}$. *Here constants* $t$ *and* $\sigma$ *are defined in Theorem* 1.1.

*Remark.* There is a misprint in Theorem 6.1 in [6]. The condition $\gamma > \frac{3}{2}$ of the theorem should be replaced by $\gamma > \max\{\frac{3}{2}, 1 + \rho_r\}$.

THEOREM 1.4. *If*

$$\bar{G}(n) = Aq^n + O(q^n n^{-\gamma}) \tag{1.21}$$

*with* $A > 0, q > 1$, *and* $\gamma > 2$, *then either*

$$\bar{\Lambda}(n) = \begin{cases} q^n(1 + O(n^{-\gamma+2})), & \text{if } \gamma \text{ is not an integer;} \\ q^n(1 + O(n^{-\gamma+2} \log n)), & \text{if } \gamma \text{ is an integer} \end{cases}$$

*or*

$$\bar{\Lambda}(n) = \begin{cases} q^n(1 - (-1)^n + O(n^{-\gamma+2} \log n)), & \text{if } \gamma \text{ is not an integer;} \\ q^n(1 - (-1)^n + O(n^{-\gamma+2} \log^2 n)), & \text{if } \gamma \text{ is an integer.} \end{cases}$$

As an application of Theorems 1.3 and 1.4, combining with Theorem 1.5 of [7], we consider mean values of functions $\lambda(a)$ and $\mu(a)$ defined on an additive arithmetic semigroup $\mathcal{G}$. Here $\lambda(a) := (-1)^{\Omega(a)}$ and $\Omega(a)$ denotes the total number of prime divisors of $a$ counted according to multiplicity. The function $\mu(a) := (-1)^{\omega(a)}$ if $a$ is squarefree and $\mu(a) := 0$ otherwise and $\omega(a)$ denotes the number of distinct prime divisors of $a$. Thus $\lambda(a), \Omega(a), \mu(a), \omega(a)$ are the respective counterparts of the classical functions $\lambda(n)$ (Liouville function), $\Omega(n)$, $\mu(n)$ (Möbius function), $\omega(n)$. Let

$$Z^{\#}(y) := \sum_{n=0}^{\infty} \bar{G}(n) y^n,$$

the generating function of $\bar{G}(n)$.

THEOREM 1.5. (1) *Assume* (1.16) *with* $q > 1$ *and* $\gamma > \max\{2 + \rho_r, 3\}$. *Then*

$$\sum_{\partial(a)=m} \lambda(a) = o(q^m m^{\rho_r - 1})$$

*and*

$$\sum_{\partial(a)=m} \mu(a) = o(q^m m^{\rho_r - 1})$$

*if* $Z^{\#}(y)$ *has no zero at* $y = -q^{-1}$ *or a zero at* $y = -q^{-1}$ *of order less than* $\rho_r$; *otherwise,* $\lambda$ *and* $\mu$ *has no mean value if* $Z^{\#}(y)$ *has a zero at* $y = -q^{-1}$ *of integer-order equal to* $\rho_r$.

(2) *Assume* (1.21) *with* $q > 1$ *and* $\gamma > 2$. *Then*

$$\sum_{\partial(a)=m} \lambda(a) = o(q^m), \qquad \sum_{\partial(a)=m} \mu(a) = o(q^m)$$

*if* $Z^{\#}(y)$ *has no zero on the circle* $|y| = q^{-1}$; *otherwise,* $\lambda$ *and* $\mu$ *has no mean value if* $Z^{\#}(y)$ *has a zero at* $y = -q^{-1}$.

*Remark.* The mean value of $\mu(a)$ is considered in [1] and [4] too. If $Z^{\#}(y)$ has a zero at $y = -q^{-1}$ of integer-order equal to $\rho_r$ then it has no other zeros on the closed disk $\{|y| \leq 1\}$ as we know from Theorems 4.1 and 5.1 of [6]. In this case, $\mu(a)$ does not have a mean value because of the dominant perturbation of the zero at $y = -q^{-1}$. This is well illustrated by Example 4.1 in [6] as we shall see from a brief discussion given at the end of this paper. We note that this phenomenon does not occur in the theory of Beurling's generalized integers [8].

Since Theorems 1.3 and 1.4 are direct consequences of Theorems 1.1 and 1.2 respectively, it is sufficient to prove Theorems 1.1, 1.2, and 1.5.

The basic idea of our proofs of Theorems 1.1 and 1.2 can be summarized as follows. In these proofs as well as in those in [6], everything boils down to an estimation of integrals of the form $\int_{-\pi}^{\pi} f(\theta) e^{in\theta} d\theta$. Formerly, in [6], we took the Riemann-Lebesgue lemma and the estimate $o(1)$ came up. Now, instead, we take a delicate analysis of the behavior of the generating function

$$Z(y) := \sum_{n=0}^{\infty} \bar{g}(n) y^n \tag{1.22}$$

of $\bar{g}(n)$ on the boundary circle $|y| = 1$ of the disk $\{|y| < 1\}$. This analysis reveals that the function $f(\theta)$ in the integrals satisfies, loosely speaking, some kind of Lipschitz condition. Using the Lipschitz condition, the estimate of the integral $\int_{-\pi}^{\pi} f(\theta)e^{in\theta}d\theta$ can be sharpened to $O(n^{-\alpha})$ with some $\alpha > 0$. Thus the analysis of the boundary behavior of $Z(y)$ will be the major part of this paper.

## 2. Proof of Theorem 1.1

We begin with rephrasing Lemma 2.1 in [6] for convenient use in further discussion.

LEMMA 2.1. *Let $\rho_1, \ldots, \rho_r$ be arbitrary real numbers with $\rho_1 < \cdots < \rho_r$ and $A_1, \ldots, A_r$ be nonzero arbitrary real numbers.*

(1) *If $\rho_1, \ldots, \rho_r$ are all positive integers then there exist positive integers $\tau_1 < \cdots < \tau_s$ with $\tau_s = \rho_r$ and $\tau_\mu = \rho_\nu - k$ for some nonnegative integer $k$, and real numbers $B_1, \ldots, B_s$ such that*

$$\sum_{\nu=1}^{r} A_\nu n^{\rho_\nu - 1} = \sum_{\mu=1}^{s} B_\mu \binom{n + \tau_\mu - 1}{n},$$

*where $B_s = A_r(\rho_r - 1)!$.*

(2) *If $\rho_1, \ldots, \rho_r$ are all non-integers, then for any positive integer $m$ there exist a positive integer $s = s(m)$, real numbers $\tau_1 < \ldots < \tau_s$ with $\tau_s = \rho_r$ and $\tau_\mu = \rho_\nu - k$ for some nonnegative integer $k$ ($\leq m + [\rho_\nu] - 2$), and real numbers $B_1, \ldots, B_s$ such that, as $n \to \infty$,*

$$\sum_{\nu=1}^{r} A_\nu n^{\rho_\nu - 1} = \sum_{\mu=1}^{s} B_\mu \binom{n + \tau_\mu - 1}{n} + O(n^{-m+\alpha}),$$

*where $\alpha = \max\{\rho_\nu - [\rho_\nu], \nu = 1, \ldots, r\}$, $B_s = A_r\Gamma(\rho_r)$, and $\Gamma$ is the Euler gamma function.*

If we write

$$\sum_{j=1}^{r} A_j n^{\rho_j - 1} = \sum_1 + \sum_2 + \sum_3,$$

where $\sum_1$, $\sum_2$, and $\sum_3$ are partial sums over $R_1$, $R_2$, and $R_3$ respectively, and apply Lemma 2.1 to $\sum_1$ and $\sum_3$, we have

$$Z(y) = \sum_{\mu=1}^{s} B_\mu (1-y)^{-\tau_\mu} + A\log(1-y) + \sum_{k=2}^{[\gamma]_1} C_k \sum_{n=1}^{\infty} \frac{y^n}{n^k} + \sum_{n=1}^{\infty} r_n y^n \tag{2.1}$$

for $|y| \leq 1, y \neq 1$, by (1.2). Here $\tau_1 < \cdots < \tau_s$ with $\tau_s = \rho_r$, $A = A_j$ if some $\rho_j = 0$ and $A = 0$ otherwise, $C_k = A_j$ for $\rho_j = -k + 1$, and $r_n = O(n^{-\beta})$ with

$\beta = \min\{\gamma, m-\alpha\}$, $\alpha = \max\{\rho_j - [\rho_j], j = 1, \ldots, r\}$ if $\rho_j$ are not all integers and $\beta = \gamma$ otherwise. Note that $Z(y)$ has no zeros in the open disk $\{|y| < 1\}$, as is proved in Theorem 3.1 [6], where the assumption $\lambda \geq 0$ is used.

Let $\alpha(\theta)$ denote the order of a zero $y = e^{2\pi i\theta}$ of $Z(y)$ on the circle $|y| = 1$ with $0 < \theta < 1$ (refer to definition (4.1) of [6]).

With $\gamma > 1, m = 2$, by Theorem 4.1 of [6], the "total number" of zeros of $Z(y)$ on the circle $|y| = 1$ is at most $\rho_r$ in the sense that

$$\alpha\left(\frac{1}{2}\right) + 2\sum_{0<\theta<\frac{1}{2}} \alpha(\theta) \leq \rho_r$$

or

$$2\sum_{0<\theta<\frac{1}{2}} \alpha(\theta) \leq \rho_r$$

according as $-1$ is or is not a zero of $Z(y)$.

Assume $\gamma > 1 + \rho_r$. Then, with the (least) integer $m \geq \gamma + \max\{\rho_j - [\rho_j], j = 1, \ldots, r\}$, $r_n = O(n^{-\gamma})$ and $\sum_{n=1}^{\infty} r_n y^n$ has continuous derivatives of order $\leq [\gamma]_1 - 1$ on the closed disk $\{|y| \leq 1\}$. We shall use this fact repeatedly later. Hence, from (2.1), $Z(y)$ has continuous derivatives of order $\leq [\gamma]_1 - 1$ on $\{|y| \leq 1, y \neq 1\}$. By Theorem 5.1 of [6], order $\alpha$ of each zero $y = e^{2\pi i\theta}$ of $Z(y)$ is a positive integer and

$$\lim_{r\to 1-} \frac{Z(re^{2\pi i\theta})}{(1-r)^{\alpha} e^{2\pi i\alpha\theta}} = \frac{(-1)^{\alpha}}{\alpha!} Z^{(\alpha)}(e^{2\pi i\theta}) \neq 0.$$

Therefore, $Z(y)$ has at most $[\rho_r]$ zeros on the circle $|y| = 1$. In the case that $-1$ is one of its zero, $Z(y)$ has $2k-1$ distinct zeros $e^{\pm 2\pi i\theta_l}$, $l = 1, \ldots, k-1$ and $-1$ with $0 < \theta_1 < \cdots < \theta_k = \frac{1}{2}$ and $k \leq (\rho_r + 1)/2$. Let the order of zero $y = e^{2\pi i\theta_l}$ be $n_l, l = 1, \ldots, k$. Then (1.6) holds. Similarly, in case that $-1$ is not a zero of $Z(y)$, (1.8) holds.

It remains to prove (1.5) and (1.7). We shall give only the proof of (1.5). The one of (1.7) is almost the same but easier. Hence we assume $\theta_k = \frac{1}{2}$.

Let

$$F(y) := \frac{(1-y)^{\rho_r} Z(y)}{(1+y)^{n_k} \prod_{l=1}^{k-1}(1-ye^{2\pi i\theta_l})^{n_l}(1-ye^{-2\pi i\theta_l})^{n_l}}. \tag{2.2}$$

Then $F(y)$ has no zeros on the closed disk $\{|y| \leq 1\}$. Also $F(y)$ has continuous derivatives of order $\leq [\gamma]_1 - 1$ on $\{y : |y| \leq 1, y \neq 1, y \neq e^{\pm 2\pi i\theta_l}, l = 1, \ldots, k\}$ since so does $Z(y)$ on $\{y : |y| \leq 1, y \neq 1\}$. Let $\theta_0 = 0$. To analyze the boundary behavior of $F(y)$, it is sufficient to consider the upper half of the circle $|y| = 1$; by symmetry, the conclusion is applicable to the lower half. Thus the boundary behavior of $F(y)$ is determined by

$$Z_l(y) := \frac{Z(y)}{(1-ye^{-2\pi i\theta_l})^{n_l}} \tag{2.3}$$

on the arc $\{y = e^{2\pi i\theta} : (\theta_l + \theta_{l-1})/2 \le \theta \le (\theta_l + \theta_{l+1})/2\}$ for $1 \le l \le k-1$, by

$$Z_k(y) := \frac{Z(y)}{(1+y)^{n_k}} \tag{2.4}$$

on the arc $\{y = e^{2\pi i\theta} : (\theta_k + \theta_{k-1})/2 \le \theta \le (\theta_k + (1-\theta_{k-1}))/2\}$, and by

$$Z_0(y) := (1-y)^{\rho_r} Z(y) \tag{2.5}$$

on the arc $\{y = e^{2\pi i\theta} : -\theta_1/2 \le \theta \le \theta_1/2\}$.

We need the following elementary estimate in the analysis of the boundary behavior of $F(y)$ on the circle $|y| = 1$.

LEMMA 2.2. *Let $H(y) = \sum_{n=0}^{\infty} a_n y^n$, $|y| \le 1$ with $a_n = O(n^{-\gamma})$ and $\gamma > 2$. Then*

$$\left|H^{([\gamma]_1-1)}(y) - H^{([\gamma]_1-1)}(y_0)\right| \ll \begin{cases} |y-y_0|^{\gamma-[\gamma]}, & \text{if } \gamma \text{ is not an integer;} \\ |y-y_0|\log\dfrac{1}{|y-y_0|}, & \text{if } \gamma \text{ is an integer.} \end{cases}$$

*Proof.* We may write

$$H^{([\gamma]_1-1)}(y) = \sum_{n=0}^{\infty} b_n y^n,$$

where

$$b_n = a_{n+[\gamma]_1-1}(n+1)\cdots(n+[\gamma]_1-1) = O(n^{-\gamma+[\gamma]_1-1}).$$

Then, if $\gamma$ is not an integer, $[\gamma]_1 = [\gamma]$,

$$\begin{aligned} H^{([\gamma]_1-1)}(y) - H^{([\gamma]_1-1)}(y_0) &= \sum_{n\le M} b_n(y^n - y_0^n) + \sum_{n>M} b_n(y^n - y_0^n) \\ &\ll |y-y_0|\sum_{n\le M}|b_n|n + \sum_{n>M}|b_n| \\ &\ll |y-y_0|\sum_{n\le M} n^{-\gamma+[\gamma]} + \sum_{n>M} n^{-\gamma+[\gamma]-1} \\ &\ll |y-y_0|M^{-\gamma+[\gamma]+1} + M^{-\gamma+[\gamma]}. \end{aligned}$$

For $y \ne y_0$, let $M = |y-y_0|^{-1}$; then

$$H^{([\gamma]_1-1)}(y) - H^{([\gamma]_1-1)}(y_0) \ll |y-y_0|^{\gamma-[\gamma]}.$$

If $\gamma$ is an integer, $\gamma - [\gamma]_1 = 1$, a similar argument gives

$$H^{([\gamma]_1-1)}(y) - H^{([\gamma]_1-1)}(y_0) \ll |y-y_0|\log M + M^{-1} \ll |y-y_0|\log\frac{1}{|y-y_0|}. \quad \square$$

LEMMA 2.3. *The derivatives of $Z_l(y)$ of order $\le [\gamma]_1 - 1 - n_l$ have continuous extension on the region $D_l := \{y = re^{2\pi i\theta} : 0 \le r \le 1, (\theta_l + \theta_{l-1})/2 \le \theta \le (\theta_l + \theta_{l+1})/2\}$ if $1 \le l \le k-1$ and on the region $D_k := \{y = re^{2\pi i\theta} : 0 \le r \le 1, (\theta_k + \theta_{k-1})/2 \le \theta \le (\theta_k + (1 - \theta_{k-1}))/2\}$ if $l = k$. Moreover, let $t = \min\{[\gamma]_1 - 3, [\gamma]_1 - 1 - n_l\}$. Then*

$$Z_l^{(t+1)}(e^{2\pi i\theta}) \ll \begin{cases} |\theta - \theta_l|^{\gamma - [\gamma] - 1}, & \textit{if } \gamma \textit{ is not an integer;} \\ \log \dfrac{1}{|\theta - \theta_l|}, & \textit{if } \gamma \textit{ is an integer} \end{cases} \tag{2.6}$$

*and*

$$Z_l^{(t+2)}(e^{2\pi i\theta}) \ll \begin{cases} |\theta - \theta_l|^{\gamma - [\gamma] - 2}, & \textit{if } \gamma \textit{ is not an integer;} \\ |\theta - \theta_l|^{-1} \log \dfrac{1}{|\theta - \theta_l|}, & \textit{if } \gamma \textit{ is an integer} \end{cases} \tag{2.7}$$

*for $(\theta_l + \theta_{l-1})/2 \le \theta \le (\theta_l + \theta_{l+1})/2, \theta \ne \theta_l$ if $1 \le l \le k-1$ and for $(\theta_k + \theta_{k-1})/2 \le \theta \le (\theta_k + (1 - \theta_{k-1}))/2, \theta \ne \theta_k$ if $l = k$.*

*Proof.* For convenience, let $y_0 = e^{2\pi i\theta_l}$. On the region $D_l$, we may express

$$\begin{aligned} Z(y) &= \frac{1}{n_l!} Z^{(n_l)}(y_0)(y - y_0)^{n_l} + \frac{1}{(n_l + 1)!} Z^{(n_l+1)}(y_0)(y - y_0)^{n_l+1} \\ &\quad + \cdots + \frac{1}{([\gamma]_1 - 1)!} Z^{([\gamma]_1 - 1)}(y_0)(y - y_0)^{[\gamma]_1 - 1} \\ &\quad + \frac{1}{([\gamma]_1 - 2)!} \int_{y_0}^{y} \left(Z^{([\gamma]_1 - 1)}(u) - Z^{([\gamma]_1 - 1)}(y_0)\right)(y - u)^{[\gamma]_1 - 2} du. \end{aligned}$$

Thus, for $y \ne y_0$,

$$Z_l(y) = (-1)^{n_l} y_0^{n_l} \left\{ P(y) + \frac{1}{([\gamma]_1 - 2)!} \frac{I(y)}{(y - y_0)^{n_l}} \right\}, \tag{2.8}$$

where $P(y)$ is a polynomial of degree $\le [\gamma]_1 - 1 - n_l$ and

$$I(y) := \int_{y_0}^{y} \left(Z^{([\gamma]_1 - 1)}(u) - Z^{([\gamma]_1 - 1)}(y_0)\right)(y - u)^{[\gamma]_1 - 2} du.$$

We have

$$\begin{aligned} I^{(m)}(y) &= ([\gamma]_1 - 2)([\gamma]_1 - 3) \ldots ([\gamma]_1 - m - 1) \\ &\quad \times \int_{y_0}^{y} \left(Z^{([\gamma]_1 - 1)}(u) - Z^{([\gamma]_1 - 1)}(y_0)\right)(y - u)^{[\gamma]_1 - 2 - m} du \end{aligned} \tag{2.9}$$

for $1 \le m \le [\gamma]_1 - 2$ and

$$I^{([\gamma]_1-1)}(y) = ([\gamma]_1 - 2)!\left(Z^{([\gamma]_1-1)}(y) - Z^{([\gamma]_1-1)}(y_0)\right).$$

By integration by substitution, the integral on the right-hand side of (2.9) equals

$$(y-y_0)^{[\gamma]_1-m-1}\int_0^1 \left(Z^{([\gamma]_1-1)}(y_0+s(y-y_0)) - Z^{([\gamma]_1-1)}(y_0)\right)(1-s)^{[\gamma]_1-2-m}ds.$$

Hence

$$\begin{aligned}\left(\frac{I(y)}{(y-y_0)^{n_l}}\right)^{(m)} &= \sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k n_l(n_l+1)\dots(n_l+k-1)}{(y-y_0)^{n_l+k}}I^{(m-k)}(y)\\ &= \sum_{k=0}^{m} d_k(y-y_0)^{[\gamma]_1-m-1-n_l}\\ &\quad\times\int_0^1\left(Z^{([\gamma]_1-1)}(y_0+s(y-y_0)) - Z^{([\gamma]_1-1)}(y_0)\right)(1-s)^{[\gamma]_1-2-m+k}ds. \qquad (2.10)\end{aligned}$$

Therefore, for $1 \le m \le [\gamma]_1 - 1 - n_l$ ($\le [\gamma]_1 - 2$ !), $(I(y)(y-y_0)^{-n_l})^{(m)}$ has limit zero as $y \to y_0$ in the region $D_l$, since $Z^{([\gamma]_1-1)}(y)$ is continuous. By (2.8), this shows that the derivatives of $Z_l(y)$ of order $\le [\gamma]_1 - 1 - n_l$ have continuous extension on region $D_l$.

For $t = \min\{[\gamma]_1 - 3, [\gamma]_1 - 1 - n_l\}$ and $y = e^{2\pi i\theta} \ne y_0$, from (2.8) and (2.10),

$$\begin{aligned}Z_l^{(t+1)}(e^{2\pi i\theta}) &\ll 1 + \sum_{k=0}^{t+1}\left|e^{2\pi i\theta} - e^{2\pi i\theta_l}\right|^{[\gamma]_1-2-t-n_l}\\ &\quad\times\int_0^1\left|Z^{([\gamma]_1-1)}(e^{2\pi i\theta_l} + s(e^{2\pi i\theta} - e^{2\pi i\theta_l}))\right.\\ &\quad\left.-Z^{([\gamma]_1-1)}(e^{2\pi i\theta_l})\right|(1-s)^{[\gamma]_1-3-t+k}ds\\ &\ll 1 + |\theta-\theta_l|^{-1}|\theta-\theta_l|^{\gamma-[\gamma]} \ll |\theta-\theta_l|^{\gamma-[\gamma]-1}\end{aligned}$$

if $\gamma$ is not an integer (equivalently, if $[\gamma] = [\gamma]_1$) and

$$\ll 1 + |\theta-\theta_l|^{-1}|\theta-\theta_l|\log\frac{1}{|\theta-\theta_l|} \ll \log\frac{1}{|\theta-\theta_l|}$$

if $\gamma$ is an integer, since, by (2.1) and Lemma 2.2,

$$Z^{([\gamma]_1-1)}(y) - Z^{([\gamma]_1-1)}(y_0) \ll \begin{cases}|y-y_0|^{\gamma-[\gamma]}, & \text{if } \gamma \text{ is not an integer;}\\ |y-y_0|\log\dfrac{1}{|y-y_0|}, & \text{if } \gamma \text{ is an integer.}\end{cases}$$

Also,

$$Z_l^{(t+2)}(e^{2\pi i\theta}) \ll 1+\sum_{k=0}^{t+2}\left|e^{2\pi i\theta}-e^{2\pi i\theta_l}\right|^{[\gamma]_1-3-t-n_l}$$
$$\times\int_0^1 \left|Z^{([\gamma]_1-1)}(e^{2\pi i\theta_l}+s(e^{2\pi i\theta}-e^{2\pi i\theta_l}))\right.$$
$$\left.-Z^{([\gamma]_1-1)}(e^{2\pi i\theta_l})\right|(1-s)^{[\gamma]_1-4-t+k}ds$$

if $t<[\gamma]_1-3$ and

$$Z_l^{(t+2)}(e^{2\pi i\theta}) \ll 1+\sum_{k=0}^{t+2}\left|e^{2\pi i\theta}-e^{2\pi i\theta_l}\right|^{-n_l}$$
$$\times\int_0^1 \left|Z^{([\gamma]_1-1)}(e^{2\pi i\theta_l}+s(e^{2\pi i\theta}-e^{2\pi i\theta_l}))\right.$$
$$\left.-Z^{([\gamma]_1-1)}(e^{2\pi i\theta_l})\right|(1-s)^{k-1}ds$$
$$+\left|e^{2\pi i\theta}-e^{2\pi i\theta_l}\right|^{-n_l}\left|Z^{([\gamma]_1-1)}(e^{2\pi i\theta})-Z^{([\gamma]_1-1)}(e^{2\pi i\theta_l})\right|$$

if $t=[\gamma]_1-3$. We note that if $t<[\gamma]_1-3$ then $t+n_l=[\gamma]_1-1$ and if $t=[\gamma]_1-3$ then $n_l\le 2$. Therefore,

$$Z_l^{(t+2)}(e^{2\pi i\theta}) \ll 1+|\theta-\theta_l|^{-2}|\theta-\theta_l|^{\gamma-[\gamma]} \ll |\theta-\theta_l|^{\gamma-[\gamma]-2}$$

if $\gamma$ is not an integer and

$$Z_l^{(t+2)}(e^{2\pi i\theta}) \ll 1+|\theta-\theta_l|^{-2}|\theta-\theta_l|\log\frac{1}{|\theta-\theta_l|} \ll |\theta-\theta_l|^{-1}\log\frac{1}{|\theta-\theta_l|}$$

if $\gamma$ is an integer. □

LEMMA 2.4. *Let* $m=\min\{m_1,m_2\}$ *and* $\sigma=\min\{\rho_r-\rho_{j_0}-[\rho_r-\rho_{j_0}],\ \rho_r-[\rho_r]_1\}$ *if* $(R_1\cup R_3)-\{\rho_r\}\ne\emptyset$ *and if* $\rho_r-\rho_j$ *are not all positive integers for* $\rho_j\in(R_1\cup R_3)-\{\rho_r\}$ *and let* $m=m_2$ *and* $\sigma=\rho_r-[\rho_r]_1$ *otherwise. The derivatives of* $Z_0(y)$ *of order* $\le m$ *have continuous extension on the region* $D_0:=\{y=re^{2\pi i\theta}:0\le r\le 1,\ -\theta_1/2\le\theta\le\theta_1/2\}$. *Moreover, let* $t=\min\{[\gamma]_1-3,m\}$. *Then*

$$Z_0^{(t+\nu)}(e^{2\pi i\theta}) \ll \begin{cases} |\theta|^{\sigma-\nu}\log\dfrac{1}{|\theta|}, & \textit{if } R_2\ne\emptyset \textit{ and if } t=m_2;\\ |\theta|^{\sigma-\nu}, & \textit{otherwise,}\end{cases} \qquad \nu=1,2 \quad (2.11)$$

*for* $0<|\theta|\le\theta_1/2$.

*Proof.* By (2.1),

$$Z_0(y) = Z_{01}(y) + Z_{02}(y) + Z_{03}(y) + Z_{04}(y), \tag{2.12}$$

where

$$Z_{01}(y) := B_s + \sum_{\mu=1}^{s-1} B_\mu (1-y)^{\rho_r - \tau_\mu},$$

$$Z_{02}(y) := (1-y)^{\rho_r} \left( \sum_{k=2}^{[\gamma]_1} C_k \sum_{n=1}^{\infty} \frac{y^n}{n^k} \right),$$

$$Z_{03}(y) := A(1-y)^{\rho_r} \log(1-y),$$

and

$$Z_{04}(y) := (1-y)^{\rho_r} \sum_{n=1}^{\infty} r_n y^n.$$

First, consider $Z_{01}(y)$. If $\rho_r - \rho_j$ is an integer, $(1-y)^{\rho_r-\tau_\mu}$ is a polynomial for $\tau_\mu = \rho_j - k$ with a nonnegative integer $k$ (see Lemma 2.1). If $\rho_r - \rho_j$ is not an integer, $(1-y)^{\rho_r-\tau_\mu}$ has continuous derivatives of order $\leq [\rho_r - \rho_j] + k$ in $D_0$ for $\tau_\mu = \rho_j - k$. Hence, if $\rho_r - \rho_j$ are not all positive integers for $\rho_j \in (R_1 \cup R_3) - \{\rho_r\}$, $Z_{01}(y)$ has continuous derivatives of order $\leq m_1$ or, otherwise, $Z_{01}(y)$ is a polynomial and has continuous derivatives of all orders.

If $(R_1 \cup R_3) - \{\rho_r\} \neq \emptyset$ and if $\rho_r - \rho_j$ are not all positive integers for $\rho_j \in (R_1 \cup R_3) - \{\rho_r\}$, we have

$$\begin{aligned} Z_{01}^{(m_1+1)}(e^{2\pi i\theta}) &= \sum_{\mu=1}^{s-1} E_\mu (1-e^{2\pi i\theta})^{\rho_r-\tau_\mu-m_1-1} \\ &\ll \sum_{\mu=1}^{s-1} |1-e^{2\pi i\theta}|^{\rho_r-\tau_\mu-m_1-1} \\ &\ll \sum_{\rho_j\in(R_1\cup R_3)-\{\rho\}} |1-e^{2\pi i\theta}|^{\rho_r-\rho_j-[\rho_r-\rho_{j_0}]-1} \\ &\ll \sum_{[\rho_r-\rho_j]=m_1} |1-e^{2\pi i\theta}|^{\rho_r-\rho_j-[\rho_r-\rho_{j_0}]-1} \\ &\ll |1-e^{2\pi i\theta}|^{\rho_r-\rho_{j_0}-[\rho_r-\rho_{j_0}]-1} \\ &\ll |\theta|^{\rho_r-\rho_{j_0}-[\rho_r-\rho_{j_0}]-1}, \end{aligned} \tag{2.13}$$

where $j_0$ is defined in (1.11). Similarly,

$$Z_{01}^{(m_1+2)}(e^{2\pi i\theta}) \ll |\theta|^{\rho_r-\rho_{j_0}-[\rho_r-\rho_{j_0}]-2}. \tag{2.14}$$

Secondly, consider $Z_{02}(y)$. We note that

$$\begin{aligned}
\left(\sum_{n=1}^{\infty} \frac{y^n}{n^k}\right)^{(l)} &= \sum_{n=1}^{\infty} \frac{n(n-1)\dots(n-l+1)}{n^k} y^{n-l} \\
&= \frac{1}{y^l}\left(\sum_{n=1}^{\infty} n^{l-k} y^n + d_1 \sum_{n=1}^{\infty} n^{l-k-1} y^n + \cdots + d_{l-1} \sum_{n=1}^{\infty} n^{-k+1} y^n\right) \\
&= \begin{cases} O(1), & \text{if } l \le k-2; \\ O\left(\log \frac{1}{|1-y|}\right), & \text{if } l = k-1; \\ O(|1-y|^{k-l-1}), & \text{if } l \ge k \end{cases} \qquad (2.15)
\end{aligned}$$

uniformly for $|y| < 1$, since, by (i) of Lemma 2.1,

$$\sum_{n=1}^{\infty} \frac{y^n}{n^k} = e_{k+1} + \sum_{u=0}^{k} e_u (1-y)^{-u-1}$$

for a positive integer $k$. We also note that $\sum_{n=1}^{\infty} y^n n^{-k}$ has an analytic extension, denoted by $W_k(y)$, say, on the domain $D$ formed by cutting the complex plane along the real axis from 1 to $+\infty$. Hence the estimate (2.15) holds for $W_k^{(l)}(y)$ on the region $D_0$ except $y = 1$.

Then $(1-y)^{\rho_r} \sum_{n=1}^{\infty} y^n n^{-k}$ has an analytic extension $(1-y)^{\rho_r} W_k(y)$ on $D$. If $\rho_r$ is not an integer or if $\rho_r$ is an integer and $m \le \rho_r$, we have

$$\begin{aligned}
((1-y)^{\rho_r} W_k(y))^{(m)} &= \sum_{l=0}^{m} b_l W_k^{(l)}(y)(1-y)^{\rho_r - m + l} \\
&= O\left(|1-y|^{\rho_r - m + 1} \log \frac{1}{|1-y|}\right) \\
&\quad + b_0 W_k(y)(1-y)^{\rho_r - m}. \qquad (2.16)
\end{aligned}$$

If $\rho_r$ is an integer and $m > \rho_r$,

$$\begin{aligned}
((1-y)^{\rho_r} W_k(y))^{(m)} &= \sum_{l=m-\rho_r}^{m} b_l W_k^{(l)}(y)(1-y)^{\rho_r - m + l} \\
&\ll 1 + |1-y|^{\rho_r - m + k - 1} \log \frac{1}{|1-y|} \\
&\quad + \sum_{k \le l \le m} |1-y|^{\rho_r - m + k - 1} \\
&\ll |1-y|^{\rho_r - m + k - 1} \log \frac{1}{|1-y|}. \qquad (2.17)
\end{aligned}$$

Hence, for $m \le [\rho_r]$, by (2.16), $Z_{02}^{(m)}(y)$ has a limit as $y \to 1$ in the region $D_0$. This shows that the derivatives of $Z_{02}(y)$ of order $\le [\rho_r]$ has continuous extension on $D_0$ if $C_k$ are not all zero.

If $\rho_r$ is not an integer, by (2.16),

$$\begin{aligned} &Z_{02}^{([\rho_r]+1)}(e^{2\pi i\theta}) \\ &= O\left(|1-e^{2\pi i\theta}|^{\rho_r-[\rho_r]}\log\frac{1}{|1-e^{2\pi i\theta}|}\right) + O(|1-e^{2\pi i\theta}|^{\rho_r-[\rho_r]-1}) \\ &\ll |\theta|^{\rho_r-[\rho_r]-1} \end{aligned} \tag{2.18$_1$}$$

and, similarly,

$$Z_{02}^{([\rho_r]+2)}(e^{2\pi i\theta}) \ll |\theta|^{\rho_r-[\rho_r]-2}. \tag{2.19$_1$}$$

If $\rho_r$ is an integer, by (2.17),

$$Z_{02}^{(\rho_r+1)}(e^{2\pi i\theta}) \ll \log\frac{1}{|1-e^{2\pi i\theta}|} \ll \log\frac{1}{|\theta|} \tag{2.18$_2$}$$

and

$$Z_{02}^{(\rho_r+2)}(e^{2\pi i\theta}) \ll |1-e^{2\pi i\theta}|^{-1}\log\frac{1}{|1-e^{2\pi i\theta}|} \ll |\theta|^{-1}\log\frac{1}{|\theta|} \tag{2.19$_2$}$$

since $k \ge 2$ ! Therefore, from (2.18) and (2.19),

$$Z_{02}^{([\rho_r]+\nu)}(e^{2\pi i\theta}) \ll \begin{cases} |\theta|^{\sigma-\nu}, & \text{if } \rho_r \text{ is not an integer;} \\ |\theta|^{\sigma-\nu}\log\dfrac{1}{|\theta|}, & \text{otherwise,} \end{cases} \qquad \nu = 1,2 \tag{2.20}$$

for $\sigma = \rho_r - [\rho_r]_1$.

Then, consider $Z_{03}(y)$. It has an analytic extension on the domain $D$ and has continuous derivatives of order $\le [\rho_r]_1$ on $\{|y| \le 1\}$. For $y \in D_0 - \{1\}$,

$$\begin{aligned} ((1-y)^{\rho_r}\log(1-y))^{([\rho_r]_1+1)} &= d_0(1-y)^{\rho_r-[\rho_r]_1-1}\log(1-y) \\ &\quad + d_1(1-y)^{\rho_r-[\rho_r]_1-1} \\ &\ll |1-y|^{\rho_r-[\rho_r]_1-1}\log\frac{1}{|1-y|} \end{aligned}$$

and

$$\begin{aligned} &((1-y)^{\rho_r}\log(1-y))^{([\rho_r]_1+2)} \\ &= d_1(1-y)^{\rho_r-[\rho_r]_1-2}\log(1-y) + d_2(1-y)^{\rho_r-[\rho_r]_1-2} \\ &\ll |1-y|^{\rho_r-[\rho_r]_1-2}\log\frac{1}{|1-y|}. \end{aligned}$$

Therefore

$$Z_{03}^{([\rho_r]_1+\nu)}(e^{2\pi i\theta}) \ll |\theta|^{\sigma-\nu}\log\frac{1}{|\theta|}, \qquad \nu = 1, 2 \tag{2.21}$$

for $\sigma = \rho_r - [\rho_r]_1$.

Finally, $Z_{04}(y)$ has on the disk $\{|y| \leq 1\}$ continuous derivatives of order $\leq [\gamma]_1 - 1$ if $\rho_r$ is an integer or of order $\leq [\rho_r]$ if $\rho_r$ is not an integer since $\gamma > 1 + \rho_r$. For $|y| \leq 1$ and $y \neq 1$, if $\rho_r$ is an integer and $m \leq \rho_r$ or if $\rho_r$ is not an integer and $m \leq [\gamma]_1 - 1$,

$$\begin{aligned} Z_{04}^{(m)} &= h_0\left(\sum_{n=1}^{\infty} r_n y^n\right)(1-y)^{\rho_r - m} + O(|1-y|^{\rho_r - m+1}) \\ &\ll |1-y|^{\rho_r - m} + 1. \end{aligned}$$

If $\rho_r$ is an integer and $\rho_r < m \leq [\gamma]_1 - 1$,

$$Z_{04}^{(m)} \ll 1.$$

Therefore, for $w = \min\{[\gamma]_1 - 3, [\rho_r]\}$,

$$Z_{04}^{(w+1)}(y) \ll 1, \qquad Z_{04}^{(w+2)}(y) \ll 1$$

if $\rho_r$ is an integer or

$$Z_{04}^{(w+1)}(y) \ll 1 + |1-y|^{\rho_r - w - 1}, \qquad Z_{04}^{(w+2)}(y) \ll 1 + |1-y|^{\rho_r - w - 2}$$

if $\rho_r$ is not an integer. It follows that

$$Z_{04}^{(w+1)}(e^{2\pi i\theta}) \ll |\theta|^{\sigma-1}, \qquad Z_{04}^{(w+2)}(e^{2\pi i\theta}) \ll |\theta|^{\sigma-2} \tag{2.22}$$

for $\sigma = \rho_r - [\rho_r]_1$.

It now follows that the derivatives of $Z_0(y)$ of order $\leq m$ have continuous extension on the region $D_0$. Also, for $t = \min\{[\gamma]_1 - 3, m\}$, (2.11) follows from (2.12), (2.13), (2.14), (2.20), (2.21), and (2.22). □

Combining Lemmas 2.3 and 2.4 yields an analysis of the boundary behavior of $F(y)$ on the boundary circle $|y| = 1$.

LEMMA 2.5. *Let $m$ be defined in Lemma* 2.4 *and $t$ defined in* (1.9). *The derivatives of $F(y)$ of order $\leq \min\{m, m_3\}$ have continuous extension on the closed disk $\{|y| \leq 1\}$. Moreover,*

$$F^{(t+1)}(e^{2\pi i\theta}) \ll \max\{|\theta - \theta_l|^{\sigma-1}, |\theta_{l+1} - \theta|^{\sigma-1}\} \tag{2.23}$$

*and*

$$F^{(t+2)}(e^{2\pi i\theta}) \ll \max\{|\theta - \theta_l|^{\sigma-2}, |\theta_{l+1} - \theta|^{\sigma-2}\} \tag{2.24}$$

*for every $\sigma < \sigma_0$ and $\theta_l < \theta < \theta_{l+1}$, $l = 1, \ldots, k-1$, where $\sigma_0$ is defined in* (1.12).

*Proof.* It is easily seen, from Lemmas 2.3 and 2.4, that the derivatives of $F(y)$ of order $\le \min\{m, m_3\}$ has continuous extension on the closed disk $\{|y| \le 1\}$.

To prove (2.23) and (2.24), we first assume that $(R_1 \cup R_3) - \{\rho_r\} \ne \emptyset$ and that $\rho_r - \rho_j$ are not all positive integers for $\rho_j \in (R_1 \cup R_3) - \{\rho_r\}$. Let $t_1 = \min\{[\gamma]_1 - 3, m_1, m_2\}$. From (2.11),

$$F^{(t_1+\nu)}(e^{2\pi i\theta}) \ll \begin{cases} |\theta|^{\sigma_1-\nu} \log \dfrac{1}{|\theta|}, & \text{if } R_2 \ne \emptyset \text{ and if } t_1 = m_2; \\ |\theta|^{\sigma_1-\nu}, & \text{otherwise,} \end{cases} \qquad \nu = 1, 2$$

for $0 < |\theta| \le \theta_1/2$, where constants $\sigma_1 = \min\{\rho_r - \rho_{j_0} - [\rho_r - \rho_{j_0}], \rho_r - [\rho_r]_1\}$. Let $t_2 = \min\{[\gamma]_1 - 3, m_3\}$. From (2.6) and (2.7),

$$F^{(t_2+\nu)}(e^{2\pi i\theta}) \ll \begin{cases} |\theta - \theta_l|^{\sigma_2-\nu} \log \dfrac{1}{|\theta - \theta_l|}, & \text{if } \gamma \text{ is an integer;} \\ |\theta - \theta_l|^{\sigma_2-\nu}, & \text{if } \gamma \text{ is not an integer,} \end{cases} \qquad \nu = 1, 2$$

for $(\theta_l + \theta_{l-1})/2 \le \theta \le (\theta_l + \theta_{l+1})/2, \theta \ne \theta_l$ if $1 \le l \le k-1$ and for $(\theta_k + \theta_{k-1})/2 \le \theta \le (\theta_k + (1 - \theta_{k-1}))/2, \theta \ne \theta_k$ if $l = k$, where constant $\sigma_2 = \gamma - [\gamma]_1$. Thus, if $t_1 = t_2$ then (2.23) and (2.24) hold on the interval $\theta_l < \theta < \theta_{l+1}$ for every $\sigma < \min\{\gamma - [\gamma]_1, \rho_r - [\rho_r]_1, \rho_r - \rho_{j_0} - [\rho_r - \rho_{j_0}]\}$. If $t_1 < t_2$, (2.23) and (2.24) hold on the interval for every $\sigma < \min\{\rho_r - [\rho_r]_1, \rho_r - \rho_{j_0} - [\rho_r - \rho_{j_0}]\}$. Finally, if $t_1 > t_2$, for $\sigma < \gamma - [\gamma]_1$. This proves that (2.23) and (2.24) hold on the interval $\theta_l < \theta < \theta_{l+1}$ for every $\sigma < \sigma_0$ with $\sigma_0$ defined in $(1.12)_1$.

Then assume that either$(R_1 \cup R_3) - \{\rho_r\} = \emptyset$ or all $\rho_r - \rho_j$ are positive integers for $\rho_j \in (R_1 \cup R_3) - \{\rho_r\}$. In the same way, we can prove that (2.23) and (2.24) hold on the interval $\theta_l < \theta < \theta_{l+1}$ for every $\sigma < \sigma_0$ with $\sigma_0$ defined in $(1.12)_2$. □

LEMMA 2.6. *Let $f(\theta)$ be defined on the interval $[a, b]$. Suppose there exist a partition $a = \theta_0 < \theta_1 < \cdots < \theta_k = b$ of $[a, b]$ and a constant $\delta$ with $0 < \delta \le 1$ such that $f(\theta)$ has continuous first order derivative on each open subinterval $(\theta_l, \theta_{l+1})$, $l = 0, 1, \ldots, k-1$ and such that*

$$|f(\theta)| \ll \max\{|\theta - \theta_l|^{\delta-1}, |\theta_{l+1} - \theta|^{\delta-1}\}$$

*and*

$$|f'(\theta)| \ll \max\{|\theta - \theta_l|^{\delta-2}, |\theta_{l+1} - \theta|^{\delta-2}\}$$

*for $\theta_l < \theta < \theta_{l+1}$, $l = 0, 1, \ldots, k-1$. Then*

$$\int_a^b f(\theta) e^{in\theta} d\theta \ll \begin{cases} n^{-\delta}, & \text{if } 0 < \delta < 1; \\ \dfrac{\log n}{n}, & \text{if } \delta = 1. \end{cases} \tag{2.25}$$

*Proof.* Let

$$0 < \eta < \min\left\{\frac{\theta_{l+1}-\theta_l}{2}, l = 0, 1, \ldots, k-1\right\}.$$

Then

$$\int_a^b f(\theta)e^{in\theta}d\theta = \left(\sum_{l=0}^{k-1}\int_{\theta_l}^{\theta_l+\eta} + \sum_{l=1}^{k}\int_{\theta_l-\eta}^{\theta_l} + \sum_{l=0}^{k-1}\int_{\theta_l+\eta}^{\theta_{l+1}-\eta}\right) f(\theta)e^{in\theta}d\theta.$$

Clearly

$$\int_{\theta_l}^{\theta_l+\eta} f(\theta)e^{in\theta}d\theta \ll \int_{\theta_l}^{\theta_l+\eta}(\theta-\theta_l)^{\delta-1}d\theta = \frac{\eta^\delta}{\delta},$$

$$\int_{\theta_l-\eta}^{\theta_l} f(\theta)e^{in\theta}d\theta \ll \int_{\theta_l-\eta}^{\theta_l}(\theta_l-\theta)^{\delta-1}d\theta = \frac{\eta^\delta}{\delta}.$$

By integration by parts, we have

$$\begin{aligned}\int_{\theta_l+\eta}^{\theta_{l+1}-\eta} f(\theta)e^{in\theta}d\theta &= \frac{1}{in}\left[f(\theta_{l+1}-\eta)e^{in(\theta_{l+1}-\eta)} - f(\theta_l+\eta)e^{in(\theta_l+\eta)}\right] \\ &\quad -\frac{1}{in}\int_{\theta_l+\eta}^{\theta_{l+1}-\eta} f'(\theta)e^{in\theta}d\theta.\end{aligned}$$

Plainly,

$$|f(\theta_{l+1}-\eta)e^{in(\theta_{l+1}-\eta)}| = |f(\theta_{l+1}-\eta)| \ll \eta^{\delta-1},$$

$$|f(\theta_l+\eta)e^{in(\theta_l+\eta)}| = |f(\theta_l+\eta)| \ll \eta^{\delta-1}.$$

Also, if $\delta < 1$,

$$\begin{aligned}\left|\int_{\theta_l+\eta}^{\theta_{l+1}-\eta} f'(\theta)e^{in\theta}d\theta\right| &\le \int_{\theta_l+\eta}^{\theta_{l+1}-\eta}|f'(\theta)|d\theta \\ &\ll \int_{\theta_l+\eta}^{\theta_{l+1}-\eta}(\theta-\theta_l)^{\delta-2}d\theta + \int_{\theta_l+\eta}^{\theta_{l+1}-\eta}(\theta_{l+1}-\theta)^{\delta-2}d\theta \\ &\le \frac{2}{1-\delta}\eta^{\delta-1}.\end{aligned}$$

Then we obtain

$$\left|\int_a^b f(\theta)e^{in\theta}d\theta\right| \ll \eta^\delta + \frac{\eta^{\delta-1}}{n}.$$

If $\delta = 1$, then

$$\left|\int_{\theta_l+\eta}^{\theta_{l+1}-\eta} f'(\theta)e^{in\theta}d\theta\right| \ll \log\frac{1}{\eta}$$

and

$$\left|\int_a^b f(\theta)e^{in\theta}d\theta\right| \ll \eta + \frac{1}{n}\left(1+\log\frac{1}{\eta}\right).$$

By taking $\eta = n^{-1}$, (2.25) follows. □

*Proof of Theorem* 1.1. As in the proof of Theorem 6.2 of [6], we have

$$\bar{\lambda}(n) = \rho_r - 2\sum_{n=1}^{k-1} n_l \cos 2n\pi\theta_l - (-1)^n n_k + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{F'(e^{i\theta})}{F(e^{i\theta})} e^{-i(n-1)\theta} d\theta.$$

It remains to show that the integral on the right-hand side, denoted by $I_{n-1}$, say, is $O((n-1)^{-t-\sigma})$. The function $F(y)$ has continuous derivatives of order $\leq [\gamma]_1 - 1$ on $\{y : |y| \leq, y \neq 1, y \neq e^{\pm 2\pi i\theta_l}, l = 1, \ldots, k\}$. By Lemma 2.5, the derivatives of $F(y)$ of order $\leq t$ defined in (1.9) have continuous extension on the closed disk $\{|y| \leq 1\}$. Let

$$f(\theta) := \left(\frac{F'(y)}{F(y)}\right)^{(t)}\Bigg|_{y=e^{i\theta}}.$$

Then $f(\theta)$ has continuous first order derivative on each open interval $(2\pi\theta_l, 2\pi\theta_{l+1})$ and $(-2\pi\theta_{l+1}, -2\pi\theta_l)$, $l = 0, 1, \ldots, k-1$. By Lemma 2.5, for $\theta \in (2\pi\theta_l, 2\pi\theta_{l+1})$ or $\theta \in (-2\pi\theta_{l+1}, -2\pi\theta_l)$,

$$\begin{aligned} f(\theta) &= \frac{F^{(t+1)}(e^{i\theta})}{F(e^{i\theta})} + \frac{P_t(F(e^{i\theta}), F'(e^{i\theta}), \ldots, F^{(t)}(e^{i\theta}))}{F^{t+1}(e^{i\theta})} \\ &\ll \max\{|\theta - 2\pi\theta_l|^{\sigma-1}, |2\pi\theta_{l+1} - \theta|^{\sigma-1}\}, \end{aligned}$$

and

$$\begin{aligned} f'(\theta) &= \frac{F^{(t+2)}(e^{i\theta})}{F(e^{i\theta})} + \frac{P_{t+1}(F(e^{i\theta}), F'(e^{i\theta}), \ldots, F^{(t+1)}(e^{i\theta}))}{F^{t+2}(e^{i\theta})} \\ &\ll \max\{|\theta - 2\pi\theta_l|^{\sigma-2}, |2\pi\theta_{l+1} - \theta|^{\sigma-2}\}, \end{aligned}$$

where $P_k(x_0, x_1, \ldots, x_k)$ is a polynomial of $x_0, x_1, \ldots, x_k$ with integer coefficients such that the degree of $x_k$ is one. We note that $F^{(t+1)}(e^{i\theta})$ contributes toward the right-hand side of the last inequality at most $\max\{|\theta - 2\pi\theta_l|^{\sigma-1}, |2\pi\theta_{l+1} - \theta|^{\sigma-1}\}$.

Applying integration by parts $t$ times, we obtain

$$\begin{aligned} I_n &= \int_{-\pi}^{\pi} \frac{F'(e^{i\theta})}{F(e^{i\theta})} e^{-in\theta} d\theta \\ &= \frac{1}{n(n-1)\dots(n-t+1)} \int_{-\pi}^{\pi} \left(\frac{F'(y)}{F(y)}\right)^{(t)} \bigg|_{y=e^{i\theta}} e^{-i(n-t)\theta} d\theta \\ &= \frac{1}{n(n-1)\dots(n-t+1)} \int_{-\pi}^{\pi} f(\theta) e^{-i(n-t)\theta} d\theta. \end{aligned}$$

The function $f(\theta)$ satisfies the conditions of Lemma 2.6 with $0 < \sigma < 1$. Hence the last integral is $O(n^{-\sigma})$ and

$$I_n = O(n^{-t-\sigma})$$

follows. □

## 3. Proof of Theorem 1.2

The idea of the proofs of (1.13) and (1.14) is included in the proof of the following lemma, while the proof of (1.14) requires a deeper analysis.

LEMMA 3.1. *Let $f(\theta)$ be a continuous function on the interval $[a, b]$. Suppose there exist constants $0 \leq \alpha \leq 1$ and $\beta \geq 0$ such that*

$$|f(\theta_1) - f(\theta_2)| \ll |\theta_1 - \theta_2|^\alpha \log^\beta \frac{1}{|\theta_1 - \theta_2|}$$

*for all $\theta_1, \theta_2 \in [a, b]$ with $0 < |\theta_1 - \theta_2| < \frac{1}{2}$. Then*

$$\int_a^b f(\theta) e^{in\theta} d\theta \ll n^{-\alpha} \log^\beta n. \tag{3.1}$$

The inequality (3.1) must be known but we are unable to locate its proof in the literature. As a substitute we give a brief proof of it here.

*Proof.* The inequality (3.1) is trivial when $\alpha = 0$. Hence we may assume $0 < \alpha \leq 1$.

Let $\theta_k = a + \frac{k}{n}(b-a)$, $k = 0, 1, \dots, n$. Define a continuous function $g(\theta)$ on $[a, b]$ such that $g(\theta_k) = f(\theta_k)$ and such that $g(\theta)$ is linear on each interval $[\theta_{k-1}, \theta_k]$. Then we have

$$\begin{aligned} g'(\theta) &= \frac{f(\theta_k) - f(\theta_{k-1})}{\theta_k - \theta_{k-1}} \ll |\theta_k - \theta_{k-1}|^{-1+\alpha} \log^\beta \frac{1}{|\theta_k - \theta_{k-1}|} \\ &\ll n^{1-\alpha} \log^\beta n \end{aligned} \tag{3.2}$$

for $\theta_{k-1} \le \theta \le \theta_k$, where the constant implied by $\ll$ is uniform for all $k$. Also,

$$\begin{aligned}
|f(\theta) - g(\theta)| &= |(f(\theta) - f(\theta_{k-1})) - (g(\theta) - g(\theta_{k-1}))| \\
&\le |(f(\theta) - f(\theta_{k-1})| + \frac{|(f(\theta_k) - f(\theta_{k-1})|}{|\theta_k - \theta_{k-1}|}|\theta - \theta_{k-1}| \\
&\ll |\theta - \theta_{k-1}|^{\alpha} \log^{\beta} \frac{1}{|\theta - \theta_{k-1}|} \\
&\quad + \left( |\theta_k - \theta_{k-1}|^{-1+\alpha} \log^{\beta} \frac{1}{|\theta_k - \theta_{k-1}|} \right) |\theta - \theta_{k-1}| \\
&\ll n^{-\alpha} \log^{\beta} n, \qquad (3.3)
\end{aligned}$$

where the constant implied by $\ll$ is uniform for all $k$ too (The condition $\alpha > 0$ is needed here).

We have

$$\int_a^b f(\theta)e^{in\theta} d\theta = \int_a^b (f(\theta) - g(\theta))e^{in\theta} d\theta + \int_a^b g(\theta)e^{in\theta} d\theta = I_1 + I_2,$$

say. Then, by (3.3),

$$|I_1| \ll n^{-\alpha} \log^{\beta} n.$$

Also, by integration by parts,

$$I_2 = \frac{1}{in} g(\theta)e^{in\theta}\Big|_a^b - \frac{1}{in} \int_a^b g'(\theta)e^{in\theta} d\theta = O(n^{-1}) + O(n^{-\alpha} \log^{\beta} n)$$

since

$$\int_a^b g'(\theta)e^{in\theta} d\theta \ll \int_a^b |g'(\theta)| d\theta \ll n^{1-\alpha} \log^{\beta} n$$

by (3.2). □

*Proof of Theorem* 1.2. Assume that

$$\bar{g}(n) = A + O(n^{-\gamma}), \qquad n \ge 1.$$

Let $r_n = \bar{g}(n) - A$ and $Z_0(y) = (1 - y)Z(y)$. Then $r_n = O(n^{-\gamma})$ and

$$Z_0(y) = 1 + (r_1 + A - 1)y + \sum_{n=2}^{\infty} (r_n - r_{n-1})y^n = 1 + \sum_{n=1}^{\infty} r'_n y^n$$

with $r'_n = O(n^{-\gamma})$. We note that the generating function $Z(y)$ has at most one zero at $y = -1$ of order one on the disk $\{|y| \le 1\}$.

If $Z(y)$ has no zero at $y = -1$ then $F(y) = Z_0(y)$. Hence $F(y)$ has continuous derivatives of order $\le [\gamma]_1 - 1$ on the disk $\{|y| \le 1\}$ and has no zero there. As in the proof of Theorem 1.1, applying integration by parts $([\gamma]_1 - 2)$ times, we obtain

$$\begin{aligned} I_n &= \int_{-\pi}^{\pi} \frac{F'(e^{i\theta})}{F(e^{i\theta})} e^{-in\theta} d\theta \\ &= \frac{1}{n(n-1)\dots(n-[\gamma]_1+3)} \int_{-\pi}^{\pi} \left(\frac{F'(y)}{F(y)}\right)^{([\gamma]_1-2)} \Bigg|_{y=e^{i\theta}} e^{-i(n-[\gamma]_1+2)\theta} d\theta. \end{aligned}$$

Let

$$\begin{aligned} f(\theta) &:= \left(\frac{F'(y)}{F(y)}\right)^{([\gamma]_1-2)} \Bigg|_{y=e^{i\theta}} \\ &= \frac{F^{([\gamma]_1-1)}(e^{i\theta})}{F(e^{i\theta})} + \frac{P_{[\gamma]_1-2}(F(e^{i\theta}), F'(e^{i\theta}), \dots, F^{([\gamma]_1-2)}(e^{i\theta}))}{F^{[\gamma]_1-1}(e^{i\theta})}. \end{aligned} \tag{3.4}$$

Then $f(\theta)$ is continuous on the interval $[-\pi, \pi]$. Applying Lemma 2.2 to $F(y)$, we conclude, from (3.4), that

$$|f(\theta_1) - f(\theta_2)| \ll \begin{cases} |\theta_1 - \theta_2|^{\gamma - [\gamma]}, & \text{if } \gamma \text{ is not an integer;} \\ |\theta_1 - \theta_2| \log \dfrac{1}{|\theta_1 - \theta_2|}, & \text{if } \gamma \text{ is an integer.} \end{cases}$$

Then, by Lemma 3.1, we obtain

$$I_n \ll \begin{cases} n^{-\gamma+2}, & \text{if } \gamma \text{ is not an integer;} \\ n^{-\gamma+2} \log n, & \text{if } \gamma \text{ is an integer.} \end{cases}$$

This prove (1.13).

If $Z(y)$ has a zero at $y = -1$ of order one, then the argument in the proof of Lemma 2.3 shows that

$$Z_1(y) := \frac{Z(y)}{1+y} = P(y) + \frac{1}{([\gamma]_1 - 2)!} \frac{I(y)}{1+y}, \tag{3.5}$$

where $P(y)$ is a polynomial of degree $\le [\gamma]_1 - 2$ and

$$I(y) := \int_{-1}^{y} \left(Z^{([\gamma]_1-1)}(u) - Z^{([\gamma]_1-1)}(-1)\right) (y-u)^{[\gamma]_1-2} du.$$

Also,

$$\begin{aligned} \left(\frac{I(y)}{y+1}\right)^{(m)} &= \sum_{k=0}^{m} d_k (y+1)^{[\gamma]_1-m-2} \\ &\quad \times \int_0^1 \left(Z^{([\gamma]_1-1)}(-1+s(y+1)) - Z^{([\gamma]_1-1)}(-1)\right) (1-s)^{[\gamma]_1-2-m+k} ds \end{aligned}$$

for $1 \le m \le [\gamma]_1 - 2$ and

$$\left(\frac{I(y)}{y+1}\right)^{([\gamma]_1-1)} = \sum_{k=0}^{[\gamma]_1-1} d_k'(y+1)^{-1} \times \int_0^1 \left(Z^{([\gamma]_1-1)}(-1+s(y+1)) - Z^{([\gamma]_1-1)}(-1)\right)(1-s)^{k-1}ds$$
$$+(y+1)^{-1}([\gamma]_1-2)!\left(Z^{([\gamma]_1-1)}(y) - Z^{([\gamma]_1-1)}(-1)\right). \quad (3.6)$$

Therefore, the derivatives of $Z_1(y)$ of order $\le [\gamma]_1 - 2$ have continuous extension on the disk $\{|y| \le 1\}$. Also,

$$Z_1^{([\gamma]_1-1)}(e^{2\pi i\theta}) \ll 1 + \sum_{k=1}^{[\gamma]_1-1} |e^{2\pi i\theta}+1|^{-1} \times \int_0^1 \left|Z^{([\gamma]_1-1)}(-1+s(e^{2\pi i\theta}+1)) - Z^{([\gamma]_1-1)}(-1)\right|(1-s)^{k-1}ds$$
$$+|e^{2\pi i\theta}+1|^{-1}\left|Z^{([\gamma]_1-1)}(e^{2\pi i\theta}) - Z^{([\gamma]_1-1)}(-1)\right|$$
$$\ll \begin{cases} |\theta - \frac{1}{2}|^{\gamma-[\gamma]-1}, & \text{if } \gamma \text{ is not an integer;} \\ \log \dfrac{1}{|\theta-\frac{1}{2}|}, & \text{if } \gamma \text{ is an integer} \end{cases} \quad (3.7)$$

for $\theta \ne \frac{1}{2}$.

Then

$$F(y) = (1-y)Z_1(y) = \frac{Z_0(y)}{1+y}$$

has continuous derivatives of order $\le [\gamma]_1 - 2$ on the disk $\{|y| \le 1\}$ and $F^{([\gamma]_1-1)}(y)$ is continuous there too except at the point $y = -1$. Let

$$f(\theta) = f_1(\theta) + f_2(\theta),$$

where

$$f_1(\theta) = \frac{F^{([\gamma]_1-1)}(e^{i\theta})}{F(e^{i\theta})}$$

and

$$f_2(\theta) = \frac{P_{[\gamma]_1-2}(F(e^{i\theta}), F'(e^{i\theta}), \ldots, F^{([\gamma]_1-2)}(e^{i\theta}))}{F^{[\gamma]_1-1}(e^{i\theta})}.$$

Note that $P_k(x_0, x_1, \ldots, x_k)$ is a polynomial of $x_0, x_1, \ldots, x_k$ with integer coefficients such that the degree of $x_k$ is one. Hence $f_2(\theta)$ is continuous on $[0, 2\pi]$ and satisfies

$$f_2'(\theta) \ll \begin{cases} |\theta-\pi|^{\gamma-[\gamma]-1}, & \text{if } \gamma \text{ is not an integer;} \\ \log \dfrac{1}{|\theta-\pi|}, & \text{if } \gamma \text{ is an integer} \end{cases}$$

for $\theta \neq \pi$, by (3.7). By integration by parts,

$$\int_{-\pi}^{\pi} f_2(\theta)e^{-in\theta}d\theta = -\frac{1}{in}\int_0^{2\pi} f_2'(\theta)e^{-in\theta}d\theta = O(n^{-1}).$$

To prove (1.14), it remains to show that

$$\int_{-\pi}^{\pi} f_1(\theta)e^{-in\theta}d\theta \ll \begin{cases} n^{-\gamma+[\gamma]}\log n, & \text{if } \gamma \text{ is not an integer;} \\ n^{-1}\log^2 n, & \text{if } \gamma \text{ is an integer.} \end{cases}$$

We shall give the computation in detail for $\gamma$ not an integer and in sketch only for $\gamma$ an integer.

Thus we first consider $\gamma$ not an integer. Let $\theta_k = \frac{k\pi}{n}, k = 0, \pm 1, \pm 2, \dots, \pm n$. We have

$$\int_{\theta_{n-1}}^{\theta_n} f_1(\theta)e^{-in\theta}d\theta \ll \int_{\theta_{n-1}}^{\theta_n} |\theta - \pi|^{\gamma-[\gamma]-1}d\theta \ll n^{-\gamma+[\gamma]}$$

by (3.7). Then define a continuous function $g(\theta)$ on $[0, \theta_{n-1}]$ such that $g(\theta_k) = f_1(\theta_k)$ and such that $g(\theta)$ is linear on each interval $[\theta_{k-1}, \theta_k], k = 0, 1, \dots, n-1$. For $\theta_{k-1} \leq \theta \leq \theta_k, k \geq 1$,

$$\begin{aligned}
&f_1(\theta) - f_1(\theta_{k-1}) \\
&= \frac{F^{([\gamma]_1-1)}(e^{i\theta}) - F^{([\gamma]_1-1)}(e^{i\theta_{k-1}})}{F(e^{i\theta})} - F^{([\gamma]_1-1)}(e^{i\theta_{k-1}})\frac{F(e^{i\theta}) - F(e^{i\theta_{k-1}})}{F(e^{i\theta})F(e^{i\theta_{k-1}})} \\
&= O(|F^{([\gamma]_1-1)}(e^{i\theta}) - F^{([\gamma]_1-1)}(e^{i\theta_{k-1}})|) + O(|\theta_{k-1}-\pi|^{\gamma-[\gamma]-1}|\theta-\theta_{k-1}|) \\
&= O(|F^{([\gamma]_1-1)}(e^{i\theta}) - F^{([\gamma]_1-1)}(e^{i\theta_{k-1}})|) + O(n^{-\gamma+[\gamma]}(n-k+1)^{\gamma-[\gamma]-1}), \quad (3.8)
\end{aligned}$$

by (3.7). Also, by (3.7) and the differentiation mean value theorem,

$$\begin{aligned}
&\left|F^{([\gamma]_1-1)}(e^{i\theta}) - F^{([\gamma]_1-1)}(e^{i\theta_{k-1}})\right| \\
&= \Big|(e^{i\theta_{k-1}} - e^{i\theta})Z_1^{([\gamma]_1-1)}(e^{i\theta}) \\
&\quad +(1 - e^{i\theta_{k-1}})\left(Z_1^{([\gamma]_1-1)}(e^{i\theta}) - Z_1^{([\gamma]_1-1)}(e^{i\theta_{k-1}})\right) \\
&\quad +([\gamma]_1 - 1)\left(Z_1^{([\gamma]_1-2)}(e^{i\theta}) - Z_1^{([\gamma]_1-2)}(e^{i\theta_{k-1}})\right)\Big| \\
&= O\left(|\theta - \pi|^{\gamma-[\gamma]-1}|\theta - \theta_{k-1}| + |\theta_{k-1} - \pi|^{\gamma-[\gamma]-1}|\theta - \theta_{k-1}|\right) \\
&\quad +O\left(\left|Z_1^{([\gamma]_1-1)}(e^{i\theta}) - Z_1^{([\gamma]_1-1)}(e^{i\theta_{k-1}})\right|\right) \\
&= O(n^{-\gamma+[\gamma]}(n-k+1)^{\gamma-[\gamma]-1}) \\
&\quad +O\left(\left|Z_1^{([\gamma]_1-1)}(e^{i\theta}) - Z_1^{([\gamma]_1-1)}(e^{i\theta_{k-1}})\right|\right). \quad (3.9)
\end{aligned}$$

From (3.5) and (3.6), we have

$$
\begin{aligned}
Z_1^{([\gamma]_1-1)}(e^{i\theta}) &- Z_1^{([\gamma]_1-1)}(e^{i\theta_{k-1}}) \\
= \sum_{k=1}^{[\gamma]_1-1} d_k'' &\Bigg[\left(\frac{1}{e^{i\theta}+1}-\frac{1}{e^{i\theta_{k-1}}+1}\right) \\
&\times \int_0^1 \left(Z^{([\gamma]_1-1)}(-1+s(e^{i\theta}+1)) - Z^{([\gamma]_1-1)}(-1)\right)(1-s)^{k-1}ds \\
&+\frac{1}{e^{i\theta_{k-1}}+1}\int_0^1 \left(Z^{([\gamma]_1-1)}(-1+s(e^{i\theta}+1))\right. \\
&\qquad \left. -Z^{([\gamma]_1-1)}(-1+s(e^{i\theta_{k-1}}+1))\right)(1-s)^{k-1}ds\Bigg] \\
+([\gamma]_1-2)! &\left[\left(\frac{1}{e^{i\theta}+1}-\frac{1}{e^{i\theta_{k-1}}+1}\right)\left(Z^{([\gamma]_1-1)}(e^{i\theta})-Z^{([\gamma]_1-1)}(-1)\right)\right. \\
&\left. + \frac{1}{e^{i\theta_{k-1}}+1}\left(Z^{([\gamma]_1-1)}(e^{i\theta})-Z^{([\gamma]_1-1)}(e^{i\theta_{k-1}})\right)\right] \\
= \Big(|\theta-\pi|^{\gamma-[\gamma]-1}&|\theta-\theta_{k-1}||\theta_{k-1}-\pi|^{-1}+|\theta_{k-1}-\pi|^{-1}|\theta-\theta_{k-1}|^{\gamma-[\gamma]}\Big) \\
= O(n^{1-\gamma+[\gamma]}&(n-k+1)^{-1}). \qquad (3.10)
\end{aligned}
$$

Combining (3.8), (3.9), and (3.10), we obtain

$$
f_1(\theta)-f_1(\theta_{k-1}) \ll n^{-\gamma+[\gamma]}(n-k+1)^{\gamma-[\gamma]-1}+n^{1-\gamma+[\gamma]}(n-k+1)^{-1}. \qquad (3.11)
$$

Then

$$
\begin{aligned}
g(\theta)-g(\theta_{k-1}) &= \frac{f_1(\theta_k)-f_1(\theta_{k-1})}{\theta_k-\theta_{k-1}}(\theta-\theta_{k-1}) \\
&\ll n^{-\gamma+[\gamma]}(n-k+1)^{\gamma-[\gamma]-1}+n^{1-\gamma+[\gamma]}(n-k+1)^{-1}. \qquad (3.12)
\end{aligned}
$$

for $\theta_{k-1}\le\theta\le\theta_k, k\ge 1$. We now have

$$
\begin{aligned}
\int_0^{\theta_{n-1}} f_1(\theta)e^{-in\theta}d\theta &= \int_0^{\theta_{n-1}}(f_1(\theta)-g(\theta))e^{-in\theta}d\theta+\int_0^{\theta_{n-1}} g(\theta)e^{-in\theta}d\theta \\
&= I_1+I_2, \qquad (3.13)
\end{aligned}
$$

say. From (3.11) and (3.12),

$$
|I_1| \le \sum_{k=1}^{n-1}\int_{\theta_{k-1}}^{\theta_k}\left(|f_1(\theta)-f_1(\theta_{k-1})|+|g(\theta)-g(\theta_{k-1})|\right)d\theta
$$

$$\ll \sum_{k=1}^{n-1}\left(n^{-\gamma+[\gamma]}(n-k+1)^{\gamma-[\gamma]-1}+n^{1-\gamma+[\gamma]}(n-k+1)^{-1}\right)n^{-1}$$
$$\ll n^{-\gamma+[\gamma]}\log n. \tag{3.14}$$

Also, by integration by parts,

$$\begin{aligned} I_2 &= \frac{1}{in}g(\theta)e^{-in\theta}\Big|_0^{\theta n-1}-\frac{1}{in}\int_0^{\theta_{n-1}}g'(\theta)e^{-in\theta}d\theta \\ &= O(n^{-\gamma+[\gamma]})-\frac{1}{in}\int_0^{\theta_{n-1}}g'(\theta)e^{-in\theta}d\theta, \end{aligned} \tag{3.15}$$

since

$$\begin{aligned} g(\theta_{n-1}) &= f_1(\theta_{n-1}) \ll |F^{([\gamma]_1-1)}(e^{i\theta_{n-1}})| \\ &\ll |(1-e^{i\theta_{n-1}})Z_1^{([\gamma]_1-1)}(e^{i\theta_{n-1}})|+|Z_1^{([\gamma]_1-2)}(e^{i\theta_{n-1}})| \\ &= O(|\theta_{n-1}-\pi|^{\gamma-[\gamma]-1})+O(1) \\ &= O(n^{1-\gamma+[\gamma]}) \end{aligned}$$

by (3.7). The last integral in (3.15) is

$$\leq \sum_{k=1}^{n-1}\int_{\theta_{k-1}}^{\theta_k}|g_k'|d\theta \ll \sum_{k=1}^{n-1}|g_k'|n^{-1},$$

where

$$\begin{aligned} g_k' &= \frac{f_1(\theta_k)-f_1(\theta_{k-1})}{\theta_k-\theta_{k-1}} \\ &\ll n^{1-\gamma+[\gamma]}(n-k+1)^{\gamma-[\gamma]-1}+n^{2-\gamma+[\gamma]}(n-k+1)^{-1} \end{aligned}$$

by (3.11). Hence, we obtain

$$\begin{aligned} \int_0^{\theta_{n-1}}g'(\theta)e^{-in\theta}d\theta &\ll \sum_{k=1}^{n-1}(n^{-\gamma+[\gamma]}(n-k+1)^{\gamma-[\gamma]-1}+n^{1-\gamma+[\gamma]}(n-k+1)^{-1}) \\ &\ll n^{1-\gamma+[\gamma]}\log n. \end{aligned} \tag{3.16}$$

By (3.15) and (3.16),

$$I_2 \ll n^{-\gamma+[\gamma]}\log n$$

and, combining with (3.14), we obtain

$$\int_0^{\theta_{n-1}}f_1(\theta)e^{-in\theta}d\theta \ll n^{-\gamma+[\gamma]}\log n.$$

This proves

$$\int_0^{\pi} f_1(\theta)e^{-in\theta}d\theta \ll n^{-\gamma+[\gamma]}\log n.$$

In a similar way, we can prove the same estimate for the integral on $[-\pi, 0]$.

We finally consider $\gamma$ an integer. We have

$$\int_{\theta_{n-1}}^{\theta_n} f_1(\theta)e^{-in\theta}d\theta \ll \int_{\theta_{n-1}}^{\theta_n} \log\frac{1}{\pi-\theta}d\theta \ll n^{-1}\log n.$$

For $\theta_{k-1} \le \theta \le \theta_k, k \ge 1$,

$$\begin{aligned} f_1(\theta) - f_1(\theta_{k-1}) &= O\left(\left|F^{([\gamma]_1-1)}(e^{i\theta}) - F^{([\gamma]_1-1)}(e^{i\theta_{k-1}})\right|\right) \\ &\quad + O\left(n^{-1}\left(1+\log\frac{n}{n-k+1}\right)\right) \end{aligned}$$

and

$$\begin{aligned} &\left|F^{([\gamma]_1-1)}(e^{i\theta}) - F^{([\gamma]_1-1)}(e^{i\theta_{k-1}})\right| \\ &= O\left(|\theta-\theta_{k-1}|\log\frac{1}{|\pi-\theta|} + |\theta-\theta_{k-1}|\log\frac{1}{|\pi-\theta_{k-1}|}\right) \\ &\quad + O\left(\left|Z_1^{([\gamma]_1-1)}(e^{i\theta}) - Z_1^{([\gamma]_1-1)}(e^{i\theta_{k-1}})\right|\right) \\ &= O\left(n^{-1}\left(1+\log\frac{n}{n-k+1}\right)\right) + O\left(\left|Z_1^{([\gamma]_1-1)}(e^{i\theta}) - Z_1^{([\gamma]_1-1)}(e^{i\theta_{k-1}})\right|\right). \end{aligned}$$

From (3.5) and (3.6),

$$\begin{aligned} &Z_1^{([\gamma]_1-1)}(e^{i\theta}) - Z_1^{([\gamma]_1-1)}(e^{i\theta_{k-1}}) \\ &= O\left(|\theta-\theta_{k-1}||\pi-\theta_{k-1}|^{-1}\log\frac{1}{|\pi-\theta|} + |\theta-\theta_{k-1}||\pi-\theta_{k-1}|^{-1}\log\frac{1}{|\pi-\theta_{k-1}|}\right) \\ &= O\left((n-k+1)^{-1}\left(1+\log\frac{n}{n-k+1}\right)\right). \end{aligned}$$

It follows that

$$f_1(\theta) - f_1(\theta_{k-1}) \ll (n-k+1)^{-1}\left(1+\log\frac{n}{n-k+1}\right).$$

Then the same estimate holds for $g(\theta) - g(\theta_{k-1})$. Now

$$\int_0^{\theta_{n-1}} (f_1(\theta)-g(\theta))e^{-in\theta}d\theta \ll \sum_{k=1}^{n-1}(n-k+1)^{-1}\left(1+\log\frac{n}{n-k+1}\right) \ll n^{-1}\log^2 n.$$

Also,

$$\int_0^{\theta_{n-1}} g(\theta)e^{-in\theta}d\theta = O(n^{-1}\log n) - \frac{1}{in}\int_0^{\theta_{n-1}} g'(\theta)e^{-in\theta}d\theta$$

and

$$\int_0^{\theta_{n-1}} g'(\theta)e^{-in\theta}d\theta \ll \sum_{k=1}^{n-1} n^{-1}|g_k'| \ll \sum_{k=1}^{n-1}(n-k+1)^{-1}\left(1+\log\frac{n}{n-k+1}\right)$$
$$\ll \log^2 n.$$

Hence, we obtain

$$\int_0^{\theta_{n-1}} f_1(\theta)e^{-in\theta}d\theta \ll n^{-1}\log^2 n$$

and

$$\int_0^{\pi} f_1(\theta)e^{-in\theta}d\theta \ll n^{-1}\log^2 n$$

follows. □

## 4. Proof of Theorem 1.5

*Proof.* It is sufficient to give only the proof of part (1). The proof of part (2) is exactly the same.

As in the classical theory, we have

$$\begin{aligned}
&Z^3(r)|Z(re^{i\theta})|^4|Z(re^{2i\theta})| \\
&= \prod_1^\infty \left[(1-r^m)|1-r^m e^{im\theta}|^4|1-r^m e^{2im\theta}|\right]^{-\bar{P}(m)} \\
&= \exp\left\{\sum_{m=1}^\infty \bar{P}(m)\sum_{k=1}^\infty \frac{r^{km}}{k}(3+4\cos km\theta+\cos 2km\theta)\right\} \\
&\geq 1
\end{aligned}$$

for $r < q^{-1}$ and all $\theta \in \mathbf{R}$ since $3 + 4\cos km\theta + \cos 2km\theta \geq 0$. Hence, for $\theta \neq (2n+1)\pi, n \in \mathbf{Z}$,

$$Z(r)|Z(re^{i\theta})| \to \infty$$

or, equivalently,

$$\log(Z(r)|Z(re^{i\theta})|) \to \infty$$

as $r \to q^{-1}-$. We note that

$$\log(Z(r)|Z(re^{i\theta})|) = \sum_{m=1}^{\infty} \bar{P}(m) \sum_{k=1}^{\infty} \frac{r^{km}}{k}(1 + \Re e^{im\theta})$$
$$= \sum_{m=1}^{\infty} \bar{P}(m) r^m (1 + \Re e^{im\theta}) + O(1)$$

since $\bar{P}(m) \ll q^m/m$. Therefore, we have

$$\sum_{m=1}^{\infty} \bar{P}(m) r^m (1 + \Re e^{im\theta}) \to \infty$$

as $r \to q^{-1}-$ for all $\theta \neq (2n+1)\pi$. If we now take $f(a) = \lambda(a)$ or $f(a) = \mu(a)$ in Theorem 1.5 of [7], then we find that

$$S := \sum_{p} q^{-\partial(p)} \left(1 - \Re(f(p) q^{-i\theta\partial(p)})\right) = \sum_{m=1}^{\infty} \bar{P}(m) r^m \left(1 + \Re e^{-im\theta \log q}\right) = \infty$$

holds for all $\theta \in \mathbf{R}$ and $\theta \neq \frac{(2n+1)\pi}{\log q}$, $n \in \mathbf{Z}$ because $f(p) = -1$.

For $\theta = \frac{(2n+1)\pi}{\log q}$,

$$S = \sum_{m=1}^{\infty} \bar{P}(m) q^{-m} \left(1 + e^{-im(2n+1)\pi}\right) = 2 \sum_{m=1}^{\infty} \bar{P}(2m) q^{-2m}.$$

Assume (1.14) with $\gamma > \max\{2 + \rho_r, 3\}$, by Theorem 1.3, there exist $k$ real numbers $0 < \theta_1 < \cdots < \theta_k \leq \frac{1}{2}$, and $k$ positive integers $n_1, \ldots, n_k$, where nonnegative integer $k \leq (\rho_r + 1)/2$, such that (i)

$$S = \sum_{m=1}^{\infty} \frac{1}{m} \left(\rho_r - 2\sum_{l=1}^{k} n_l \cos 4m\pi\theta_l + O(m^{-t-\sigma})\right)$$

if $\theta_k < \frac{1}{2}$ or (ii)

$$S = \sum_{m=1}^{\infty} \frac{1}{m} \left(\rho_r - 2\sum_{l=1}^{k-1} n_l \cos 4m\pi\theta_l - n_k + O(m^{-t-\sigma})\right)$$

if $\theta_k = \frac{1}{2}$, where $t$ is a nonnegative integer and $0 < \sigma < 1$, since

$$\bar{P}(n) = \frac{1}{n} \sum_{r|n} \mu(r) \bar{\Lambda}\left(\frac{n}{r}\right)$$

(here $\mu(r)$ is the classical Möbius function). In case (i), $S = \infty$ since

$$\sum_{m=1}^{\infty} \frac{1}{m} \cos 4m\pi\theta_l, \qquad l = 1, \dots, k$$

are convergent trigonometric series for $0 < \theta_1 < \cdots < \theta_k < \frac{1}{2}$. Hence, $S$ diverges for all real number $\theta$ and

$$\sum_{\partial(a)=m} f(a) = o(q^m m^{\rho_r - 1}) \tag{4.1}$$

by Theorem 1.5 of [7]. In case (ii), $\theta_k = \frac{1}{2}$, i.e., $Z^{\#}(y)$ has a zero at $y = -q^{-1}$. If $\rho_r > n_k$, then $S = \infty$ and (3.1) follows. If $\rho_r = n_k$, then $Z^{\#}(y)$ has no zeros but $y = -q^{-1}$ of integer-order $n_k$ on the circle $|y| = q^{-1}$ and $S$ converges for $\alpha = \frac{(2n+1)\pi}{\log q}$. In the last case, $f$ does not have mean value 0. It remains to determine whether $f$ has a nonzero mean value. If $f(a) = \lambda(a)$, for each prime $p$, we have

$$\begin{aligned} 1 + \sum_{k=1}^{\infty} q^{-k\partial(p)(1+i\alpha)} f(p^k) &= 1 + \sum_{k=1}^{\infty} q^{-k\partial(p)} (-1)^{k(\partial(p)+1)} \\ &= \frac{1}{1 - (-1)^{\partial(p)+1} q^{-\partial(p)}} \\ &> 0 \end{aligned}$$

and if $f(a) = \mu(a)$, we have

$$1 + \sum_{k=1}^{\infty} q^{-k\partial(p)(1+i\alpha)} f(p^k) = 1 - (-1)^{\partial(p)} q^{-\partial(p)} > 0$$

since $q > 1$, $\partial(p) \geq 1$. Moreover,

$$\sum_{p} q^{-\partial(p)}(1 - f(p)) = 2 \sum_{m=1}^{\infty} \bar{P}(m) q^{-m} \geq \sum_{m=1}^{\infty} \bar{P}(2m-1) q^{-2m+1} = \infty.$$

Therefore, by Theorem 1.5 of [**7**], $f$ does not have a nonzero mean value and the mean value $m_f$ does not exist. □

Theorem 1.5 is well illustrated by Example 4.1 in [6]. Let $q$ be a positive integer and $q \geq 2$. Let

$$2q^k \equiv r_k \pmod{k}, \qquad 0 \leq r_k < k$$

for $k = 1, 2, \dots$. We set

$$\bar{P}(k) = \begin{cases} \frac{1}{k}(2q^k - r_k) + 1, & \text{if } k \text{ is odd;} \\ 1, & \text{if } k \text{ is even} \end{cases}$$

(we can even put $\bar{P}(k) = 0$ for $k$ even). Then $\bar{P}(k), k = 1, 2, \ldots$ are all positive integers and

$$Z^{\#}(y) = \prod_{m=1}^{\infty}(1 - y^m)^{-\bar{P}(m)}$$

converges absolutely in the disk $\{|y| < q^{-1}\}$. It is shown in [6] that if we write

$$Z^{\#}(y) = 1 + \sum_{n=1}^{\infty} \bar{G}(n) y^n$$

then

$$\bar{G}(n) = Aq^n + O(q^{\frac{n}{2}} n^{-\frac{1}{2}}).$$

Also, it is shown in [6] that $Z^{\#}(y)$ has an analytic extension

$$Z^{\#}(y) = \frac{1 + qy}{1 - qy}\left(\frac{1 + qy^2}{1 - qy^2}\right)^{\frac{1}{2}} e^{F(y)} \tag{4.2}$$

in the domain $D$ formed by cutting the complex plane along the real axis from $-\infty$ to $-q^{-\frac{1}{2}}$ and from $q^{-\frac{1}{2}}$ to $+\infty$ and along the imaginary axis from $-i\infty$ to $-iq^{-\frac{1}{2}}$ and from $iq^{-\frac{1}{2}}$ to $i\infty$. Here, in (4.2), the function $F(y)$ is holomorphic in the disk $\{|y| < q^{-\frac{1}{3}}\}$ and the function

$$H(y) := \left(\frac{1 + qy^2}{1 - qy^2}\right)^{\frac{1}{2}}$$

is the single-valued branch with $H(0) = 1$ of the associated multiple-valued function in the domain $D$.

Consider the generating function

$$M(y) := \sum_{a \in \mathcal{G}} \mu(a) y^{\partial(a)} = \sum_{m=0}^{\infty} \left( \sum_{\partial(a)=m} \mu(a) \right) y^m, \qquad |y| < q^{-1}$$

of $\sum_{\partial(a)=m} \mu(a)$. It is easily seen that

$$\begin{aligned} M(y) &= \prod_{p} (1 - y^{\partial(p)}) = \prod_{m=1}^{\infty} (1 - y^m)^{\bar{P}(m)} \\ &= \frac{1}{Z^{\#}(y)} = \frac{1 - qy}{1 + qy}\left(\frac{1 - qy^2}{1 + qy^2}\right)^{\frac{1}{2}} e^{-F(y)}. \end{aligned}$$

Therefore, we have

$$\sum_{\partial(a)=m} \mu(a) = \frac{1}{2\pi i} \int_{|y|=r} \frac{1}{Z^{\#}(y) y^{m+1}} dy,$$

where $0 < r < q^{-1}$. If we shift the integration contour to the circle $|y| = q^{-\frac{1}{2}-\epsilon}$ then we obtain

$$\begin{aligned}\sum_{\partial(a)=m} \mu(a) &= \mathrm{Res}_{y=-q^{-1}} \frac{1}{Z^{\#}(y)y^{m+1}} + \frac{1}{2\pi i}\int_{|y|=q^{-\frac{1}{2}-\epsilon}} \frac{1}{Z^{\#}(y)y^{m+1}}dy \\ &= (-1)^{m+1}2q^m\left(\frac{q-1}{q+1}\right)^{\frac{1}{2}} e^{-F(-q^{-1})} + O_\epsilon(q^{(\frac{1}{2}+\epsilon)m}).\end{aligned}$$

It is clear that, roughly speaking, $q^{-m}\sum_{\partial(a)=m}\mu(a)$ alternates between values

$$2\left(\frac{q-1}{q+1}\right)^{\frac{1}{2}} e^{-F(-q^{-1})}, \qquad -2\left(\frac{q-1}{q+1}\right)^{\frac{1}{2}} e^{-F(-q^{-1})}$$

as $m \to \infty$. Hence $\mu(a)$ does not have a mean value because of the dominant perturbation of the zero of $Z^{\#}(y)$ at $y = -q^{-1}$.

## REFERENCES

1. K.-H. Indlekofer and E. Manstavicius, *Additive and multiplicative functions on arithmetical semigroups*, Publ. Math. Debrecen **45** (1994), 1–17.
2. K.-H. Indlekofer, E. Manstavicius and R. Warlimont, *On a certain class of infinite product with an application to arithmetical semigroups*, Arch. Math. (Basel) **56** (1991), 446–453.
3. J. Knopfmacher, *Analytic arithmetics of algebraic function fields*, Lecture Notes in Pure and Applied Mathematics, no. 50, Marcel Dekker, New York, 1979.
4. R. Warlimont, *Arithmetical semigroups V: Multiplicative functions*, Manuscripta Math. **77** (1992), 361–383.
5. W.-B. Zhang, "The abstract prime number theorem for algebraic function fields" in *Analytic Number Theory* (*Berndt et al. eds.*), Prog. Math., no. 85, Birkhäuser, 1990, pp. 529–558, MR92d:11111.
6. ———, *The prime element theorem in additive arithmetic semigroups. I*, Illinois J. Math., **40** (1996), 245–280.
7. ———, *Mean-value theorems of multiplicative functions on additive arithmetic semigroups*, to appear in Math. Z.
8. ———, *A generalization of Halász's theorem to Beurling's generalized integers and its application*, Illinois J. Math., **31** (1987), 645–664.

Current address: Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, Illinois 61801