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A GENERALIZED H^∞ FUNCTIONAL CALCULUS FOR OPERATORS ON SUBSPACES OF L^p AND APPLICATION TO MAXIMAL REGULARITY

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Let (Ω, μ) be a measure space and let $1 < p < \infty$ be a number. Consider a closed operator A on $L^p(\Omega)$ and assume that it admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus (in the sense of McIntosh [10]) for some sector

$$\Sigma_\theta = \{z \in \mathbb{C}^*\colon\ |\arg z| < \theta\}. \tag{1}$$

Let H be a Hilbert space and let $\mathcal{A}$ be the closure, which exists, of $A \otimes I_H$ on $L^p(\Omega; H)$. In a recent joint work with F. Lancien [8], we showed that for any $\nu > \theta$, the bounded holomorphic functional calculus of A naturally extends to a bounded $H^\infty(\Sigma_\nu; B(H))$ functional calculus for $\mathcal{A}$. As a consequence, we could deduce abstract maximal regularity results on spaces of the form $L^p(\Omega; H)$, for operators which are the sum of an operator acting on $L^p(\Omega)$ and another one acting on H. The purpose of this paper is to extend these results to the case $p = 1$ and to the situation where $L^p(\Omega)$ is replaced by one of its closed subspaces. As a consequence, we get a new class of operators satisfying the L_p-maximal regularity property for the first order Cauchy problem on intervals. As a matter of fact, the present work yields a new proof of Theorem 5.2 in [8] which is somewhat simpler than the original one.

Let us now explain the framework of this paper and fix some terminology and notation. All Banach spaces considered here will be complex ones. The n-dimensional Hilbert space will be denoted by ℓ_2^n for every integer $n \geq 1$. The Banach algebra of all bounded operators on a Banach space X will be denoted by $B(X)$. We shall use the notation (1) to denote open sectors around the half-line of positive real numbers. For any $\theta \in (0, \pi)$ and any Banach space E, we denote by $H^\infty(\Sigma_\theta; E)$ the Banach space of all bounded analytic functions from Σ_θ into E, equipped with the supremum norm. When $E = \mathbb{C}$, we simply denote this space by $H^\infty(\Sigma_\theta)$. Given a linear operator A on a Banach space X, we shall denote by $D(A)$, $\sigma(A)$ and $\rho(A)$ the domain, the spectrum and the resolvent set of A respectively. We will assume the reader familiar with McIntosh's H^∞ functional calculus on Banach spaces for which we refer to

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[10], [3]. Let A be a closed linear operator on a Banach space X and assume that A admits a bounded $H^\infty(\Sigma_\nu)$ functional calculus for some $\nu \in (0,\pi)$. As usual, for any $f \in H^\infty(\Sigma_\nu)$, we denote by $f(A)$ the bounded operator corresponding to this functional calculus. Let us denote by

$$E_A = \{T \in B(X)\colon\ T(\lambda - A)^{-1} = (\lambda - A)^{-1}T,\ \lambda \in \rho(A)\} \tag{2}$$

the commutant of A. In [8] (see also [1]), we introduced a generalization of McIntosh's functional calculus for which scalar-valued analytic functions are replaced by E_A-valued ones. For any $F \in H^\infty(\Sigma_\nu; E_A)$, we defined a closed and densely defined operator $u_A(F)$ in a way that preserves reasonable algebraic and continuity properties. Moreover u_A is given on the algebraic tensor product $H^\infty(\Sigma_\nu) \otimes E_A$ by the simple formula

$$u_A\left(\sum f_i \otimes T_i\right) = \sum f_i(A)T_i \qquad ((f_i)_i \subset H^\infty(\Sigma_\nu),\ (T_i)_i \subset E_A).$$

It turns out that given any closed subspace $E \subset E_A$, the operator $u_A(F)$ is bounded for any $F \in H^\infty(\Sigma_\nu; E)$ if and only if u_A is bounded on $H^\infty(\Sigma_\nu) \otimes E$, i.e., there exists a constant $K \geq 0$ such that for any finite families $(f_i)_i \subset H^\infty(\Sigma_\nu)$ and $(T_i)_i \subset E$, we have

$$\left\|\sum f_i(A)T_i\right\| \leq K \left\|\sum f_i \otimes T_i\right\|_{H^\infty(\Sigma_\nu; E)}. \tag{3}$$

In this case we say that A admits a bounded $H^\infty(\Sigma_\nu; E)$ functional calculus. We refer the reader to [8] for further details.

The spaces on which we shall work can be described as follows. We give ourselves a number $1 \leq p < \infty$ and a measure space (Ω, μ). Let $S \subset L^p(\Omega)$ be a closed subspace and let X be any Banach space. We denote by $S(X)$ the closed subspace of $L^p(\Omega; X)$ spanned by $S \otimes X$. It should be noticed that this definition relies on the embedding of S into $L^p(\Omega)$ and does not only depend on the Banach space structures of S and X. In the sequel, a closed subspace S of some L^p-space will be called an SL^p-space and a Banach space of the form $S(X)$ will be called a vector valued SL^p-space. This definition includes vector valued Hardy spaces and vector valued Sobolev spaces for example. The following tensor extension results for vector valued SL^p-spaces are elementary.

LEMMA 1. *Let S be an SL^p-space for some $1 \leq p < \infty$ and let X be a Banach space.*

(i) Let A be any closable linear operator on X. Then $I_S \otimes A$, defined on $S \otimes D(A)$, is closable on $S(X)$.

(ii) Let A be any closable linear operator on S. Then $A \otimes I_X$, defined on $D(A) \otimes X$, is closable on $S(X)$.

Proof. We only prove (i), the proof for (ii) being the same. For any $\xi \in S^*$, let us denote by $\widehat{\xi}\colon S(X) \to X$ the bounded extension of $\xi \otimes I_X$. Let $(z_n)_n$ be a sequence in $S \otimes D(A)$ which converges to 0 and such that $(I_S \otimes A)(z_n)$ converges to some $w \in S(X)$. Then $\lim \widehat{\xi}(z_n) = 0$ and $\lim \widehat{\xi}(I_S \otimes A)(z_n) = \widehat{\xi}(w)$ for any $\xi \in S^*$. Clearly $\widehat{\xi}(I_S \otimes A)(z) = A\widehat{\xi}(z)$ for any z in $S \otimes D(A)$ hence the closability of A shows that $\widehat{\xi}(w) = 0$ for any $\xi \in S^*$. This implies that $w = 0$ and shows the closability of $I_S \otimes A$. □

LEMMA 2. *Let S be an SL^p-space for some $1 \leq p < \infty$ and let X be a Banach space.*

(i) For any $T \in B(X)$, $I_S \otimes T$ extends to a (unique) bounded operator on $S(X)$ with norm equal to $\|T\|$.

(ii) Assume that $X = H$ is a Hilbert space. Then for any $T \in B(S)$, $T \otimes I_H$ extends to a (unique) bounded operator on $S(H)$ with norm equal to $\|T\|$.

Proof. We only prove (ii), the proof for (i) being obvious. So we let $T \in B(S)$, with $S \subset L^p(\Omega, \mu)$ and fix an integer $n \geq 1$. Let $(g_i)_{1 \leq i \leq n}$ be a finite sequence of independent Gaussian normal random variables on a probability space (Ω', μ'). Then for any complex numbers $t_1, \ldots, t_n$, we have

$$\Big\|\sum t_i g_i\Big\|_{L^p(\Omega')} = \delta_p \Big(\sum |t_i|^2\Big)^{1/2}$$

with $\delta_p = \|g_1\|_{L^p(\Omega')}$. Therefore given any $s_1, \ldots, s_n$ in S, we have

$$\begin{aligned}
\Big\|\Big(\sum_i |T(s_i)|^2\Big)^{1/2}\Big\|_{L^p(\Omega)}^p &= \delta_p^{-p} \int \Big\|\sum_i (T(s_i))(\omega) g_i\Big\|_{L^p(\Omega')}^p \, d\mu(\omega) \\
&= \delta_p^{-p} \int \Big\|\sum_i g_i(\omega')T(s_i)\Big\|_{L^p(\Omega)}^p \, d\mu'(\omega') \quad \text{by Fubini} \\
&\leq \|T\|^p \, \delta_p^{-p} \int \Big\|\sum_i g_i(\omega') s_i\Big\|_{L^p(\Omega)}^p \, d\mu'(\omega') \\
&\leq \|T\|^p \Big\|\Big(\sum_i |s_i|^2\Big)^{1/2}\Big\|_{L^p(\Omega)}^p
\end{aligned}$$

This shows the result when $H = \ell_2^n$ is finite-dimensional and the general case follows by approximation. □

Remark 3. Let S be an SL^p-space for some $1 \leq p < \infty$ and let H be a Hilbert space. The following observations, which readily follow from above, will be used in the sequel.

(i) Let A be an operator on S which admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (0, \pi)$ and let $\mathcal{A}$ be the closure of $A \otimes I_H$ provided by Lemma 1 (ii).

Then it follows from Lemma 2 (ii) that $\mathcal{A}$ also admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus. Furthermore for any f in $H^\infty(\Sigma_\theta)$, $f(\mathcal{A})$ is the bounded extension of $f(A) \otimes I_H$.

(ii) Let $(U_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$ be two bounded C_0-semigroups on S and H respectively. We denote by $-B$ and $-C$ the infinitesimal generators of $(U_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$ respectively. By Lemma 2, $U_t \otimes V_t$ extends to a bounded operator on $S(H)$ for any $t \geq 0$. Letting W_t the resulting extension, it is clear that $(W_t)_{t\geq 0}$ is a bounded C_0-semigroup as well. Now let $\mathcal{B}$ and $\mathcal{C}$ be the closures of $B \otimes I_H$ and $I_S \otimes C$ provided by Lemma 1. Then $D(\mathcal{B}) \cap D(\mathcal{C})$ is a core for the generator of $(W_t)_{t\geq 0}$, and the restriction of this generator to $D(\mathcal{B}) \cap D(\mathcal{C})$ is $-(\mathcal{B} + \mathcal{C})$.

We now turn to the main result of this paper. Before stating it, we recall that any operator which admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus automatically admits a bounded $H^\infty(\Sigma_\nu)$ functional calculus for all $\theta < \nu < \pi$. We also observe that given any Hilbert space H and any SL^p-space S, with $1 \leq p < \infty$, it easily follows from Lemma 2 (i) that $B(H)$ can be regarded as a subspace of $B(S(H))$. Namely we identify $T \in B(H)$ with the bounded extension of $I_S \otimes T$. Furthermore if A is a linear operator on S, then $B(H)$ is actually included into the commutant algebra $E_{\mathcal{A}}$ (see (2)) of the closure $\mathcal{A}$ of $A \otimes I_H$.

THEOREM 4. *Let H be a Hilbert space and S be an SL^p-space, with $1 \leq p < \infty$. Let A be an operator on S which admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some θ in $(0, \pi)$.*

Then the closure $\mathcal{A}$ of $A \otimes I_H$ admits a bounded $H^\infty(\Sigma_\nu;\, B(H))$ functional calculus for any $\nu > \theta$.

As noted in our introduction, the above result generalizes Theorem 5.2 of [8]. Roughly speaking, the proof of the latter relies upon integral quadratic estimates from [3]. Here we shall appeal to a deep decomposition result for analytic functions established by Franks and McIntosh in [7] and which lead them to discrete analogues of the integral quadratic estimates mentioned above.

LEMMA 5 (FRANKS-MCINTOSH [7]). *Fix two numbers $\nu > \theta$ in $(0, \pi)$. Then there exist a constant C and two sequences $(\psi_k)_{k\geq 1}$ and $(\widetilde{\psi}_k)_{k\geq 1}$ in $H^\infty(\Sigma_\theta)$ such that the following hold.*

(i) For any $z \in \Sigma_\theta$ we have $\sum_{k\geq 1} |\psi_k(z)| \leq C$ and $\sum_{k\geq 1} |\widetilde{\psi}_k(z)| \leq C$.

(ii) For any Banach space E and any function F in $H^\infty(\Sigma_\nu; E)$, there exists a bounded sequence $(\alpha_k)_{k\geq 1}$ in E with $\sup_k \|\alpha_k\|_E \leq C\|F\|_{H^\infty(\Sigma_\nu;E)}$ and

$$\forall z \in \Sigma_\theta, \quad F(z) = \sum_{k\geq 1} \alpha_k \psi_k(z) \widetilde{\psi}_k(z). \tag{4}$$

Proof. This is established for scalar-valued functions by combining Proposition 2.1 with (6) and (7) in [7]. The proof for vector-valued functions is identical. □

Proof of Theorem 4. Let $S \subset L^p(\Omega)$ and A satisfying the assumption of Theorem 4 and let us fix $\nu > \theta$. We let φ_θ (resp. φ_ν) denote the bounded homomorphism from $H^\infty(\Sigma_\theta)$ (resp. $H^\infty(\Sigma_\nu)$) into $B(S)$ defined by $f \mapsto f(A)$. We will show that there is a constant $K \geq 0$ such that for any integer $n \geq 1$,

$$\|\varphi_\nu \otimes I_{M_n}\colon H^\infty(\Sigma_\nu; M_n) \to B(S(\ell_2^n))\| \leq K. \tag{5}$$

By an obvious approximation argument, this implies that

$$\|\varphi_\nu \otimes I_{B(H)}\colon H^\infty(\Sigma_\nu) \otimes B(H) \to B(S(H))\| \leq K,$$

where the space $H^\infty(\Sigma_\nu) \otimes B(H)$ is regarded as included in $H^\infty(\Sigma_\nu; B(H))$. By Remark 3 (i), this means that $\mathcal{A}$ satisfies (3) with $E = B(H)$, whence the result.

From now on, we fix two sequences $(\varepsilon_i)_{i\geq 1}$ and $(\varepsilon'_i)_{i\geq 1}$ of independent ± 1-valued random variables on a probability space $(D, \mathbb{P})$, with $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = \mathbb{P}(\varepsilon'_i = 1) = \mathbb{P}(\varepsilon'_i = -1) = 1/2$. Moreover we assume that these two sequences are mutually independent. We will use the following well known consequence of the two-variable version of Khintchine's inequality. For any finite family $(y_{ij})_{i,j}$ in $L^p(\Omega)$, for some constant c_p only depending on p we have

$$\Big\|\sum_{i,j} \varepsilon_i\, \varepsilon'_j\, y_{ij}\Big\|_{L^2(D;L^p(\Omega))} \leq c_p \left\|\Big(\sum_{i,j} |y_{ij}|^2\Big)^{1/2}\right\|_{L^p(\Omega)} \tag{6}$$

$$\left\|\Big(\sum_{i,j} |y_{ij}|^2\Big)^{1/2}\right\|_{L^p(\Omega)} \leq c_p \Big\|\sum_{i,j} \varepsilon_i\, \varepsilon'_j\, y_{ij}\Big\|_{L^2(D;L^p(\Omega))} \tag{7}$$

We now fix an integer $n \geq 1$. We let Γ be the set of all subsets of $\{1, \ldots, n\}$ and for any $\gamma \in \Gamma$, we let $w_\gamma = \prod_{i\in\gamma} \varepsilon'_i$. By convention $\omega_\gamma = 1$ for $\gamma = \emptyset$.

We give ourselves a function F in $H^\infty(\Sigma_\nu; M_n)$ to which we apply Lemma 5. We thus dispose of (4) for some bounded sequence $(\alpha_k)_{k\geq 1}$ in M_n. For any k, we denote by $\alpha_k(i, j)$ (with $1 \leq i, j \leq n$) the entries of α_k. Furthermore for any integer $m \geq 1$, we let

$$F_m = \sum_{k=1}^m \alpha_k \psi_k \widetilde{\psi}_k \quad \in H^\infty(\Sigma_\theta; M_n).$$

Let $\sigma = (s_1, \ldots, s_n)$ and $\sigma^* = (s_1^*, \ldots, s_n^*)$ be n-tuples in S and S^* respectively.

Then

$$(8)\ \langle(\varphi_\theta \otimes I_{M_n})(F_m)(\sigma), \sigma^*\rangle = \sum_{k=1}^{m} \langle(\psi_k(A)\widetilde{\psi}_k(A) \otimes \alpha_k)(\sigma), \sigma^*\rangle$$
$$= \sum_{k=1}^{m} \sum_{1\le i,j\le n} \alpha_k(i,j)\langle \psi_k(A)\widetilde{\psi}_k(A)s_j, s_i^*\rangle$$
$$= \sum_{k,i} \left\langle \left(\sum_j \alpha_k(i,j)\psi_k(A)(s_j)\right), \widetilde{\psi}_k(A)^*(s_i^*)\right\rangle.$$

We now introduce an arbitrary family $(s_\gamma^*)_{\gamma\in\Gamma}$ in S^* which completes σ^*, in the sense that when $\gamma = \{i\}$ is a one point set, we simply have $s_\gamma^* = s_i^*$. Since $\{\varepsilon_k, w_\gamma : 1 \le k \le n,\ \gamma \in \Gamma\}$ is an orthonormal family in $L^2(D, \mathbb{P})$, from (8) we derive

$$\langle(\varphi_\theta \otimes I_{M_n})(F_m)(\sigma), \sigma^*\rangle$$
$$= \int_D \left\langle \left(\sum_{k,i}\left(\sum_j \alpha_k(i,j)\psi_k(A)(s_j)\right)\varepsilon_k\,\varepsilon_i'\right), \left(\sum_{k,\gamma} \widetilde{\psi}_k(A)^*(s_\gamma^*)\,\varepsilon_k\, w_\gamma\right)\right\rangle d\mathbb{P}.$$

Hence by the Cauchy-Schwarz inequality we obtain

$$(9)\qquad \left|\langle(\varphi_\theta \otimes I_{M_n})(F_m)(\sigma), \sigma^*\rangle\right|$$
$$\le \left\|\sum_{k,i}\left(\sum_j \alpha_k(i,j)\psi_k(A)(s_j)\right)\varepsilon_k\,\varepsilon_i'\right\|_{L^2(D;S)}$$
$$\times \left\|\sum_{k,\gamma} \widetilde{\psi}_k(A)^*(s_\gamma^*)\,\varepsilon_k\, w_\gamma\right\|_{L^2(D;S^*)}.$$

Now let us estimate the two norms in the right hand side of (9). On one hand, we may write $\sum_{k,\gamma} \widetilde{\psi}_k(A)^*(s_\gamma^*)\,\varepsilon_k\, w_\gamma = \left(\varphi_\theta(\sum_k \varepsilon_k\widetilde{\psi}_k)\right)^*(\sum_\gamma w_\gamma s_\gamma^*)$, whence

$$\left\|\sum_{k,\gamma} \widetilde{\psi}_k(A)^*(s_\gamma^*)\varepsilon_k w_\gamma\right\|_{L^2(D;S^*)} \le \sup_{\varepsilon_k=\pm1}\left\{\left\|\varphi_\theta\left(\sum_k \varepsilon_k\widetilde{\psi}_k\right)\right\|_{B(S)}\right\}\left\|\sum_\gamma w_\gamma s_\gamma^*\right\|_{L^2(D;S^*)}.$$

Applying Lemma 5 (i), we obtain

$$(10)\qquad \left\|\sum_{k,i} \widetilde{\psi}_k(A)^*(s_\gamma^*)\,\varepsilon_k\, w_\gamma\right\|_{L^2(D;S^*)} \le C\ \|\varphi_\theta\|\ \left\|\sum_\gamma w_\gamma s_\gamma^*\right\|_{L^2(D;S^*)}.$$

On the other hand we have

$$\begin{aligned}
&\Big\|\sum_{k,i}\Big(\sum_j \alpha_k(i,j)\psi_k(A)(s_j)\Big)\,\varepsilon_k\,\varepsilon'_i\Big\|_{L^2(D;S)}\\
&\leq c_p\,\Bigg\|\Bigg(\sum_{k,i}\Big|\Big(\sum_j \alpha_k(i,j)\psi_k(A)(s_j)\Big)\Big|^2\Bigg)^{1/2}\Bigg\|_{L^p(\Omega)} \qquad \text{by (6)}\\
&\leq c_p\,\sup_k\|\alpha_k\|\,\Bigg\|\Bigg(\sum_{k,j}\big|\psi_k(A)(s_j)\big|^2\Bigg)^{1/2}\Bigg\|_{L^p(\Omega)}\\
&\leq c_p^2\,C\,\|F\|_{H^\infty(\Sigma_\nu;M_n)}\,\Big\|\sum_{k,j}\psi_k(A)(s_j)\varepsilon_k\,\varepsilon'_j\Big\|_{L^2(D;S)} \qquad \text{by (7).}
\end{aligned}$$

Applying the same arguments as in the proof of (10), we see that

$$\Big\|\sum_{k,j}\psi_k(A)(s_j)\,\varepsilon_k\,\varepsilon'_j\Big\|_{L^2(D;S)} \leq C\,\|\varphi_\theta\|\,\Big\|\sum_j \varepsilon'_j s_j\Big\|_{L^2(D;S)}$$

Now putting all together, we deduce from (9) that

$$\begin{aligned}
(11)\qquad &\big|\langle(\varphi_\theta\otimes I_{M_n})(F_m)(\sigma),\sigma^*\rangle\big| \leq c_p^2\,C^2\,\|\varphi_\theta\|\,\|F\|_{H^\infty(\Sigma_\nu;M_n)}\\
&\qquad\times\Big\|\sum_j\varepsilon_j s_j\Big\|_{L^2(D;S)}\,\Big\|\sum_\gamma w_\gamma s^*_\gamma\Big\|_{L^2(D;S^*)}
\end{aligned}$$

Let us provisionally set $\tau = (\varphi_\theta\otimes I_{M_n})(F_m)(\sigma)$ in $S(\ell_2^n)$, and let us write $\tau = (\tau_1,\dots,\tau_n)$ with $\tau_i\in S$. Thus the left hand side of (11) is $|\sum_j\langle\tau_j,s_j^*\rangle|$. By classical duality, we have

$$(12)\quad \Big\|\sum_j\varepsilon'_j\tau_j\Big\|_{L^2(D;S)} = \sup\left\{\left|\int_D\Big\langle\sum_j\varepsilon'_j\tau_j,\,F\Big\rangle\,d\mathbb{P}\right| : F\in L^2(D;S^*),\ \|F\|\leq 1\right\}$$

By means of the conditional expectation with respect to the σ-algebra generated by $(\varepsilon'_1,\dots,\varepsilon'_n)$, we can restrict to F belonging to $\mathrm{Span}\{w_\gamma : \gamma\in\Gamma\}\otimes S^*$ in (12). Any such F is of the form $F=\sum_\gamma w_\gamma s^*_\gamma$, with $(s^*_\gamma)_{\gamma\in\Gamma}\subset S^*$ as above, and

$$\int_D\Big\langle\sum_j\varepsilon'_j\tau_j,\,\sum_\gamma w_\gamma s^*_\gamma\Big\rangle\,d\mathbb{P} = \sum_j\langle\tau_j,s_j^*\rangle.$$

We therefore deduce from (11) that

$$\Big\|\sum_j\varepsilon'_j\tau_j\Big\|_{L^2(D;S)} \leq c_p^2\,C^2\,\|\varphi_\theta\|\,\|F\|_{H^\infty(\Sigma_\nu)}\,\Big\|\sum_j\varepsilon_j s_j\Big\|_{L^2(D;S)}$$

Applying both (6) and (7), we infer

$$\|\tau\|_{S(\ell_2^n)} \leq c_p^4\, C^2\, \|\varphi_\theta\|\, \|F\|_{H^\infty(\Sigma_\nu; M_n)}\, \|\sigma\|_{S(\ell_2^n)}.$$

Letting $K = c_p^4\, C^2\, \|\varphi_\theta\|$ we have thus proved

$$\|(\varphi_\theta \otimes I_{M_n})(F_m)\| \leq K\, \|F\|_{H^\infty(\Sigma_\nu; M_n)}.$$

Since this holds for all $m \geq 1$, it follows from the so-called Convergence Lemma (see [10, Section 5] and [3, Lemma 2.1] for scalar valued functions, the proof for the vector valued case being the same [1], [8]) that in fact, $\|(\varphi_\nu \otimes I_{M_n})(F)\| \leq K\, \|F\|_{H^\infty(\Sigma_\nu; M_n)}$, whence (5). $\square$

Remark 6. Let H be a Hilbert space and let A be an operator on H which admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (0, \pi)$. Let us consider a measure space (Ω, μ) and a number $1 \leq p < \infty$. We let $\mathcal{A}$ be the closure of $I_{L^p} \otimes A$ on $L^p(\Omega; H)$. Then by Lemma 2 (i), $\mathcal{A}$ admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus. In view of Theorem 4, it is therefore tempting to ask whether $\mathcal{A}$ automatically admits a bounded $H^\infty(\Sigma_\nu; B(L^p(\Omega)))$ functional calculus for some (or for any) $\nu > \theta$. It turns out that in general, the answer is no, except when $p = 2$ where Theorem 4 can be applied.

Indeed assume that H is infinite-dimensional and separable and let $(e_n)_{n\geq 1}$ be a basis of H. Let A be defined by letting

$$A\Big(\sum t_n e_n\Big) = \sum n t_n e_n,$$

where the domain of A is the space of all $h = \sum t_n e_n$ in H such that $\sum n^2 |t_n|^2 < \infty$. Then for any $\theta > 0$, the operator A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus. Indeed, for any $f \in H^\infty(\Sigma_\theta)$, $f(A)\big(\sum t_n e_n\big) = \sum f(n) t_n e_n$. Now let $\mathbb{T}$ be the unit circle of $\mathbb{C}$, equipped with its usual Haar measure and assume that for some $\nu \in (0, \pi)$, $\mathcal{A}$ admits a bounded $H^\infty(\Sigma_\nu; B(L^p(\Omega)))$ functional calculus. Then arguing as in [8, Section 6], one can prove that $L^p(\mathbb{T})$ is both 2-concave and 2-convex (see e.g. [9] for a definition), whence $p = 2$. The details are left to the reader.

We shall now apply Theorem 4 to the problem of the closedness of the sum of two commuting closed operators and then to abstract regularity theory. The next two statements generalize Corollary 5.6 and Theorem 1.4 in [8]. We only provide brief proofs since they are quite similar to those in [8] and we refer to the latter paper for further details. We recall that by convention, the domain of a finite sum of linear operators on a Banach space is simply defined as the intersection of the domains of these operators.

COROLLARY 7. Let H be a Hilbert space and S be an SL^p-space, with $1 \leq p < \infty$. Let B be a closed and densely defined operator on H. We assume that for some

$\omega \in (0, \pi)$, it satisfies the following (sectoriality) condition:

$$(13) \qquad \sigma(B) \subset \overline{\Sigma_\omega} \qquad \text{and} \qquad \forall \theta \in (\omega, \pi), \quad \sup_{\lambda \notin \overline{\Sigma_\theta}} \left\| \lambda(\lambda - B)^{-1} \right\| < \infty.$$

Let A be an operator on S which admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some θ in $(0, \pi)$. We assume the (parabolicity) condition $\omega + \theta < \pi$.

Let $\mathcal{A}$ and $\mathcal{B}$ be the closures of $A \otimes I_H$ and $I_S \otimes B$ respectively. Then $\mathcal{A} + \mathcal{B}$ is a closed densely defined operator on $S(H)$. Moreover $\mathcal{A} + \mathcal{B}$ is one to one, has a dense range and $\mathcal{A}(\mathcal{A} + \mathcal{B})^{-1}$ is bounded.

Proof. Let $\nu > \theta$ such that $\omega + \nu < \pi$. By (13), the function F defined by $F(z) = z(z + B)^{-1}$ belongs to $H^\infty(\Sigma_\nu; B(H))$. Consequently the result follows from Theorem 4 and [8, Proposition 2.6]. □

Let us now turn to abstract regularity theory for generators of bounded analytic semigroups. We recall a classical definition. We give ourselves two numbers $1 < p < \infty$ and $0 < T < \infty$. Let X be a Banach space and let $-B$ be the generator of a bounded analytic semigroup on X. Then B is said to have the maximal regularity property provided that:

There exists $C > 0$ such that for any $f \in L^p([0, T); X)$, there exists a unique function $u \in W_0^{1,p}([0, T); X) \cap L^p([0, T); D(B))$ satisfying

$$u' + Bu = f \quad \text{on } [0, T) \quad \text{and} \quad \|u\| \leq C\|f\|.$$

In this definition, the notation $W_0^{1,p}([0, T); X)$ stands for the space of functions u belonging to the Sobolev space $W^{1,p}([0, T); X)$ which satisfy $u(0) = 0$. Note that the maximal regularity property does not depend on the choice of p and T (see [2], [4], [5]). In Theorem 8 below, we consider the problem of maximal regularity for generators of semigroups on spaces of the form $X = S(H)$ obtained as tensor products of semigroups on S and H respectively, as explained in Remark 3 (ii). Two classical results should be mentioned here. First, any B as above has the maximal regularity property when $X = H$ is a Hilbert space (De Simon, [4]). Second, if X is a UMD Banach space (in particular if X is an SL^p-space with $1 < p < \infty$) and if B admits bounded imaginary powers which generate a C_0-group of exponential type $< \pi/2$, then it has the maximal regularity property (Dore-Venni [6], see also [11]). Thus in our result below, the case $B = 0$ corresponds to the De Simon Theorem whereas the case $C = 0$ corresponds to a classical particular case of the Dore-Venni Theorem.

THEOREM 8. *Let S be an SL^p-space with $1 < p < \infty$, and let H be a Hilbert space. Let $-B$ and $-C$ be the generators of bounded analytic semigroups on S and H respectively, and let $\mathcal{B}$ (resp. $\mathcal{C}$) be the closure of $B \otimes I_H$ (resp. $I_S \otimes C$) on the space $S(H)$. Assume that B admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus on S for some $\theta < \pi/2$.*

Then the operator $\mathcal{B} + \mathcal{C}$ is closed and has the maximal regularity property.

Proof. Observe that since $-C$ generates a bounded analytic semigroup, it satisfies (13) for some $\omega < \pi/2$. Hence as a first application of Corollary 7, we obtain that $\mathcal{B} + \mathcal{C}$ is closed and

$$\mathcal{B}(\mathcal{B} + \mathcal{C})^{-1} \text{ is bounded.} \tag{14}$$

Let $X = S(H)$ and let $\mathcal{A}_1$ be the derivation operator $u \mapsto u'$ on $L^p([0, T); X)$, with domain $W_0^{1,p}([0, T); X)$. Let $\mathcal{B}_1$ and $\mathcal{C}_1$ be the closures of $I_{L^p([0,T))} \otimes \mathcal{B}$ and $I_{L^p([0,T))} \otimes \mathcal{C}$ respectively on $L^p([0, T); X)$. We may deduce from (14) that $\mathcal{B}_1 + \mathcal{C}_1$ is actually the closure of $I_{L^p([0,T))} \otimes (\mathcal{B} + \mathcal{C})$ hence by a well known characterization of maximal regularity (see e.g. [2], [5]), it suffices to check that

$$\mathcal{A}_1(\mathcal{A}_1 + \mathcal{B}_1 + \mathcal{C}_1)^{-1} \text{ is bounded.} \tag{15}$$

It follows from a known variant of Theorem 3.1 in [6] and from Theorem 1.3 in [8] that the operator $\mathcal{A}_1 + \mathcal{B}_1$ admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \pi/2$. Therefore, by Corollary 6, $(\mathcal{A}_1 + \mathcal{B}_1)(\mathcal{A}_1 + \mathcal{B}_1 + \mathcal{C}_1)^{-1}$ is bounded. Now, by [6], [11], $\mathcal{A}_1(\mathcal{A}_1 + \mathcal{B}_1)^{-1}$ is also bounded whence (15). □

Remark 9. Let (Ω, μ) be a measure space and let Λ be a Banach lattice of functions on (Ω, μ). We assume that Λ is q-concave for some $q < \infty$. Then the estimates (6) and (7) are valid with Λ instead of $L^p(\Omega)$. Indeed this follows from Maurey's Theorem (see e.g. [9, Theorem 1.d.6]) and Kahane's inequality (see e.g. [9, Theorem 1.e.13]). For any Hilbert space H, we may define $\Lambda(H)$ as the space of strongly measurable functions $f\colon \Omega \to H$ such that the scalar-valued function $\omega \mapsto \|f(\omega)\|_H$ belongs to Λ. This is a Banach space for the norm $\|f\|_{\Lambda(H)} = \big\| \|f(\cdot)\|_H \big\|_\Lambda$. For any closed subspace $S \subset \Lambda$, we let $S(H)$ be the Banach subspace of $\Lambda(H)$ spanned by $S \otimes H$.

Then it is not hard to check that Theorem 4 and Corollary 7 remain true if we allow S to be any subspace of a q-concave Banach lattice as above. Moreover, if we assume that Λ is a UMD Banach lattice, Theorem 8 remains true as well for any $S \subset \Lambda$.

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