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BERGMAN AND HARDY SPACES ON MODEL DOMAINS

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1. Introduction

Let $p\colon \mathbb{C} \longrightarrow \mathbb{R}_+$ denote a $\mathcal{C}^1$–function and define $\Omega_p \subseteq \mathbb{C}^2$ by

$$\Omega_p = \{(z_1, z_2) \in \mathbb{C}^2\colon\ \Im(z_2) > p(z_1)\}.$$

Weakly pseudoconvex domains of this kind were investigated by McNeal [McN1] and Nagel, Rosay, Stein and Wainger [NRSW1],[NRSW2]. For the case where $p(z) = |z|^k, k \in \mathbb{N}$, Greiner and Stein [GS] found an explicit expression for the Szegö kernel of Ω_p. If p is a subharmonic function, which depends only on the real or only on the imaginary part of z, then one can find analogous expressions and estimates in [N] (see also [Has1]). In [D] and in [K] properties of the Szegö projection for such domains are studied. The asymptotic behavior of the corresponding Szegö kernel was investigated in [Han] and [Has2]. There have been several recent papers obtaining explicit formulas for the Bergman kernel function on various weakly pseudoconvex domains ([D'A], [BFS], [FH2] and [FH3]).

Let $H^2(\partial\Omega_p)$ denote the subspace of $L^2(\partial\Omega_p)$ consisting of boundary values of holomorphic functions f on Ω_p such that

$$\sup_{y>0} \int_{\mathbb{C}} \int_{\mathbb{R}} |f(z, t + ip(z) + iy)|^2 \, d\lambda(z)\, dt < \infty,$$

where $d\lambda$ denotes the Lebesgue measure on $\mathbb{C}$. We identify $\partial\Omega_p$ with $\mathbb{C} \times \mathbb{R}$ and note that for each $f \in H^2(\partial\Omega_p)$ there exists a boundary function f_0 on $\partial\Omega_p$ such that $f_y(z,t) := f(z, t + ip(z) + iy)$ tends to $f_0(z,t)$ in $L^2(\mathbb{C}\times\mathbb{R})$ as y tends to 0, moreover we have

$$\int_{\mathbb{C}} \int_{\mathbb{R}} |f_0(z,t)|^2 \, dt\, d\lambda(z) = \sup_{y>0} \int_{\mathbb{C}} \int_{\mathbb{R}} |f(z, t + ip(z) + iy)|^2 \, dt\, d\lambda(z)$$

(see [M], [SW]).

We consider the tangential Cauchy-Riemann operator for $\partial\Omega_p$,

$$\overline{L} = \frac{\partial}{\partial \overline{z_1}} - 2i \frac{\partial p}{\partial \overline{z_1}}(z_1) \frac{\partial}{\partial \overline{z_2}}.$$

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After the identification of $\partial\Omega_p$ with $\mathbb{C} \times \mathbb{R}$ the tangential Cauchy–Riemann operator has the form

$$\overline{L} = \frac{\partial}{\partial \overline{z}} - i\frac{\partial p}{\partial \overline{z}}\frac{\partial}{\partial t}.$$

Distributions ϕ satisfying $\overline{L}(\phi) = 0$ are called CR-distributions (see [B]).

The main result of this paper is to show that each function $f \in L^2(\partial\Omega_p)$ which is also a CR-distribution can be extended to a function holomorphic on Ω_p which belongs to $H^2(\partial\Omega_p)$. The extension is expressed in terms of a corresponding entire function with a growth condition depending on p.

In fact the space $H^2(\partial\Omega_p)$ can be identified with the space of all functions $f \in L^2(\partial\Omega_p)$ satisfying $\overline{L}(f) = 0$ in the distribution sense (Theorem 1). We also point out how this result can be extended to corresponding domains in $\mathbb{C}^{n+1}$.

In the next result we express the Bergman and the Szegö kernel of the domains Ω_p by means of the Bergman kernel of certain weighted Hilbertspaces of entire funtions (Theorem 2 and Theorem 3).

Finally we apply these results to determine the boundary limits of the Bergman kernel on the diagonal of a bounded pseudoconvex domain Ω in $\mathbb{C}^{n+1}$, that is h-extendible at the boundary point P, using a reduction to the model case due to Boas, Straube and Yu [BSY] (Theorem 4 and Theorem 5).

2. Weighted spaces of entire functions with parameters

For the sake of simplicity we concentrate on domains Ω_p in $\mathbb{C}^2$; see Remark (b) after Theorem 1 for domains in higher dimensions.

Let E_p denote the space of measurable functions F on $\mathbb{C} \times \mathbb{R}_+$ which are entire with respect to the first variable and satisfying

$$\int_0^\infty \int_{\mathbb{C}} |F(z,t)|^2 \exp(-4\pi t p(z))\, d\lambda(z)\, dt < \infty.$$

Here and in what follows we have to make sure that the weight function p is chosen in a way that the corresponding space of entire functions E_p is nontrivial, for instance if p grows like $|z|^\alpha$ $(\alpha > 0)$ for $|z| \to \infty$.

The following lemma is a version of an important representation result for Hardy spaces (see [SW] for a special case).

LEMMA 1. *Every function in $H^2(\partial\Omega_p)$ has the representation*

$$f(z,w) = \int_0^\infty F(z,t) e^{2\pi i t w}\, dt, \tag{1}$$

where $F \in E_p$. In addition F can be recovered from the boundary value of f by

$$F(z,\tau) = \int_{\mathbb{R}} f(z, t + ip(z)) e^{-2\pi i \tau t} e^{2\pi \tau p(z)}\, dt. \tag{2}$$

Proof. For a function $g \in L^2(d\lambda(z)dt)$ let $\mathcal{F}$ denote the Fourier transform with respect to the variable $t \in \mathbb{R}$:

$$(\mathcal{F}g)(z,\tau) = \int_{\mathbb{R}} g(z,t)e^{-2\pi i t\tau}\,dt\ .$$

Then

$$\mathcal{F}\overline{L}\mathcal{F}^{-1} = \frac{\partial}{\partial \overline{z}} + \tau\frac{\partial p}{\partial \overline{z}}\ . \tag{3}$$

$\mathcal{F}$ and $\mathcal{F}^{-1}$ are to be taken in the sense of the Plancherel theorem.

Now let M denote the multiplication operator

$$M\colon\ L^2(d\lambda(z)dt) \longrightarrow L^2(e^{-4\pi t p(z)}d\lambda(z)dt)$$

defined by

$$(Mg)(z,\tau) = e^{2\pi\tau p(z)}g(z,\tau)\ ,$$

for $g \in L^2(d\lambda(z)dt)$. Then from (1) we get

$$\mathcal{F}\overline{L}\mathcal{F}^{-1} = M^{-1}\frac{\partial}{\partial \overline{z}}M\ . \tag{4}$$

If $f \in H^2(\partial\Omega_p)$, then its boundary function f_0 satisfies $\overline{L}(f_0) = 0$ in the sense of distributions (the functions f_y are holomorphic in a neighborhood of $\partial\Omega_p$, they satisfy the equation $\overline{L}(f_y) = 0$ (see [Ra]) and they converge to the boundary function f_0 in L^2). Now let F be as in (2). Then, using Plancherel's theorem, we get

$$\mathcal{F}^{-1}M^{-1}F = f_0$$

and from (4),

$$0 = \overline{L}(f_0) = \overline{L}\mathcal{F}^{-1}M^{-1}F = \mathcal{F}^{-1}M^{-1}\frac{\partial}{\partial \overline{z}}F,$$

which implies that $\frac{\partial}{\partial \overline{z}}F = 0$.

Again by Plancherel's theorem we obtain

$$\int_0^\infty |F(z,\tau)|^2 e^{-4\pi\tau p(z)}\,d\tau = \int_{\mathbb{R}} |f_0(z,t)|^2\,dt = \int_{\mathbb{R}} |f(z,t+ip(z))|^2\,dt;$$

hence

$$\int_{\mathbb{C}}\int_0^\infty |F(z,\tau)|^2 e^{-4\pi\tau p(z)}\,d\tau d\lambda(z) = \int_{\mathbb{C}}\int_{\mathbb{R}} |f(z,t+ip(z))|^2\,dt\,d\lambda(z).$$

Since

$$\int_{\mathbb{C}}\int_{\mathbb{R}} |f(z,t+ip(z))|^2\,dt\,d\lambda(z) = \sup_{y>0}\int_{\mathbb{C}}\int_{\mathbb{R}} |f(z,t+ip(z)+iy)|^2\,dt\,d\lambda(z) < \infty,$$

we get $F \in E_p$.

For $(z, w) \in \Omega_p$ we set

$$h(z,w) = \int_0^\infty F(z,\tau)e^{2\pi i\tau w}\,d\tau,$$

we can write $w = t + i(p(z) + \rho)$, $t \in \mathbb{R}$, $\rho > 0$, and from $F \in E_p$ we derive that h is holomorphic in Ω_p (see for instance [Ru], p. 404). A further application of Plancherel's theorem implies that $h = f$. □

3. Extension of CR-distributions

THEOREM 1. *Let $f \in L^2(\partial\Omega_p)$ and suppose that $\overline{L}(f) = 0$ in the sense of distributions. Then f can be extended to a function holomorphic in Ω_p, in fact belonging to $H^2(\partial\Omega_p)$, whose boundary value coincides with the original function.*

Proof. We define a function F on $\mathbb{C} \times \mathbb{R}_+$ by formula (2),

$$F(z,\tau) = \int_{\mathbb{R}} f(z,t+ip(z))e^{-2\pi i\tau t}e^{2\pi\tau p(z)}\,dt,$$

and conclude again from (4) that F is entire with respect to z and belongs to E_p and

$$h(z,w) = \int_0^\infty F(z,\tau)e^{2\pi i\tau w}\,d\tau,$$

for $(z, w) \in \Omega_p$ yields the desired extension. □

Remarks. (a) If f belongs to $H^2(\partial\Omega_p)$, then its boundary value f_0 satisfies $\overline{L}(f_0) = 0$ in the sense of distributions. Hence $H^2(\partial\Omega_p)$ can be identified with the space of all functions $f \in L^2(\partial\Omega_p)$ satisfying $\overline{L}(f) = 0$ in the sense of distributions.

The Szegö projection

$$S\colon L^2(\partial\Omega_p) \longrightarrow H^2(\partial\Omega_p)$$

can therefore be viewed as the projection onto the kernel of the tangential Cauchy-Riemann operator.

(b) If Ω_p is a domain in $\mathbb{C}^{n+1}$ of the form

$$\Omega_p = \{(z, w) \colon z \in \mathbb{C}^n \ , \ w \in \mathbb{C} \ , \ \Im w > p(z)\},$$

then the tangential Cauchy-Riemann operator has the form

$$\overline{\partial}_b(f) = \sum_{j=1}^{n} \overline{L}_j(f)\, d\overline{z}_j ,$$

where

$$\overline{L}_j = \frac{\partial}{\partial \overline{z}_j} - i \frac{\partial p}{\partial \overline{z}_j} \frac{\partial}{\partial t} .$$

The same reasoning as in Theorem 1 applies now to these domains Ω_p in $\mathbb{C}^{n+1}$.

(c) The above extension of CR-distributions to the whole domain Ω_p seems to be new even in the case of the Siegel upper half space ($p(z) = |z|^2$); see for instance [S, p. 642], where only a local extension property is mentioned. Theorem 1 gives the global extension in one step.

4. Bergman and Szegö kernels on certain unbounded domains

Now we suppose that the weight function $p \colon \mathbb{C}^n \longrightarrow \mathbb{R}_+$ is (pluri)subharmonic and with a growth behavior guaranteeing that the corresponding Bergman spaces H_τ of entire functions are nontrivial, where H_τ $(\tau > 0)$ consists of all entire functions $\phi \colon \mathbb{C}^n \longrightarrow \mathbb{C}$ such that

$$\int_{\mathbb{C}^n} |\phi(z)|^2 e^{-4\pi\tau p(z)}\, d\lambda(z) < \infty.$$

The Bergman kernels of these spaces are denoted by $K_\tau(z, w)$. A result on parameter families of Bergman kernels of pseudoconvex domains of Diederich and Ohsawa [DO] can be adapted to our case, showing that for fixed (z, w) the function $\tau \mapsto K_\tau(z, w)$ is continuous. Then we can apply a method from [Has1] to obtain the following formulas for the Szegö kernel S of the Hardy space $H^2(\partial\Omega_p)$ and the Bergman kernel B of the domain Ω_p:

THEOREM 2. *(a) If $\partial\Omega_p$ is identified with $\mathbb{C}^n \times \mathbb{R}$, the Szegö kernel on $\partial\Omega_p \times \partial\Omega_p$ has the form*

$$S((z', t), (w', s)) = \int_0^\infty K_\tau(z', w') e^{-2\tau(p(z')+p(w'))} e^{-2\pi i \tau(s-t)}\, d\tau,$$

where $z', w' \in \mathbb{C}^n$, $s, t \in \mathbb{R}$.

(*b*) *For* (z', z), $(w', w) \in \Omega_p$ $(z', w' \in \mathbb{C}^n;\ z, w \in \mathbb{C})$ *the Szegö kernel can be expressed in the form*

$$S((z', z), (w', w)) = \int_0^\infty K_\tau(z', w') e^{-2\pi i \tau(\overline{w}-z)}\, d\tau.$$

(*c*) *The Bergman kernel of* Ω_p *is written as*

$$B((z', z), (w', w)) = 4\pi \int_0^\infty \tau K_\tau(z', w') e^{-2\pi i \tau(\overline{w}-z)}\, d\tau.$$

Proof. (a) and (b) follow directly from Theorem 1 in [Has1] since the weight function satisfies the assumptions there. Special cases can be found in [GS] and [FH1]. The proof of Theorem 1 in [Has1] again uses the operator identity

$$\mathcal{F}\overline{L}\mathcal{F}^{-1} = M^{-1}\frac{\partial}{\partial \overline{z}} M.$$

For (c) one has to recall the formula

$$B((z', z), (w', w)) = 2i\frac{\partial S}{\partial \overline{w}}((z', z), (w', w))$$

from [NRSW2]. □

5. Boundary limits of the Bergman kernel

In this part we compute or at least estimate the Bergman kernel B of certain unbounded domains Ω_p at the point $((0, i), (0, i))$, $0 \in \mathbb{C}^n$, of the diagonal. Applying a theorem of Boas, Straube and Yu [BSY] this determines the boundary behavior of the Bergman kernel K_Ω on the diagonal of a bounded pseudoconvex domain Ω in $\mathbb{C}^{n+1}$ that is h-extendible at the boundary point Q with multiple type $(m_0, m_1, \dots, m_n)$ and that has Ω_p as local model at Q:

$$\lim_{\substack{z \to Q \\ z \in \Gamma}} K_\Omega(z)\, d(z)^{\sum_{j=0}^{n} 2/m_j} = B((0, i), (0, i)),$$

here Γ is a nontangential cone in Ω with vertex at Q and $d(z)$ is the distance from z to the boundary of Ω.

THEOREM 3.

$$B((0, i), (0, i)) = 4\pi \int_0^\infty \tau K_\tau(0, 0) e^{-4\pi\tau}\, d\tau,$$

where K_τ *denotes the Bergman kernel of the Hilbert space* H_τ *of entire functions*

ϕ: $\mathbb{C}^n \longrightarrow \mathbb{C}$, *such that*

$$\int_{\mathbb{C}^n} |\phi(z)|^2 e^{-4\pi\tau p(z)}\, d\lambda(z)\,, \ \tau > 0.$$

See Theorem 2(c).

Finally we compute $B((0, i), (0, i))$ for two classes of examples. Here we concentrate on the so-called decoupled domains (see [McN2]).

(a) $p(z_1, \ldots, z_n) = \sum_{j=1}^n |z_j|^{\alpha_j}$, where $\alpha_j > 1$, $j = 1, \ldots, n$.

In this (radial) case the Bergman kernel of the weighted spaces of entire functions H_τ can be represented by the orthonormalized system of the monomials and a computation using the formula

$$\int_0^\infty \tau^m \exp(-c\tau^q)\, d\tau = \frac{\Gamma\left(\frac{m+1}{q}\right)}{q c^{(m+1)/q}}\,, \ c, m, q > 0, \tag{5}$$

(see [GR]) proves the following:

THEOREM 4. *If* $p(z_1, \ldots, z_n) = \sum_{j=1}^n |z_j|^{\alpha_j}$, *where* $\alpha_j > 1$, $j = 1, \ldots, n$, *then*

$$B((0, i), (0, i)) = \frac{\Gamma\left(2 + \frac{2}{\alpha_1} + \cdots + \frac{2}{\alpha_n}\right)\prod_{j=1}^n \alpha_j}{2^{n+4}\pi^{n+2}\prod_{j=1}^n \Gamma\left(\frac{2}{\alpha_j}\right)}.$$

(b) $p(z_1, \ldots, z_n) = \sum_{j=1}^n \frac{|x_j|^{\alpha_j}}{\alpha_j}$, where $z_j = x_j + iy_j$, $\alpha_j > 1$, $j = 1, \ldots, n$.

In this (nonradial) case the computation of the Bergman kernel is more complicated. We use a formula from [Has1] for the Bergman kernels K_τ of the weighted spaces of entire functions H_τ: for $r = (r_1, \ldots, r_n)$, $r_j \in \mathbb{R}$, we write $p(r_1, \ldots, r_n) = \sum_{j=1}^n p_j(r_j)$ and $p_j(r_j) = \frac{|r_j|^{\alpha_j}}{\alpha_j}$, then

$$K_\tau(z, w) = \prod_{j=1}^n \int_{\mathbb{R}} \frac{\exp(2\pi\eta_j(z_j + \overline{w}_j))}{\int_{\mathbb{R}} \exp(4\pi(r_j\eta_j - \tau p_j(r_j)))\, dr_j}\, d\eta_j.$$

In order to apply Theorem 3 we have to compute $K_\tau(0, 0)$. We consider a single factor in the above formula for $K_\tau(z, w)$ and omit the index j. Let

$$I(\eta, \tau) = \left[\int_{\mathbb{R}} \exp(4\pi(\eta r - \tau p(r))\, dr\right]^{-1}.$$

A straightforward computation shows that

$$\int_{\mathbb{R}} I(\eta, \tau)\, d\eta = (4\pi)^{1/\alpha - 1/\alpha'}\tau^{2/\alpha} \int_{\mathbb{R}} \frac{d\eta}{\int_{\mathbb{R}} \exp(r\eta - p(r))\, dr},$$

where $1/\alpha + 1/\alpha' = 1$. We set $p^*(s) = |s|^{\alpha'}/\alpha'$, which is the Young conjugate of p. In the following we estimate

$$\int_{\mathbb{R}} \exp(r\eta - p(r))\, dr$$

from below and from above in terms of the Young conjugate: for this purpose let $\eta > 0$ and $\lambda > 1$; then from Young's inequality we get

$$r\eta - p(r) \leq r\eta - \lambda r\eta + p^*(\lambda\eta)$$

and hence

$$\int_0^\infty \exp(r\eta - p(r))\, dr \leq \frac{\exp(p^*(\lambda\eta))}{(\lambda - 1)\eta}.$$

Further we have

$$\int_0^\infty \exp(-r\eta - p(r))\, dr \leq \frac{1}{\eta},$$

and finally

$$\int_{\mathbb{R}} \exp(r\eta - p(r))\, dr \leq \frac{(\lambda - 1) + \exp(p^*(\lambda\eta))}{(\lambda - 1)\eta},$$

for $\eta > 0$. By similar argument we handle the case $\eta < 0$ and get

$$\int_{\mathbb{R}} \exp(r\eta - p(r))\, dr \leq \frac{(\lambda - 1) + \exp(p^*(\lambda\eta))}{(\lambda - 1)|\eta|}, \tag{6}$$

for $\eta \neq 0$ and $\lambda > 1$.

Next we claim that

$$\int_{\mathbb{R}} \exp(r\eta - p(r))\, dr \geq \exp(-\chi(\mu) p^*(\mu\eta)), \tag{7}$$

where $0 < \mu < 1$ and

$$\chi(\mu) = \beta\left[(\alpha\beta)^{1/(\alpha-1)} - 1\right]^{-(\alpha-1)}, \quad \beta = 1 - \frac{\mu^{\alpha'}}{\alpha'}.$$

In order to show (7) we first note that

$$p^*(\eta) = \max_{x \geq 0}\{x|\eta| - p(x)\} = \frac{|\eta|^{\alpha'}}{\alpha'},$$

and that the maximum is attained at the point

$$\kappa(\eta) = |\eta|^{1/(\alpha-1)}.$$

Hence for $\eta \geq 0$ it follows that

$$\int_{\mathbb{R}} \exp(r\eta - p(r))\,dr \geq \int_0^\infty \exp(r\eta - p(r))\,dr \geq \exp[\eta(\kappa(\eta)+1) - p(\kappa(\eta)+1)].$$

The last expression can be written in the form

$$\eta(\kappa(\eta)+1) - p(\kappa(\eta)+1) = \eta^{\alpha/(\alpha-1)} + \eta - \frac{1}{\alpha}(\eta^{1/(\alpha-1)}+1)^\alpha.$$

On the other hand, for $0 < \mu < 1$ we have

$$\mu\eta\kappa(\mu\eta) - p(\kappa(\mu\eta)) = \frac{1}{\alpha'}(\mu\eta)^{\alpha'} = p^*(\mu\eta)$$

and we show that

$$\eta^{\alpha'} + \eta - \frac{1}{\alpha}(\eta^{1/(\alpha-1)}+1)^\alpha \geq p^*(\mu\eta) - \chi(\mu),$$

which gives inequality (7). For this aim we have to prove that

$$\frac{1}{\alpha}(\eta^{1/(\alpha-1)}+1)^\alpha - \left(1 - \frac{1}{\alpha'}\mu^{\alpha'}\right)\eta^{\alpha'} - \eta \leq \chi(\mu).$$

We set $\beta = 1 - \frac{1}{\alpha'}\mu^{\alpha'}$; then $\beta > \frac{1}{\alpha}$ and we have

$$\frac{1}{\alpha}(\eta^{1/(\alpha-1)}+1)^\alpha - \beta\eta^{\alpha'} - \eta \leq \frac{1}{\alpha}(\eta^{1/(\alpha-1)}+1)^\alpha - \beta\eta^{\alpha'}.$$

A straightforward computation shows that the expression $\frac{1}{\alpha}(\eta^{1/(\alpha-1)}+1)^\alpha - \beta\eta^{\alpha'}$ attains its maximum when $\eta = [(\alpha\beta)^{1/(\alpha-1)} - 1]^{-(\alpha-1)}$ and therefore we obtain

$$\frac{1}{\alpha}(\eta^{1/(\alpha-1)}+1)^\alpha - \beta\eta^{\alpha'} \leq \beta[(\alpha\beta)^{1/(\alpha-1)} - 1]^{-(\alpha-1)},$$

which proves (7).

For $\eta < 0$ we can proceed in an analogous way arriving with the same inequality (7).

From (6) and (7), for $0 < \mu < 1$ and $\lambda > 1$ we get

$$(8)\quad \int_{\mathbb{R}} \frac{(\lambda-1)|\eta|}{(\lambda-1)+e^{p^*(\lambda\eta)}}\,d\eta \leq \int_{\mathbb{R}} \frac{d\eta}{\int_{\mathbb{R}}\exp(r\eta - p(r))\,dr} \leq e^{\chi(\mu)}\int_{\mathbb{R}} e^{-p^*(\mu\eta)}\,d\eta.$$

For the left term in inequality (8) we mention that

$$\frac{(\lambda-1)|\eta|}{\lambda}e^{-p^*(\lambda\eta)} \leq \frac{(\lambda-1)|\eta|}{(\lambda-1)+e^{p^*(\lambda\eta)}},$$

for each $\eta \in \mathbb{R}$. Using formula (5) the integration of the term on the left side yields

$$\frac{(\lambda-1)}{\lambda}\int_{\mathbb{R}}|\eta|e^{-p^*(\lambda\eta)}\,d\eta = \frac{2(\lambda-1)}{\lambda^3}\alpha'^{-1+2/\alpha'}\Gamma\left(\frac{2}{\alpha'}\right),$$

for each $\lambda > 1$ The maximal value of $2(\lambda-1)/\lambda^3$ is $8/27$.

The right term in inequality (8) is equal to

$$\frac{2e^{\chi(\mu)}}{\mu}\alpha'^{-1+1/\alpha'}\Gamma\left(\frac{1}{\alpha'}\right).$$

It is easily seen that $\frac{2e^{\chi(\mu)}}{\mu}$ tends to infinity as μ tends to 0 or to 1

Now let $\gamma_j = \min_{0<\mu<1}\{2e^{\chi_j(\mu_j)}/\mu_j\}$, where

$$\chi_j(\mu_j) = \beta_j\left[(\alpha_j\beta_j)^{1/(\alpha_j-1)} - 1\right]^{-(\alpha_j-1)}, \quad \beta_j = 1 - \frac{\mu_j^{\alpha_j'}}{\alpha_j'}.$$

Then

$$\begin{aligned}(4\pi)^{1/\alpha_j - 1/\alpha_j'}\tau^{2/\alpha_j}\frac{8}{27}\alpha_j'^{-1+2/\alpha_j'}\Gamma\left(\frac{2}{\alpha_j'}\right)\\ \leq \int_{\mathbb{R}} I_j(\eta_j,\tau)\,t\eta\\ \leq (4\pi)^{1/\alpha_j-1/\alpha_j'}\tau^{2/\alpha_j}\gamma_j\alpha_j'^{-1+1/\alpha_j'}\Gamma\left(\frac{1}{\alpha_j'}\right),\end{aligned}$$

which implies

$$\begin{aligned}\prod_{j=1}^{n}\left[(4\pi)^{1/\alpha_j-1/\alpha_j'}\tau^{2/\alpha_j}\frac{8}{27}\alpha_j'^{-1+2/\alpha_j'}\Gamma\left(\frac{2}{\alpha_j'}\right)\right]\\ \leq K_\tau(0,0)\\ \leq \prod_{j=1}^{n}\left[(4\pi)^{1/\alpha_j-1/\alpha_j'}\tau^{2/\alpha_j}\gamma_j\alpha_j'^{-1+1/\alpha_j'}\Gamma\left(\frac{1}{\alpha_j'}\right)\right].\end{aligned}$$

In the last step we integrate over τ and, from Theorem 3, we obtain:

THEOREM 5. *If $p(z_1,\dots,z_n) = \sum_{j=1}^{n}\frac{|x_j|^{\alpha_j}}{\alpha_j}$, where $z_j = x_j + iy_j$, $\alpha_j > 1$, $j = 1,\dots,n$, then the Bergman kernel in the point $((0,i),(0,i))$ can be estimated* *in the form*

$$
\begin{aligned}
&(4\pi)^{-n-1}\left(\frac{8}{27}\right)^n \Gamma\left(2+\frac{2}{\alpha_1}+\cdots+\frac{2}{\alpha_n}\right)\prod_{j=1}^{n}\left[\alpha_j'^{-1+2/\alpha_j'}\Gamma\left(\frac{2}{\alpha_j'}\right)\right] \\
&\quad \le B((0,i),(0,i)) \\
&\quad \le (4\pi)^{-n-1}\Gamma\left(2+\frac{2}{\alpha_1}+\cdots+\frac{2}{\alpha_n}\right)\prod_{j=1}^{n}\left[\gamma_j\alpha_j'^{-1+1/\alpha_j'}\Gamma\left(\frac{1}{\alpha_j'}\right)\right],
\end{aligned}
$$

with $1/\alpha_j + 1/\alpha_j' = 1$ *and* $\gamma_j = \min_{0<\mu<1}\{2e^{\chi_j(\mu_j)}/\mu_j\}$, *where*

$$
\chi_j(\mu_j) = \beta_j\left[(\alpha_j\beta_j)^{1/(\alpha_j-1)} - 1\right]^{-(\alpha_j-1)}, \quad \beta_j = 1 - \frac{\mu_j^{\alpha_j'}}{\alpha_j'}.
$$

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