

## BIFURCATION SETS OF FUNCTIONS DEFINABLE IN O-MINIMAL STRUCTURES

TA LÊ LOI AND ALEXANDRU ZAHARIA

### 1. Introduction

1.1. Let  $g: X \rightarrow Y$  be a  $C^p$  map between two manifolds, for some  $p \geq 1$ , where we allow  $p = \infty$ . Let  $1 \leq q \leq p$ . We say that  $y_0 \in Y$  is a  $C^q$  typical value of  $g$  if  $y_0$  is a regular value of  $g$  and there exists an open neighborhood  $U$  of  $y_0$  in  $Y$ , such that the restriction  $g: g^{-1}(U) \rightarrow U$  is a  $C^q$  trivial fibration, i.e., such that there exists a  $C^q$  diffeomorphism  $h: g^{-1}(y_0) \times U \rightarrow g^{-1}(U)$  satisfying  $g(h(y, u)) = u$ , for all  $(y, u) \in g^{-1}(y_0) \times U$ . If  $y_0 \in Y$  is not a  $C^q$  typical value of  $g$ , then  $y_0$  is called a  $C^q$  atypical value of  $g$ . We denote by  $\text{Bif}^q(g)$  the  $C^q$  bifurcation set of  $g$ , i.e., the set of  $C^q$  atypical values of  $g$ . In the case of a complex polynomial function  $f: \mathbf{C}^n \rightarrow \mathbf{C}$  it follows from [15] that  $\text{Bif}^\infty(f)$  is a finite set; see also [16], [12]. In [1] it is proved that the  $C^\infty$  bifurcation sets of real polynomial functions are also finite.

The aim of this note is to show that, for  $U \subseteq \mathbf{R}^n$  open, the bifurcation sets of differentiable functions  $f: U \rightarrow \mathbf{R}$  definable in an o-minimal expansion of the real field (see Definition 1.2) are finite. We proceed as in [10] and we give an upper bound for the cardinality of the bifurcation sets. We also present a new proof for the fact that the bifurcation set of a complex polynomial function of  $n$  variables is finite. Before stating our results, we give some definitions.

1.2 DEFINITION. An o-minimal structure on the real field  $(\mathbf{R}, +, \cdot)$  is a sequence  $\mathcal{D} = (\mathcal{D}_n)_{n \in \mathbf{N}}$  such that, for each  $n \in \mathbf{N}$ :

- (D1)  $\mathcal{D}_n$  is a boolean algebra of subsets of  $\mathbf{R}^n$ , i.e.,  $\mathcal{D}_n$  is closed under taking complements and finite unions.
- (D2) If  $A \in \mathcal{D}_n$ , then  $A \times \mathbf{R}$  and  $\mathbf{R} \times A \in \mathcal{D}_{n+1}$ .
- (D3) If  $A \in \mathcal{D}_{n+1}$ , then  $\pi(A) \in \mathcal{D}_n$ , where  $\pi: \mathbf{R}^{n+1} \rightarrow \mathbf{R}^n$  is the projection on the first  $n$  coordinates.
- (D4)  $\mathcal{D}_n$  contains  $\{x \in \mathbf{R}^n \mid P(x) = 0\}$  for every polynomial  $P \in \mathbf{R}[X_1, \dots, X_n]$ .
- (D5) Each set belonging to  $\mathcal{D}_1$  is a finite union of intervals and points. (This property is called o-minimality.)

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A subset  $A \subseteq \mathbf{R}^n$  will be called *definable* (in the structure  $\mathcal{D}$ ) if  $A \in \mathcal{D}_n$ . A map  $f: A \rightarrow \mathbf{R}^k$  will be called *definable* if the graph of  $f$  is a definable set.

Sets definable in o-minimal structures have many nice properties similar to those of semi-algebraic sets. We mention here only the fact that any definable set has only finitely many connected components, and each of them is path connected and definable. For more details, see for instance [3] and [4].

1.3. Let  $U \subseteq \mathbf{R}^n$  be an open set and let us fix a  $C^1$  function  $\rho: U \rightarrow \mathbf{R}$  such that

$$\text{for any } r \in \mathbf{R}, \text{ the “ball” } \mathbf{B}_r^\rho := \{ u \in U \mid \rho(u) \leq r \} \text{ is compact.} \tag{1}$$

We also let  $\mathbf{S}_r^\rho := \{ x \in U \mid \rho(x) = r \}$  denote the corresponding “sphere”. For a  $C^1$  function  $g: U \rightarrow \mathbf{R}$ , we let

$$M(g; \rho) := \{ x \in U \mid \exists \lambda \in \mathbf{R}, \text{ grad } g(x) = \lambda \text{ grad } \rho(x) \}.$$

Note that  $u \in M(g; \rho)$  if and only if either  $u$  is a critical point of  $g$ , or  $u$  is not a critical point of  $g$  (hence  $\text{grad } \rho(u) \neq 0$ ) and the level set  $g^{-1}(g(u))$  is a submanifold of  $\mathbf{R}^n$  near  $u$ , which is not transversal to the “sphere”  $\mathbf{S}_r^\rho$  at  $u$ .

For a sequence  $\{y^k\} \subseteq M(g; \rho)$  we consider the conditions

$$\lim_{k \rightarrow \infty} \rho(y^k) = \infty \text{ and } \lim_{k \rightarrow \infty} g(y^k) = c. \tag{2}$$

We denote by  $\Sigma_g$  the set of critical values of  $g$ , and we put

$$S_{g;\rho} := \{ c \in \mathbf{R} \mid \text{there exists a sequence } \{y^k\} \subseteq M(g; \rho) \text{ such that (2) is satisfied} \}.$$

If  $\rho(x) = \|x\|^2$  for all  $x \in \mathbf{R}^n$ , we write  $S_g$  instead of  $S_{g;\rho}$ . The motivation for considering the set  $S_{g;\rho}$  is given by the following result, which is similar to Theorem 1 in [10].

**PROPOSITION.** *Let  $U \subseteq \mathbf{R}^n$  be open and let  $g, \rho: U \rightarrow \mathbf{R}$  be  $C^{p+1}$  functions for some  $p \in \mathbf{N} \cup \{\infty\}$ . Assume, moreover, that  $\Sigma_\rho$  is bounded. Then for any open interval  $J \subseteq g(\mathbf{R}^n) \setminus (\Sigma_g \cup S_{g;\rho})$ , the restriction*

$$g: g^{-1}(J) \rightarrow J$$

*is a  $C^p$  trivial fibration.*

1.4. Let  $\mathcal{D}$  be a fixed, but arbitrary, o-minimal structure on  $(\mathbf{R}, +, \cdot)$ . “Definable” will mean definable in  $\mathcal{D}$ .

**PROPOSITION.** *Let  $U \subseteq \mathbf{R}^n$  be an open definable set. Then for any  $p \in \mathbf{N}$  there exists  $\rho: U \rightarrow \mathbf{R}$  definable and  $C^{p+1}$  such that condition (1) is satisfied.*

*Remark.* Sometimes it is possible to prove this proposition for  $p = \infty$ ; for example, when  $U = \mathbf{R}^n$  one can take  $\rho(x) = x_1^2 + \dots + x_n^2$ . We do not know if the conclusion of Proposition 1.4 is true for  $p = \infty$  and for any definable open set  $U \subseteq \mathbf{R}^n$ .

1.5 THEOREM. *Let  $U \subseteq \mathbf{R}^n$  be open and let  $\rho: U \rightarrow \mathbf{R}$  definable and  $C^1$  such that condition (1) is satisfied. If  $f: U \rightarrow \mathbf{R}$  is definable and  $C^1$ , then  $\Sigma_f$  and  $S_{f,\rho}$  are finite.*

As a consequence of Propositions 1.3 and 1.4, and this theorem, we have:

COROLLARY. *Let  $U \subseteq \mathbf{R}^n$  be open and let  $f: U \rightarrow \mathbf{R}$  be definable and  $C^{p+1}$ . Let  $q \in \mathbf{N} \cup \{\infty\}$  be such that  $q \leq p$ . If  $q = \infty$ , assume moreover that the conclusion of Proposition 1.4 holds for  $q$ . Then  $\text{Bif}^q(f)$  is finite.*

1.6. We will denote the cardinality of a set  $X$  by  $\#(X)$ . The following proposition provides us an upperbound for  $\#(S_{f,\rho})$ .

PROPOSITION. *Let  $U \subseteq \mathbf{R}^n$  be open and let  $f, \rho: U \rightarrow \mathbf{R}$  be definable and  $C^1$ . Suppose that condition (1) is satisfied by  $\rho$ . Then there exists  $R \geq 0$  such that  $\#(S_{f,\rho})$  is less than or equal to the number of connected components of  $M(f) \cap \mathbf{S}_r^\rho$ , for all  $r > R$ .*

1.7. In the complex case, if  $U \subseteq \mathbf{C}^n$  is open, the gradient of a holomorphic function  $f: U \rightarrow \mathbf{C}$  is defined to be

$$\text{grad } f(x) := \left( \overline{\frac{\partial f}{\partial x_1}}(x), \dots, \overline{\frac{\partial f}{\partial x_n}}(x) \right),$$

where  $x = (x_1, \dots, x_n)$  and the bar denotes conjugation.

For a polynomial  $f: \mathbf{C}^n \rightarrow \mathbf{C}$ , we take  $\rho(x) := \|x\|^2$  and we define

$$M(f) := \{ x \in \mathbf{C}^n \mid \exists \lambda \in \mathbf{C}, \text{ grad } f(x) = \lambda x \},$$

$S_f := \{ c \in \mathbf{C} \mid \text{there exists a sequence } \{y^k\} \subseteq M(f) \text{ such that (2) is satisfied} \}$ .

By Theorem 1 in [10],  $\text{Bif}^\infty(f) \subseteq \Sigma_f \cup S_f$ . The next result shows us that  $S_f$  is finite, hence  $\text{Bif}^\infty(f)$  is also finite.

THEOREM. *Let  $f: \mathbf{C}^n \rightarrow \mathbf{C}$  be a complex polynomial function. Then  $S_f$  is finite.*

The next section contains the proofs. Several examples and related remarks are given in the last section.

**2. Proofs**

2.1. We keep the notations from Section 1. We denote by  $\langle a, b \rangle$  the Euclidean scalar product of  $a, b \in \mathbf{R}^n$ . We skip the proof of Proposition 1.3, since it is a routine modification of the proof of Theorem 1 in [10], obtained by using  $\text{grad } \rho(x)$ ,  $\mathbf{B}_r^\rho$ , and  $\mathbf{S}_r^\rho$  instead of vector field  $z$ , usual ball and sphere.

*Proof of Proposition 1.4.* Let  $\varphi: \mathbf{R}^n \rightarrow (-1, 1)^n$  be a definable  $C^\infty$  diffeomorphism. Then  $\varphi(U)$  is a bounded definable open set in  $\mathbf{R}^n$ . By Theorem 4.22 in [4], there exists a definable  $C^{p+1}$  function  $h: \mathbf{R}^n \rightarrow \mathbf{R}$  such that  $h^{-1}(0) = \mathbf{R}^n \setminus \varphi(U)$ . Moreover, by squaring if necessary, one can suppose that  $h$  is non-negative. Then  $\rho: U \rightarrow \mathbf{R}$ , defined by  $\rho(x) := \frac{1}{h(\varphi(x))}$  satisfies the conclusion of Proposition 1.4. Note that if  $h$  is  $C^\infty$ , then  $\rho$  is also  $C^\infty$ .  $\square$

2.2 LEMMA. Let  $U \subseteq \mathbf{R}^n$  be open and let  $f, \rho: U \rightarrow \mathbf{R}$  be definable and  $C^1$ . Suppose that condition (1) is satisfied by  $\rho$ . Let  $c \in \mathbf{R}$  and suppose that  $\rho(f^{-1}(c) \cap M(f; \rho))$  is unbounded. Then  $c \in \Sigma_f$ .

*Proof.* By hypothesis and using Definable Choice and Monotonicity Theorems from [4], there exist a definable  $C^1$  path  $\gamma: (a, \infty) \rightarrow \mathbf{R}^n$  and definable  $\lambda: (a, \infty) \rightarrow \mathbf{R}$ , such that

$$f \circ \gamma = c, \quad (\text{grad } f) \circ \gamma = \lambda \cdot (\text{grad } \rho) \circ \gamma \quad \text{and} \quad \frac{d(\rho \circ \gamma)}{dt} > 0.$$

Then

$$0 = (f \circ \gamma)' = \langle (\text{grad } f) \circ \gamma, \gamma' \rangle = \lambda \cdot \langle (\text{grad } \rho) \circ \gamma, \gamma' \rangle = \lambda \cdot \frac{d(\rho \circ \gamma)}{dt}.$$

So  $\lambda = 0$ , every point of  $\gamma$  is a critical point, and  $c \in \Sigma_f$ .  $\square$

2.3 Proof of Theorem 1.5. Of course,  $\Sigma_f$  is finite by Cell Decomposition Theorem 4.2 in [4]. So it suffices to show that  $S_{f;\rho} \setminus \Sigma_f$  is finite. Note that

$$A := \{(c, y, \lambda, t) \in \mathbf{R} \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} \mid c = f(y), \text{grad } f(y) = \lambda \text{grad } \rho(y), t\rho(y) > 1\}$$

is a definable set. We have

$$c \in S_{f;\rho} \iff \forall \varepsilon > 0 \forall \delta > 0 \exists (c', y, \lambda, t) \in A, |t| < \varepsilon, |c' - c| < \delta.$$

Using the interpretation of the logical symbols in terms of operations on sets, one can see that  $S_{f;\rho}$  is definable. Hence  $S_{f;\rho}$  is a finite union of points and intervals.

Suppose, to the contrary, that  $S_{f;\rho} \setminus \Sigma_f$  contains an open interval  $I$ . Using Lemma 2.2, define  $\eta: I \rightarrow \mathbf{R}$  by  $\eta(r) := \sup\{ \rho(x) \mid x \in f^{-1}(r) \cap M(f; \rho) \}$ . Then  $\eta$

is definable. Monotonicity Theorem 4.1 in [4], applied to  $\eta$ , provides us a nonempty open interval  $J \subseteq I$  such that  $\rho(f^{-1}(J) \cap M(f; \rho))$  is bounded. But given  $c \in J$ , there is a definable continuous path  $\gamma: (a, \infty) \rightarrow \mathbf{R}^n$ , whose image is contained in  $M(f; \rho)$ , with  $\rho(\gamma(t)) \rightarrow +\infty$  and  $f(\gamma(t)) \rightarrow c$  as  $t \rightarrow +\infty$ . So there exists  $R > 0$  such that if  $t > R$ , then  $f(\gamma(t)) \in J$ . This contradicts the fact that  $\rho(f^{-1}(J) \cap M(f; \rho))$  is bounded.  $\square$

**2.4 Proof of Proposition 1.6.** Since  $S_{f;\rho}$  is finite, we can choose  $\delta > 0$  such that  $\delta < |s - s'|/2$  for every  $s, s' \in S_{f;\rho}$  with  $s \neq s'$ . For  $r > 0$  and  $s \in S_{f;\rho}$ , put

$$A(r, s) := \{ x \in M(f) \cap \mathbf{S}_r^\rho \mid |f(x) - s| < \delta \}.$$

By Definable Choice Theorem 4.5 in [4], we get  $R > 0$  such that  $A(r, s) \neq \emptyset$  for all  $r > R$  and  $s \in S_{f;\rho}$ ; moreover,  $s \neq s'$  implies that  $A(r, s)$  and  $A(r, s')$  are disjoint.  $\square$

**2.5 LEMMA.** Let  $U \subseteq \mathbf{R}^k$  be open and let  $F: U \times (0, \varepsilon) \rightarrow \mathbf{R}^m$  be a  $C^1$  definable map. Suppose that there exists a constant  $K > 0$  such that  $\|F(s, t)\| \leq K$  for all  $(s, t) \in U \times (0, \varepsilon)$ . Then there exist a definable set  $V$ , closed in  $U$  and with  $\dim V < k$ , and continuous definable functions  $\kappa, \tau: U \setminus V \rightarrow (0, \infty)$ , such that for all  $s \in U \setminus V$  and  $t \in (0, \tau(s))$ ,

$$\left\| \frac{\partial F}{\partial s}(s, t) \right\| \leq \kappa(s).$$

For the proof of this lemma, see [6].

**2.6 LEMMA.** Let  $F: (0, \varepsilon) \rightarrow \mathbf{R}$  be a  $C^1$  definable function with  $\lim_{t \rightarrow 0} F(t) = 0$ . Then  $\lim_{t \rightarrow 0} tF'(t) = 0$ .

*Proof.* By Monotonicity Theorem 4.1 in [4],  $F$  is either constant, or strictly monotone near 0. So, it is sufficient to consider the case when  $F$  and  $F'$  are strictly monotone on  $(0, \varepsilon)$  and  $F > 0$ . Then  $F' > 0$ . By Mean Value Theorem, we have  $F(t) = F'(\zeta(t))t$  for  $\zeta(t) \in (0, t)$ . It is easy to see that  $\zeta: (0, \varepsilon) \rightarrow (0, \varepsilon)$  is definable and  $\lim_{t \rightarrow 0} \zeta(t) = 0$ . Therefore,

$$0 \leq \lim_{\zeta \rightarrow 0} \zeta F'(\zeta) = \lim_{t \rightarrow 0} \zeta(t)F'(\zeta(t)) \leq \lim_{t \rightarrow 0} tF'(t) = 0. \quad \square$$

**2.7 Proof of Theorem 1.7.** In this proof we will denote by  $\langle a, b \rangle$  the Hermitian product of  $a, b \in \mathbf{C}^n$ .

Since the class of all semi-algebraic sets is an o-minimal structure on  $(\mathbf{R}, +, \cdot)$ , it follows, as in the proof of Theorem 1.5, that  $S_f$  is definable, i.e., a semi-algebraic subset of  $\mathbf{R}^2 \cong \mathbf{C}$ . We will show that  $\dim S_f \leq 0$ .

If  $\dim S_f \geq 1$ , then there exist an interval  $J \subseteq \mathbf{R}$  and a definable and  $C^1$  curve  $\ell: J \rightarrow S_f$  whose derivative satisfies  $\ell'(s) \neq 0$ , for all  $s \in J$ . Therefore, for all  $s \in J$  and  $t > 0$  there exist  $p(s, t) \in \mathbf{C}^n$  and  $\lambda(s, t) \in \mathbf{C}$  such that

$$|f(p(s, t)) - \ell(s)| < t \tag{3}$$

$$\text{grad } f(p(s, t)) = \lambda(s, t)p(s, t) \tag{4}$$

and

$$\|p(s, t)\| > \frac{1}{t}.$$

By Definable Choice [4] and [11], after eventually shrinking the interval  $J$ , we get  $\varepsilon > 0$  such that

$$\begin{aligned} p(s, t) &= a(s)t^\alpha + a_1(s, t)t^{\alpha_1} \\ f(p(s, t)) &= \ell(s) + b(s)t^\beta + b_1(s, t)t^{\beta_1} \\ \lambda(s, t) &= c(s)t^\gamma + c_1(s, t)t^{\gamma_1} \end{aligned}$$

where  $\alpha < 0$ ,  $\alpha < \alpha_1$ ,  $\beta < \beta_1$ ,  $\gamma < \gamma_1$ , and  $a: J \rightarrow \mathbf{C}^n$  is  $C^1$  with  $a(s) \neq 0$ ,  $\forall s \in J$ ;  $b, c: J \rightarrow \mathbf{C}$  are  $C^1$  with  $b(s), c(s) \neq 0$ ,  $\forall s \in J$ ;  $a_1: J \times (0, \varepsilon) \rightarrow \mathbf{C}^n$  is  $C^1$  and bounded, and  $b_1, c_1: J \times (0, \varepsilon) \rightarrow \mathbf{C}$  are  $C^1$  and bounded. Moreover, using Lemma 2.5, after shrinking  $J$  and reducing  $\varepsilon$ , if necessary, we may assume that  $\frac{\partial a_1}{\partial s}$ ,  $\frac{\partial b_1}{\partial s}$  and  $\frac{\partial c_1}{\partial s}$  are bounded on  $J \times (0, \varepsilon)$ . By (3), we have  $\beta \geq 1$ . Therefore

$$\frac{\partial f(p(s, t))}{\partial s} \rightarrow \ell'(s) \text{ when } t \rightarrow 0.$$

Relation (4) implies

$$\frac{\partial f(p(s, t))}{\partial s} = \left\langle \frac{\partial p(s, t)}{\partial s}, \text{grad } f(p(s, t)) \right\rangle = \overline{\lambda(s, t)} \left\langle \frac{\partial p(s, t)}{\partial s}, p(s, t) \right\rangle.$$

Hence

$$\gamma + 2\alpha \leq 0. \tag{5}$$

On the other hand, by Lemma 2.6,  $t \frac{\partial f(p(s, t))}{\partial t} \rightarrow 0$  when  $t \rightarrow 0$ . Relation (4) implies

$$t \frac{\partial f(p(s, t))}{\partial t} = t \overline{\lambda(s, t)} \left\langle \frac{\partial p(s, t)}{\partial t}, p(s, t) \right\rangle \rightarrow 0 \text{ when } t \rightarrow 0.$$

Hence,  $1 + \gamma + (2\alpha - 1) > 0$ ; i.e.,  $\gamma + 2\alpha > 0$ . This contradicts (5).  $\square$

### 3. Examples and remarks

3.1. The Pfaffian functions (see [7] for the definition), for example, all functions  $f \in \mathbf{R}[x_1, \dots, x_n, \exp(x_1), \dots, \exp(x_n)]$ , are shown to be definable in a suitable o-minimal structure. This is a consequence of a general result of Wilkie; see [18].

3.2 *Remark.* In Section 1.3, suppose moreover that the functions  $g$  and  $\rho$  are definable. Then the trivialization obtained in Proposition 1.3 is not necessarily definable. In general, it is not known even if definable  $C^1$  trivializations can be obtained; compare with [2].

3.3 *Remark.* Let  $f: \mathbf{C}^n \rightarrow \mathbf{C}$  be a complex polynomial.

(a) The finiteness of  $S_f$  was also proved in [14].

(b) If  $n = 2$ , it follows from [17] that  $\text{Bif}^\infty(f) = S_f \cup \Sigma_f$ .

(c) If  $n \geq 3$ , we do not know if  $S_f \subseteq \text{Bif}^\infty(f)$  or not.

3.4 *Remark.* (a) The bifurcation values of a polynomial  $f: \mathbf{R}^2 \rightarrow \mathbf{R}$  are characterized in [5].

(b) Let  $g: \mathbf{R} \rightarrow \mathbf{R}$  be defined by  $g(x) := x^3$ . Then  $g$  is a  $C^0$  trivial fibration, but  $\text{Bif}^1(g) = \Sigma_g = \{0\}$ . More generally, for each  $q \in \mathbf{N}$  there exists a semi-algebraic  $C^q$  diffeomorphism  $h: \mathbf{R} \rightarrow \mathbf{R}$  which is not a  $C^{q+1}$  diffeomorphism. Therefore, for such an  $h$  we have  $\text{Bif}^q(h) = \emptyset$  and  $\text{Bif}^{q+1}(h) \neq \emptyset$ .

(c) The  $C^1$  bifurcation set of  $h(x) = x \sin x$  is infinite. Obviously,  $h$  is not definable in any o-minimal structure.

(d) In general, the inclusions  $\Sigma_f \subseteq \text{Bif}^q(f) \subseteq \Sigma_f \cup S_f$  cannot be replaced by equalities, as the following examples show us.

3.5 EXAMPLE. (a) Let  $f: \mathbf{R} \rightarrow \mathbf{R}$  be defined by  $f(x) := x \exp x$ . Then  $\Sigma_f = \{-1/e\}$ ,  $S_f = \{0\}$  and  $\text{Bif}^q(f) = \{0, -1/e\}$ .

(b) Let  $g: \mathbf{R}^2 \rightarrow \mathbf{R}$  be defined by  $g(x, y) := x^2 y^2 + 2xy$ . Then  $\Sigma_g = \{0\}$ ,  $S_g = \{-1\}$  and  $\text{Bif}^q(g) = \{0, -1\}$ .

(c) Let  $h: \mathbf{R}^2 \rightarrow \mathbf{R}$  be defined by  $h(x, y) = y \exp(2x) + \exp x$ . Then  $S_h = \{0\}$ , but  $\text{Bif}^\infty(h) = \Sigma_h = \emptyset$  (one can check that  $H: \mathbf{R}^2 \rightarrow h^{-1}(0) \times \mathbf{R}$  defined by  $H(x, y) = (x, -\exp(-x), h(x, y))$  is a  $C^\infty$  trivialization of  $h$ ).

Note that in (a) and (b) we have  $\text{Bif}^q(f) = \Sigma_f \cup S_f$ ; i.e., the equality can be attained.

3.6 *Remark.* Theorem 2 in [10] describes an approximation from above of the  $C^\infty$  bifurcation sets of complex polynomial functions, using the Newton polyhedron at infinity. With the same proof as in [10] one can obtain a similar result for the case of real polynomial functions.

**3.7 Remark.** If  $f$  is a Pfaffian function, then using Proposition 1.6 and Khovanskii's theory on Fewnomials, see [7], one can estimate from above the cardinalities of  $S_f$ ,  $\Sigma_f$  and  $\text{Bif}^q(f)$ . Note also that the conclusion of Proposition 1.6 is still true when  $g: \mathbf{C}^n \rightarrow \mathbf{C}$  is a complex polynomial, and hence, Khovanskii's theory can be applied to get a (rough) estimate of  $\#(\text{Bif}^\infty(g))$  in this case (when  $n = 2$ , a better estimate of  $\#(\text{Bif}^\infty(g))$  was obtained in [13]).

**3.8 Remark.** The referee kindly pointed out that, with minor modifications, Theorem 1.7 and its proof are also valid when  $U \subseteq \mathbf{C}^n$  is open and  $f: U \rightarrow \mathbf{C}$  is holomorphic and definable in a polynomially bounded o-minimal expansion of the real field. (See [4] and [9] for polynomially bounded o-minimal expansions of the real field.) Namely, we have to use a  $C^1$  definable function  $\rho: U \rightarrow \mathbf{R}$  satisfying (1) and we have to replace  $M(f)$  by

$$M(f; \rho) := \{ x \in U \mid \exists \lambda \in \mathbf{C}, \text{grad } f(x) = \lambda \text{grad } \rho(x) \}.$$

Here,  $\text{grad } f(x)$  is as in 1.7, while for  $\text{grad } \rho(x)$ , we identify  $\mathbf{C}^n$  with  $\mathbf{R}^{2n}$  in the usual way. We also have to replace  $S_f$  by

$$S_{f;\rho} := \{ c \in \mathbf{C} \mid \text{there exists a sequence } \{y^k\} \subseteq M(f; \rho) \text{ such that (2) is fulfilled} \}$$

and to refer at Proposition 5.2 in [8] instead of [11]. We do not know if the assumption of polynomial bounds is necessary, or only convenient, for obtaining the conclusion.

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#### REFERENCES

- [1] F. Chazal, *Un théorème de fibration pour les feuilletages algébriques de codimension un de  $\mathbf{R}^n$* , C. R. Acad. Sci. Paris Série I **321** (1995), 327–330.
- [2] M. Coste and M. Shiota, *Nash trivality in families of Nash manifolds*, Invent. Math. **108** (1992), 349–368.
- [3] L. van den Dries, *Tame topology and o-minimal structures*, LMS Lecture Notes, Cambridge University Press, to appear.
- [4] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. **84** (1996), 497–540.
- [5] J. Ferrera and M. J. de la Puente, *The asymptotic values of a polynomial function on the real plane*, J. Pure Applied Algebra **106** (1996), 263–273.
- [6] T. L. Loi, *Verdier and strict Thom stratifications in o-minimal structures*, Illinois J. Math., **42** (1998), 347–356.
- [7] A. G. Khovanskii, *Fewnomials*, Trans. Math. Monographs, Vol. 88, Amer. Math. Soc., Providence, RI, 1991.
- [8] C. Miller, *Expansions of the real field with power functions*, Ann. Pure Appl. Logic **68** (1994), 79–94.



- [9] ———, *Infinite differentiability in polynomially bounded o-minimal structures*, Proc. Amer. Math. Soc. **123** (1995), 2551–2555.
- [10] A. Némethi and A. Zaharia, *On the bifurcation set of a polynomial function and Newton boundary*, Publ. RIMS, Kyoto Univ. **26** (1990), 681–689.
- [11] W. Pawlucki, *Le théorème de Puiseux pour une application sous-analytique*, Bull. Polish Acad. Sci. Math. **32** (1984), 555–560.
- [12] F. Pham, *Vanishing homologies and the  $n$  variables saddlepoint method*, Proc. Symp. Pure Math., vol. 40, Part 2, 1983, 319–333.
- [13] L. V. Thanh and M. Oka, *Note on estimation of the number of the critical values at infinity*, Kodai Math. J. **17** (1994), 409–419.
- [14] M. Tibăr, *On the monodromy fibration of polynomial functions with singularities at infinity*, C. R. Acad. Sci. Paris Série I **324** (1997), 1031–1035.
- [15] R. Thom, *Ensembles et morphismes stratifiés*, Bull. Amer. Math. Soc. **75** (1969), 240–284.
- [16] A. N. Varchenko, *Theorems on topological equisingularity of families of algebraic varieties and families of polynomial maps*, Izvestiya Akad. Nauk **36** (1972), 957–1019.
- [17] H. H. Vui, *Nombres de Lojasiewicz et singularités à l'infini des polynômes de deux variables complexes*, C. R. Acad. Sci. Paris Série I **311** (1990), 429–432.
- [18] A. Wilkie, *A general theorem of the complement and some new o-minimal structures*, manuscript, 1996.

Ta Lê Loi, Department of Mathematics, University of Dalat, Dalat, Vietnam

Alexandru Zaharia, Institute of Mathematics of the Romanian Academy, Bucharest, Romania