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FAILURE OF F-PURITY AND F-REGULARITY IN CERTAIN RINGS OF INVARIANTS

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1. Introduction

Let $\mathbb{F}_q$ be a finite field of characteristic p, K a field containing it, and $R = K[X_1, \ldots, X_n]$ a polynomial ring in n variables. The general linear group $GL_n(\mathbb{F}_q)$ has a natural action on R by degree preserving ring automorphisms. L. E. Dickson showed that the subring of elements which are fixed by this group action is a polynomial ring [Di], though for an arbitrary subgroup G of $GL_n(\mathbb{F}_q)$, the structure of the ring of invariants R^G may be rather mysterious. If the order of the group $|G|$ is relatively prime to the characteristic p of the field, there is an R^G-linear retraction $\rho\colon R \to R^G$, the *Reynolds operator*. This retraction makes R^G a direct summand of R as an R^G-module, and so R^G is F-regular. However when the characteristic p divides $|G|$, this method no longer applies, and the ring of invariants R^G need not even be Cohen-Macaulay. M.-J. Bertin showed that when R is a polynomial ring in four variables and G is the cyclic group with four elements which acts by permuting the variables in cyclic order, then the ring of invariants R^G is a unique factorization domain which is not Cohen-Macaulay, providing the first example of such a ring, [Be]. Related work and bounds on the depth of R^G can be found in the work of R. M. Fossum and P. A. Griffith; see [FG]. More recently D. Glassbrenner studied the invariant subrings of the action of the alternating group A_n on a polynomial ring in n variables over a field of characteristic p, constructing examples of F-pure rings which are not F-regular [G1], [G2]. Both these families of examples study rings of invariants of $K[X_1, \ldots, X_n]$ under the action of a subgroup G of the symmetric group on n elements, i.e., an action which permutes the variables, and Glassbrenner shows that for such a group the ring of invariants is F-pure, see [G1, Proposition 0.6.7].

We shall construct examples which demonstrate that the ring of invariants for the natural action of a subgroup G of $GL_n(\mathbb{F}_q)$ need not be F-pure. We shall obtain such examples with the group G being the symplectic group over a finite field. These non F-pure invariant subrings are always complete intersections, and are actually hypersurfaces in the case of $G = Sp_4(\mathbb{F}_q) < GL_4(\mathbb{F}_q)$ acting on the polynomial ring $R = K[X_1, X_2, X_3, X_4]$. These examples are particularly interesting if one is attempting to interpret the Frobenius closures and tight closures of ideals as contractions from certain extension rings, since we have an ideal generated by a system of

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parameters and the socle element modulo this ideal is being forced into the expansion of the ideal to a module-finite extension ring which is a separable, in fact Galois, extension. This element is also forced into an expanded ideal in a linearly disjoint purely inseparable extension, being in the Frobenius closure of the ideal. It is noteworthy that the element can be forced into expanded ideals in two such different ways.

Our results depend on the work of D. Carlisle and P. Kropholler where they show that the ring of invariants under the natural action of the symplectic group on a polynomial ring is a complete intersection [CK]. We obtain the precise equations defining these complete intersections in some examples using the program Macaulay, and in some other cases collect enough information to display that the invariant subrings are not F-pure.

The second part of this paper deals with the alternating group A_n acting on the polynomial ring $R = K[X_1, \dots, X_n]$ by permuting the variables. We shall assume that the characteristic p of K is an odd prime, and denote by R^{A_n}, the invariant subring of this action. Since R^{A_2} is a polynomial ring we shall always assume $n \geq 3$. If the order of the group $|A_n| = \frac{1}{2}(n!)$ is relatively prime to the characteristic p of the field, the Reynolds operator makes R^{A_n} a direct summand of R as an R^{A_n}-module, and in the language of tight closure, the existence of such a retraction is equivalent to the ring R^{A_n} being F-regular; see Lemma 5.1. When p divides n or $n-1$, Glassbrenner has shown that the invariant subring R^{A_n} is no longer F-regular; see [G1, Proposition 1.2.5]. We shall extend this result by showing that R^{A_n} is F-regular if and only if p does not divide $|A_n|$.

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2. F-purity and F-regularity

We recall some basic notation and definitions from [HH1], [HH2], [HH3].

Let R be a Noetherian ring of characteristic $p > 0$. We shall always use the letter e to denote a variable nonnegative integer, and q to denote the e th power of p, i.e., $q = p^e$. For an ideal $I = (x_1, \dots, x_n) \subseteq R$ we let $I^{[q]} = (x_1^q, \dots, x_n^q)$.

For an element x of R, we say that $x \in I^F$, the *Frobenius closure* of I, if there exists $q = p^e$ such that $x^q \in I^{[q]}$. We shall say that the ring R is *F-pure* if for all ideals I of R we have $I^F = I$.

We shall denote by R° the complement of the union of the minimal primes of R. For an ideal $I \subseteq R$ and an element x of R, we say that $x \in I^*$, the *tight closure* of I, if there exists $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all $q = p^e \gg 0$. If $I = I^*$ for all ideals I of R, we say R is *weakly F-regular*. R is called *F-regular* if every localization is weakly F-regular. These two notions are known to be the same if R is Gorenstein [HH2, Corollary 4.7].

3. Symplectic invariants

We shall summarize in this section the results of Carlisle and Kropholler as presented in [B]. Let $\mathbb{F}_q$ be a finite field of characteristic p, and K an infinite field containing it. L. E. Dickson showed that the ring of invariant forms under the natural action of $GL_n(\mathbb{F}_q)$ on the polynomial ring $R = K[X_1, \ldots, X_n]$ is a graded polynomial algebra on the algebraically independent generators $c_{n,i}$, where the $c_{n,i}$ are the coefficients in the equation

$$\prod_{v \in \mathbb{F}_q[X_1,\ldots,X_n]_1} (T - v) = T^{q^n} - c_{n,n-1}T^{q^{n-1}} + c_{n,n-2}T^{q^{n-2}} - \cdots + (-1)^n c_{n,0}T.$$

When working with a fixed polynomial ring $R = K[X_1, \ldots, X_n]$, we shall drop the first index, and write the generators of $R^{GL_n(\mathbb{F}_q)}$ as $c_0, \ldots, c_{n-1}$, the *Dickson invariants*. It is clear that for any subgroup G of $GL_n(\mathbb{F}_q)$, the ring of invariants R^G is a module-finite extension of the polynomial ring $R^{GL_n(\mathbb{F}_q)} = K[c_0, \ldots, c_{n-1}]$.

Let V be a vector space of dimension $2n$ over the field $\mathbb{F}_q$, on which we have a non-degenerate alternating bilinear form B. We may choose a basis $e_1, \ldots, e_{2n}$ for V, such that B is given by

$$B\left(\sum a_i e_i, \sum b_j e_j\right) = a_1 b_2 - a_2 b_1 + \cdots + a_{2n-1}b_{2n} - a_{2n}b_{2n-1}.$$

The symplectic group $G = Sp_{2n}(\mathbb{F}_q)$ is the subgroup of $GL_{2n}(\mathbb{F}_q)$ consisting of the elements which preserve B. We consider the natural action of G on $R = K[X_1, \ldots, X_{2n}]$. In addition to the Dickson invariants, it is easily seen that R^G must contain

$$\xi_i = X_1 X_2^{q^i} - X_2 X_1^{q^i} + \cdots + X_{2n-1}X_{2n}^{q^i} - X_{2n}X_{2n-1}^{q^i}.$$

Carlisle and Kropholler show that the Dickson invariants $c_0, \ldots, c_{2n-1}$ along with the above $\xi_1, \ldots, \xi_{2n}$ form a generating set for R^G, and that there are $2n$ relations, i.e., that R^G is a complete intersection. One may eliminate $c_0, \ldots, c_{n-1}$ and ξ_{2n} using $n+1$ of these relations, after which the remaining $n-1$ relations are

$$\sum_{j=0}^{i-1} (-1)^j \xi_{i-j}^{q^j} c_j = \sum_{j=i+1}^{2n} (-1)^j \xi_{j-i}^{q^i} c_j$$

where $1 \leq i \leq n-1$ and $c_{2n} = 1$. Their results furthermore show that $c_0 \in K[\xi_1, \ldots, \xi_{2n-1}]$ which is, in fact, a polynomial ring.

4. Rings of invariants which are not F-pure

We shall first show that the ring of invariants of $G = Sp_4(\mathbb{F}_q)$ acting on the polynomial ring $R = K[X_1, X_2, X_3, X_4]$ is not F-pure when $q = 2$ or 3. Note that $Sp_2(\mathbb{F}_q)$ is the same as $SL_2(\mathbb{F}_q)$, and so the ring of invariants in that case is a polynomial ring.

EXAMPLE 4.1. Let $R = K[X_1, X_2, X_3, X_4]$ and $G = Sp_4(\mathbb{F}_q)$ be the symplectic group with its natural action on R. In the notation of the previous section, $R^G = K[c_2, c_3, \xi_1, \xi_2, \xi_3]$, where the only relation is

$$\xi_1 c_0 = \xi_1^q c_2 - \xi_2^q c_3 + \xi_3^q.$$

We need to determine c_0 as an element of $K[\xi_1, \xi_2, \xi_3]$. When $q = 2$, it can be verified that $c_0 = \xi_1^5 + \xi_2^3 + \xi_3 \xi_1^2$, and so

$$\xi_3^2 = \xi_1^6 + \xi_1 \xi_2^3 + \xi_1^3 \xi_3 + \xi_1^2 c_2 + \xi_2^2 c_3,$$

by which $\xi_3 \in ((\xi_1, \xi_2)R^G)^F$. Since $\xi_3 \notin (\xi_1, \xi_2)R^G$, the ring R^G is not F-pure.

In the case $q = 3$, c_0 can be expressed as an element of $K[\xi_1, \xi_2, \xi_3]$ by the equation

$$c_0 = \xi_2^8 + \xi_3 \xi_1^3 \xi_2^4 + \xi_1^6 \xi_3^2 + \xi_1^{10} \xi_2^4 - \xi_1^{13} \xi_3 + \xi_1^{20}.$$

Once again we see that $\xi_3 \in ((\xi_1, \xi_2)R^G)^F$, and so R^G is not F-pure.

Computations with Macaulay helped us determine the precise equations in these examples.

THEOREM 4.2. *Let $\mathbb{F}_q$ be a finite field of characteristic p, and K an infinite field containing it. Let $G = Sp_{2n}(\mathbb{F}_q)$ be the symplectic group with its natural action on the polynomial ring $R = K[X_1, \dots, X_{2n}]$. If $n \geq 2$ and $q \geq 4n - 4$, then the ring of invariants R^G is not F-pure.*

Proof. In the notation of the previous section, the ring of invariants is $R^G = K[c_n, \dots, c_{2n-1}, \xi_1, \dots, \xi_{2n-1}]$, where there are exactly $n - 1$ relations, as stated before. Using the relation with $i = 1$, we see that

$$\xi_{2n-1}^q \in (\xi_1^q, \dots, \xi_{2n-2}^q, \xi_1 c_0)R^G,$$

whereas $\xi_{2n-1} \notin (\xi_1, \dots, \xi_{2n-2})R^G$.

If R^G is indeed F-pure, $\xi_{2n-1}^q \notin (\xi_1^q, \dots, \xi_{2n-2}^q)R^G$, and so the expression of c_0 as an element of $K[\xi_1, \dots, \xi_{2n-1}]$ must have a monomial of the form $\xi_1^{a_1} \xi_2^{a_2} \cdots \xi_{2n-1}^{a_{2n-1}}$, with $a_1 \leq q - 2$ and $a_2, \dots, a_{2n-1} \leq q - 1$. Equating degrees, we have

$$\begin{aligned} \deg c_0 = q^{2n} - 1 &= a_1(q + 1) + a_2(q^2 + 1) + \cdots + a_{2n-1}(q^{2n-1} + 1) \\ &= \sum_{i=1}^{2n-1} a_i + \sum_{i=1}^{2n-1} a_i q^i. \end{aligned}$$

Examining this modulo q, we get that $\sum_{i=1}^{2n-1} a_i = \lambda q - 1$, where the bounds on a_i show that $1 \leq \lambda \leq 2n - 2 < q$. Substituting this, we get $q^{2n} = \lambda q + \sum_{i=1}^{2n-1} a_i q^i$.

Working modulo q^2, we see that $a_1 = q - \lambda$, and continuing this way we get that $a_2, \ldots, a_{2n-1} = q - 1$. Hence

$$q^{2n} - 1 = (q - \lambda)(q + 1) + (q - 1)(q^2 + 1) + \cdots + (q - 1)(q^{2n-1} + 1),$$

which simplifies to give $\lambda(q+1) = 2nq - 2n - q + 3$. Since $\lambda \leq 2n - 2$, this implies that $q \leq 4n - 5$, a contradiction.

Hence R^G is not F-pure. In particular $\xi_{2n-1} \in ((\xi_1, \ldots, \xi_{2n-2})R^G)^F$, the Frobenius closure. □

COROLLARY 4.3. *The ring of invariants of the symplectic group* $G = Sp_4(\mathbb{F}_q)$ *acting on the polynomial ring* $R = K[X_1, X_2, X_3, X_4]$ *is not F-pure.*

Proof. We have, in the examples above, treated the case where $q = 2$ or 3. When $q \geq 4$, the result follows from the previous theorem. □

5. Rings of invariants of the alternating group

The invariant subring under the natural action of the alternating group A_n is $R^{A_n} = K[e_1, \ldots, e_n, \Delta]$ where e_i is the elementary symmetric function of degree i in $X_1, \ldots, X_n$, and $\Delta = \prod_{i>j}(X_i - X_j)$. The element Δ is easily seen to be fixed by all even permutations of $X_1, \ldots, X_n$, though not by odd permutations. However its square, Δ^2, is fixed by all permutations, and so is a polynomial in the algebraically independent elements $e_1, \ldots, e_n$. Consequently the invariant subring R^{A_n} is a hypersurface, in particular it is Gorenstein. The elements $e_1, \ldots, e_n$ are an obvious choice as a homogeneous system of parameters for R^{A_n}, and the one-dimensional socle modulo this system of parameters is generated by Δ.

LEMMA 5.1. *With the above notation, the following are equivalent:*

(1) $R^{A_n} = K[e_1, \ldots, e_n, \Delta]$ *is F-regular.*
(2) R^{A_n} *is a direct summand of* $R = K[X_1, \ldots, X_n]$ *as an* R^{A_n}*-module.*
(3) $\Delta \notin (e_1, \ldots, e_n)R$.

Proof. (1) $\Rightarrow$ (2). By [HH3, Theorem 5.25], an F-regular ring is a direct summand of any module-finite extension ring.

(2) $\Rightarrow$ (3). Since R^{A_n} is a direct summand of R, we have

$$(e_1, \ldots, e_n)R \cap R^{A_n} = (e_1, \ldots, e_n)R^{A_n}.$$

(3) $\Rightarrow$ (1). The elements $e_1, \ldots, e_n$ form a system of parameters for the Gorenstein ring R^{A_n} and Δ is the socle generator modulo this system of parameters. If Δ is in the tight closure of $(e_1, \ldots, e_n)R^{A_n}$, then $\Delta \in (e_1, \ldots, e_n)R^* = (e_1, \ldots, e_n)R$. Hence Δ cannot be in the tight closure of $(e_1, \ldots, e_n)R^{A_n}$, by which R^{A_n} is F-regular. □

Consequently our aim is to establish that $\Delta \in (e_1, \dots, e_n)R$, whenever p divides $|A_n|$. We shall henceforth denote this ideal by $I = (e_1, \dots, e_n)R$.

LEMMA 5.2. *Let T_j^i denote the sum of all monomials of degree i in the variables $X_j, \dots, X_n$. Then $T_j^i \in I$ whenever $i \geq j \geq 1$. In particular, $T_i^i \in I$ for all $1 \leq i \leq n$.*

Proof. Observe that $T_j^i = T_{j-1}^i - X_{j-1}T_{j-1}^{i-1}$. Given T_j^i with $i \geq j \geq 1$, we may use this formula to rewrite T_j^i as a sum of terms which are multiples of T_1^i. Since T_1^i is the sum of all the monomials of degree i in $X_1, \dots, X_n$, it is certainly an element of I, and so $T_j^i \in I$. □

LEMMA 5.3. *The ideal $I = (e_1, \dots, e_n)R$ generated by the elementary symmetric functions contains the elements X_n^n, $X_n^{n-1}X_{n-1}^{n-1}$, $X_n^{n-1}X_{n-1}^{n-2}X_{n-2}^{n-2}, \dots,$ $X_n^{n-1}X_{n-1}^{n-2}\cdots X_i^{i-1}X_{i-1}^{i-1}, \dots, X_n^{n-1}X_{n-1}^{n-2}\cdots X_2X_1$.*

Proof. We shall use the fact that $T_i^i \in I$ for $1 \leq i \leq n$, Lemma 5.2. This already says that $X_n^n = T_n^n \in I$, and since I is symmetric in the X_i, we also have $X_{n-1}^n \in I$. Next, $X_{n-1}^{n-1}T_{n-1}^{n-1} \in I$, but examining this using $X_{n-1}^n \in I$ we see that $X_n^{n-1}X_{n-1}^{n-1} \in I$. We proceed by induction.

Since $T_{i-1}^{i-1} \in I$, we know that $X_n^{n-1}X_{n-1}^{n-2}\cdots X_i^{i-1}T_{i-1}^{i-1} \in I$, but using the inductive hypothesis this gives

$$X_n^{n-1}X_{n-1}^{n-2}\cdots X_i^{i-1}X_{i-1}^{i-1} \in I. \qquad \square$$

LEMMA 5.4. *In the above notation, $\Delta \equiv (n!)X_n^{n-1}X_{n-1}^{n-2}\cdots X_2 \pmod{I}$.*

Proof. Let $\delta_r = (X_r - X_1)(X_r - X_2)\cdots(X_r - X_{r-1})$. Then $\Delta = \delta_n\delta_{n-1}\cdots\delta_2$. We shall show that $\delta_r \equiv rX_r^{r-1} \pmod{I + (X_{r+1}, \dots, X_n)R}$ for $2 \leq r \leq n$. Note that for $r = n$, this says $\delta_n \equiv nX_n^{n-1} \pmod{I}$.

Fix r, where $2 \leq r \leq n$. Let f_i be the elementary symmetric function of degree i in the variables $X_1, \dots, X_{r-1}$. Then

$$f_i \equiv (-X_r)f_{i-1} \pmod{I + (X_{r+1}, \dots, X_n))R},$$

and using this repeatedly, we see

$$f_i \equiv (-X_r)^i \pmod{J} \quad \text{where } J = I + (X_{r+1}, \dots, X_n)R.$$

Consequently

$$\begin{aligned}
\delta_r &= (X_r - X_1)(X_r - X_2)\cdots(X_r - X_{r-1}) \\
&= X_r^{r-1} - X_r^{r-2}(X_1 + \cdots + X_{r-1}) + \cdots + (-1)^{r-1}X_1\cdots X_{r-1} \\
&\equiv X_r^{r-1} - X_r^{r-2}f_1 + \cdots + (-1)^{r-1}f_{r-1} \pmod{J} \\
&\equiv X_r^{r-1} - X_r^{r-2}(-X_r) + \cdots + (-1)^{r-1}(-X_r)^{r-1} \pmod{J} \\
&\equiv rX_r^{r-1} \pmod{J}.
\end{aligned}$$

Since $X_n^n \in I$, when evaluating the term $\delta_n\delta_{n-1} \pmod{I}$, it is enough to consider $\delta_{n-1} \pmod{I + X_n R}$, and using this we get

$$\delta_n\delta_{n-1} \equiv n(n-1)X_n^{n-1}X_{n-1}^{n-2} \pmod{I}.$$

Proceeding in this manner, one obtains from the above calculations that $\Delta = \delta_n\delta_{n-1}\cdots\delta_2 \equiv (n!)X_n^{n-1}X_{n-1}^{n-2}\cdots X_2 \pmod{I}$. The point is that since $\delta_n\delta_{n-1}\cdots\delta_r \equiv n(n-1)\cdots(r)X_n^{n-1}X_{n-1}^{n-2}\cdots X_r^{r-1} \pmod{I}$, we have $\delta_n\delta_{n-1}\cdots\delta_r(X_r,\ldots,X_n) \subseteq I$, by Lemma 5.3 and so when evaluating the product $\delta_n\delta_{n-1}\cdots\delta_{r-1} \pmod{I}$, one need only consider the element δ_{r-1} modulo the ideal $I + (X_r,\ldots,X_n)R$. □

We are now ready to prove the main result of this section.

THEOREM 5.5. *Let $R = K[X_1,\ldots,X_n]$ be a polynomial ring in n variables over a field k of characteristic p, an odd prime, and let the alternating group A_n act on R by permuting the variables. Then the invariant subring R^{A_n} is F-regular (equivalently, R^{A_n} is a direct summand of R) if and only if the order of the group $|A_n| = \frac{1}{2}(n!)$ is relatively prime to p.*

Proof. As we noted, it suffices to show that $\Delta \in I = (e_1,\ldots,e_n)R$. By Lemma 5.4, $\Delta \equiv (n!)X_n^{n-1}X_{n-1}^{n-2}\cdots X_2 \pmod{I}$, and so the result follows. □

Remark 5.6. Proposition 0.6.7 in [G1] shows that R^{A_n} is always F-pure. Consequently when the characteristic p of the field K is an odd prime dividing $|A_n|$, R^{A_n} is an F-pure ring which is not F-regular.

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