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GOLOMB'S SELF-DESCRIBED SEQUENCE AND FUNCTIONAL DIFFERENTIAL EQUATIONS

Y.-F.S. PÉTERMANN AND JEAN-LUC RÉMY

A sequence (word) W of positive integers is *self-described* or *self-generating* if $\tau(W) = W$, where $\tau(W)$ is the sequence consisting of the numbers of consecutive equal entries of W. A famous self-generating bounded sequence is Kolakoski's $\underbrace{1,}_{1,}\ \underbrace{2,2,}_{2,}\ \underbrace{1,1,}_{2,}\ \underbrace{2,}_{1,}\ \underbrace{1,}_{1,}\ \underbrace{2,2,}_{2,}\cdots$ (see [Ch]). In this paper we consider Golomb's sequence F, which is the only nondecreasing self-generating sequence taking all positive integral values, $\underbrace{1,}_{1,}\ \underbrace{2,2,}_{2,}\ \underbrace{3,3,}_{2,}\ \underbrace{4,4,4}_{3,}\ \underbrace{5,5,5,}_{3,}\ \underbrace{6,6,6,6,}_{4,}\cdots$. Let ϕ denote the golden number. We prove that

$$F(n) = \phi^{2-\phi}n^{\phi-1} + \frac{n^{\phi-1}}{\log n}h\left(\frac{\log\log n}{\log\phi}\right) + O\left(\frac{n^{\phi-1}}{\log^2 n}\log\log n\right),$$

where the real function h is continuous and satisfies $h(x) = -h(x+1)$ $(x \geq 0)$. The method of proof is intimately connected with the more general problem of characterising the solution E_1 of an approximate functional integral equation of the type

$$E_1(t) = -\phi^{1-\phi}t^{\phi-2}\int_2^{\phi^{2-\phi}t^{\phi-1}} E_1(u)\,du + O\left(\frac{t^{\phi-1}}{\log^2 t}\right),$$

which we discuss in the second part of the paper.

1. Introduction

In the problem section of the American Mathematical Monthly in 1966, S.W. Golomb [Go] considered the unique nondecreasing sequence $\{F(n)\}_{n\geq 1} = \{1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, 7\ldots\}$ "self-described" by the two conditions $F(1) = 1$ and $F(n) = |\{m : F(m) = n\}|$ $(n \geq 1)$. At the time he only requested an asymptotic expression for $F(n)$ as $n \to \infty$. We have

$$F(n) = cn^{\phi-1} + E(n), \tag{1.1}$$

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where $c = \phi^{2-\phi}$, $\phi = (\sqrt{5}+1)/2$ is the golden number, and $E(n) = o(n^{\phi-1})$. The first published complete proof of (1.1) is due to N.J. Fine [Fi]; D. Marcus [Ma] proposed a clever heuristical argument: see [Pé2] for a proof based on Marcus' idea.

More recently, I. Vardi [Va] asked for a more precise estimate for the error term $E(n)$ of (1.1). On the one hand he could establish

$$E(n) = O\left(\frac{n^{\phi-1}}{\log n}\right); \tag{1.2}$$

on the other hand he conjectured that estimate (1.2) is optimal.

CONJECTURE 1. *We have*

$$E(n) = \Omega_{\pm}\left(\frac{n^{\phi-1}}{\log n}\right). \tag{1.3}$$

Vardi's Conjecture 1 is based on a heuristic argument, which led him to be more precise.

CONJECTURE 2. *For $n \geq 2$ we have*

$$E(n) = \frac{n^{\phi-1} h\left(\frac{\log\log n}{\log\phi}\right)}{\log n} + O\left(\frac{n^{\phi-1}}{\log^2 n}\right), \tag{1.4}$$

where $h(x)$ satisfies

$$h(x+1) = -h(x) \quad \textit{for } x \geq 0, \tag{1.5}$$

and

$$|h(x)| > 0 \quad \textit{for } x \in (0,1). \tag{1.6}$$

However, as Vardi himself pointed out, he was "not even able to show that $\limsup_{n\to\infty} |E(n)| = \infty$". This was proved by Y.-F.S. Pétermann [Pé1], who showed that $E(n) = \Omega_{\pm}(n^{\phi-1-\epsilon})$ for every $\epsilon > 0$. Recently J.-L. Rémy [Ré] succeeded in verifying the truth of Conjecture 1.

In this paper we are concerned with Conjecture 2. We prove:

THEOREM 1. *We have*

$$E(n) = \frac{n^{\phi-1} h\left(\frac{\log\log n}{\log\phi}\right)}{\log n} + O\left(\frac{n^{\phi-1}\log\log n}{\log^2 n}\right), \tag{1.7}$$

where $h(x)$ satisfies

$$h(x+1) = -h(x) \quad \textit{for } x \geq 0, \tag{1.5}$$

and

$$h \textit{ is continuous} \tag{1.8}$$

Remark 1. Conjecture 1, which we know is true from Rémy's result [Ré], implies that the function h in Theorem 1 is not identically zero. It should also be noted that assertion (1.6) in Conjecture 2 is false: we can indeed prove that $h(0) \neq 0$ (see Remark 2 in Section 3), and this shows with (1.5) and (1.8) that there is a number α with $0 < \alpha < 1$ and $h(\alpha) = 0$.

Hence we propose to modify assertion (1.6).

CONJECTURE 3. *If h is as in Theorem* 1 *there is a number α with* $0 < \alpha < 1$ *and*

$$|h(x)| > 0 \quad \text{for } x \in (\alpha, 1+\alpha). \tag{1.6'}$$

The rest of Section 3 is devoted to four other remarks.

The proof of Theorem 1 contains in fact an almost complete treatment of the approximate functional integral equation

$$E(t) = -c^{-\phi} t^{\phi-2} \int_2^{ct^{\phi-1}} E(u)du + O\left(\frac{t^{\phi-1}}{\log^2 t}\right). \tag{1.9}$$

and its summatory counterpart (2.1) in Section 2 just below: this is discussed in Remarks 3 and 4.

But the error term bound of Theorem 1 is not as good as the bound conjectured in Conjecture 2: we extensively discuss in Remark 5 the exact and non trivial functional equation

$$d(t) = -t^{\phi-2} \int_0^{t^{\phi-1}} d(u)\,du + 1, \tag{1.10}$$

which is of a type similar to (1.9), and for the solution of which the better error term bound holds. (By "non trivial" we mean that the function h associated to the solution is not identically zero).

In Theorem 2 of Remark 3 we show that the solution of an equation of type (1.9) satisfies (1.7), with (1.5) and (1.8): a sort of restricted converse to this is obtained in Theorem 4 of Remark 6, showing that a function $E(t) = t^{\phi-1} h(\log\log t/\log\phi)/\log t$ satisfies (1.9) for many choices of the function h.

Notation. In the sequel we shall frequently appeal without comment to classical properties of the golden number ϕ. Although all these properties can easily be inferred from the definition of ϕ, we state below a few of them for convenience.

$$\phi^2 = \phi + 1, \qquad \phi(\phi-1) = 1, \qquad (\phi-1)^2 = 2-\phi, \qquad (\phi-1)^3 = 2\phi - 3.$$

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2. Proof of Theorem 1

Our starting point is the heuristic formula that led Vardi to his Conjecture 2.

LEMMA 1. *We have*

$$E(n) = -c^{-\phi} n^{\phi-2} \sum_{m \le cn^{\phi-1}} E(m) + O\left(\frac{n^{\phi-1}}{\log^2 n}\right). \tag{2.1}$$

Proof. Every $n \geq 1$ can be written uniquely as $n = G(m) - r$, where $0 \leq r < F(m)$, and where $G(m)$ denotes the position of the last occurence of the integer m in the sequence F. We have

$$F(n) = F(G(m) - r) = m. \tag{2.2}$$

Now if we put $R(m) := \sum_{k \le m} E(k)$, then, from Vardi's result (1.2), we clearly have

$$R(m) = O\left(\frac{m^{\phi}}{\log m}\right). \tag{2.3}$$

Moreover,

$$E(n) = E(G(m) - r) = -\frac{R(m)}{cm^{\phi-1}}\left(1 + O\left(\frac{|R(m)|}{m^{\phi}}\right)\right) + O(1). \tag{2.4}$$

(Equation (2.4) can easily be derived from Vardi's paper [Va]: see (10) of that paper; or see Lemma 3 in [Pé1].) Thus, with (2.3), (2.2) and (1.2), we may write

$$\begin{aligned} E(n) &= -\frac{R(m)}{cm^{\phi-1}} + O\left(\frac{m}{\log^2 m}\right) = -\frac{R(F(n))}{c(F(n))^{\phi-1}} + O\left(\frac{F(n)}{\log^2(F(n))}\right) \\ &= -\frac{R(cn^{\phi-1} + O(n^{\phi-1}/\log n))}{c(cn^{\phi-1} + O(n^{\phi-1}/\log n))^{\phi-1}} + O\left(\frac{n^{\phi-1}}{\log^2 n}\right) \\ &= -c^{-\phi} n^{\phi-2} R(cn^{\phi-1}) + O\left(\frac{n^{\phi-1}}{\log^2 n}\right). \end{aligned}$$

This proves Lemma 1. Now we replace the sum in the approximate functional equation of Lemma 1 by an integral, which is smoother (differentiable) and thus easier to handle. For this we use a result of Segal's [Se]. □

LEMMA A. *Let $f(n)$ be a function of a positive integral variable, and suppose*

$$\sum_{n \le x} f(n) = z(x) + E(x),$$

where $z(x)$ is twice continuously differentiable, and $z''(x)$ is of constant sign for $x \geq 1$. Then

$$\sum_{n \leq x} E(n) = \frac{1}{2} z(x) + (1 - \{x\}) E(x) + \int_1^x E(t)\, dt + O(|z'(x)|) + O(1).$$

LEMMA 2. *If the definitions of the functions F and E of (1.1) are extended to the real arguments $t \geq 1$ by putting $F(t) := F([t]) =: ct^{\phi-1} + E(t)$, then we have*

$$E(t) = -c^{-\phi} t^{\phi-2} \int_2^{ct^{\phi-1}} E(u)\, du + O\left(\frac{t^{\phi-1}}{\log^2 t}\right). \tag{2.5}$$

Proof. If we put $f(m) = 1$ when $m = 1$ or $m = G(k) + 1$ for some k, and $f(m) = 0$ otherwise in Lemma A, then $F(x) = \sum_{m \leq x} f(m) = z(x) + E(x)$ where $z(x) := cx^{\phi-1}$. The hypotheses of Lemma A are satisfied and Lemma 2 is proved.

Now if we let

$$D_0(t) := -c^{-\phi} t^{\phi-2} \int_2^{ct^{\phi-1}} E(u)\, du, \tag{2.6}$$

we have of course

$$E(t) = D_0(t) + O\left(\frac{t^{\phi-1}}{\log^2 t}\right), \tag{2.7}$$

so that from (2.6) and (2.7) we have

$$D_0(t) = -c^{-\phi} t^{\phi-2} \int_2^{ct^{\phi-1}} D_0(u) du + O\left(\frac{t^{\phi-1}}{\log^2 t}\right). \tag{2.8}$$

Similarly if we put

$$D(t) := -c^{-\phi} t^{\phi-2} \int_2^{ct^{\phi-1}} D_0(u)\, du, \tag{2.9}$$

we may write

$$D(t) = D_0(t) + O\left(\frac{t^{\phi-1}}{\log^2 t}\right) \tag{2.10}$$

and

$$D(t) = -c^{-\phi} t^{\phi-2} \int_2^{ct^{\phi-1}} D(u)\, du + O\left(\frac{t^{\phi-1}}{\log^2 t}\right). \tag{2.11}$$

We shall work with the approximate functional-differential equation (2.11), rather than with (2.5) or (2.8), which are "functional" but not "differential". By (2.7) and (2.10) it is indeed sufficient to prove the theorem for D instead of E. If we write

$$D(t) = \frac{t^{\phi-1}}{\log t} K(t), \tag{2.12}$$

this defines a function $K(t) = O(1)$ which is differentiable for every $t \geq 2$. If similarly we write

$$D_0(t) = \frac{t^{\phi-1}}{\log t} K_0(t), \tag{2.13}$$

then by (2.10) we have

$$K_0(t) = K(t) + O(1/\log t). \tag{2.14}$$

$\square$

LEMMA 3. *We have*

$$tK'(t) + K(t) + K(ct^{\phi-1}) = O(1/\log t). \tag{2.15}$$

Proof. First we differentiate expression (2.12).

$$D'(t) = \frac{t^{\phi-1}}{\log t} K'(t) + \frac{t^{\phi-2}}{\log t}(\phi - 1)K(t) - \frac{t^{\phi-2}}{\log^2 t} K(t). \tag{2.16}$$

Then we differentiate (2.9).

$$\begin{aligned} D'(t) &= -c^{-\phi}(\phi-2)t^{\phi-3} \int_2^{ct^{\phi-1}} D_0(u)\,du \\ &\quad - c^{-\phi}t^{\phi-2}\frac{(ct^{\phi-1})^{\phi-1}}{\log(ct^{\phi-1})} K_0(ct^{\phi-1})c(\phi-1)t^{\phi-2} \\ &= (\phi-2)\frac{D(t)}{t} - \frac{t^{\phi-2}}{\log t} K_0(ct^{\phi-1}) + O\left(\frac{t^{\phi-2}}{\log^2 t}\right). \end{aligned} \tag{2.17}$$

Equating (2.16) and (2.17) (with a use of definition (2.12)) yields

$$\frac{t^{\phi-2}}{\log t} tK'(t) + \frac{t^{\phi-2}}{\log t} K(t) + \frac{t^{\phi-2}}{\log t} K_0(ct^{\phi-1}) = O\left(\frac{t^{\phi-2}}{\log^2 t}\right),$$

and a division by $t^{\phi-2}/\log t$ with an appeal to (2.14) finishes the proof. $\square$

A key step in the proof of Theorem 1 is to ensure now that $K(t)$ is sufficiently near $K(ct^{\phi-1})$. This will be done by an induction argument on m for t in intervals of the form $[N_m, N_{m+1}]$, where $N_0, N_1, N_2, \ldots$ is a sequence of positive real numbers with

$$N_m = cN_{m+1}^{\phi-1}, \quad \text{that is} \quad N_{m+1} = \frac{c}{\phi} N_m^{\phi}, \tag{2.18}$$

for $m \geq 0$. We need a lower bound on N_m.

LEMMA 4. *If N_0 is chosen large enough we have*

$$N_m > 3^{\phi^m}. \tag{2.19}$$

Proof. If $Q := \phi^{\phi-2} = 1/c$ we have

$$N_1 = \frac{c}{\phi} N_0^{\phi} = \phi^{1-\phi} N_0^{\phi} = (\phi^{\phi-2} N_0)^{\phi} = (Q N_0)^{\phi},$$

and in general, if $Q_0 := Q^{\phi^2}$ $(= 1/\phi)$, we have

$$\begin{aligned} N_m &= (QN_{m-1})^{\phi} = (Q(QN_{m-2})^{\phi})^{\phi} = \cdots = Q^{\phi+\phi^2+\cdots+\phi^m} N_0^{\phi^m} \\ &= Q^{\phi\frac{\phi^m-1}{\phi-1}} N_0^{\phi^m} = Q_0^{\phi^m-1} N_0^{\phi^m} > (Q_0 N_0)^{\phi^m-1} > \left(3^{\phi^2}\right)^{\phi^m-1}, \end{aligned}$$

provided N_0 is chosen larger than $3^{\phi^2}/Q_0 = \phi 3^{\phi^2}$. And we have

$$\phi^2(\phi^m - 1) = \phi^m(\phi^2(1 - \phi^{-m})) \geq \phi^m$$

if $m \geq 1$. This concludes the proof if we note that since $N_0 > 3$, (2.19) also holds for $m = 0$. □

We are now in position to prove:

LEMMA 5. *For $t \geq 3$ we have*

$$K(t) + K(ct^{\phi-1}) = O\left(\frac{\log\log t}{\log t}\right). \tag{2.20}$$

Proof. Let $N_0, N_1, N_2, \ldots$ be a sequence of positive real numbers satisfying (2.18) for $m \geq 0$, and with N_0 large enough to ensure the validity of (2.19). We show that there is an absolute positive constant C and a sequence of real positive numbers $M_0, M_1, M_2, \ldots$ with

$$M_{m+1} = (C + M_m)\left(1 + \frac{C}{\log N_m}\right) \quad (m \geq 0) \tag{2.21}$$

and

$$|K(t) + K(ct^{\phi-1})| \leq \frac{M_m}{\log t} \tag{2.22}$$

for $2 \leq t \leq N_m$. The proof is by induction on m. Estimate (2.22) holds for $m = 0$ if we choose M_0 large enough. Assume it is satisfied for some $m \geq 0$, and let $N_m \leq s \leq N_{m+1}$. Then we have $s = c\phi^{-1}t^{\phi}$ and $t = cs^{\phi-1}$ for some t with $t \leq N_m$. And by (2.11) and (2.12) we may write

$$D(s) = -c^{-\phi} s^{\phi-2} \int_2^t K(u) \frac{u^{\phi-1}}{\log u}\, du + \frac{s^{\phi-1}}{\log^2 s} B(s), \tag{2.23}$$

where $|B(s)| \le B$ for an absolute constant B. First we show that there is an absolute constant C_1 such that

$$\int_2^t K(u)\frac{u^{\phi-1}}{\log u}\,du = \phi c^{-1}K(t)\frac{s}{\log s} + \vartheta_1(s)\phi c^{-1}\frac{s}{\log^2 s}(C_1+M_m)\left(1+\frac{C_1}{\log s}\right) \tag{2.24}$$

for some function ϑ_1 with $|\vartheta_1(s)| \le 1$. Integrating by parts in (2.23) and replacing $\log t$ by $\phi^{-1}\log s + O(1)$ we obtain

$$\begin{aligned}\int_2^t K(u)\frac{u^{\phi-1}}{\log u}\,du &= \phi c^{-1}K(t)\frac{s}{\log s} + O\left(\frac{s}{\log^2 s}\right)\\ &\quad -\frac{1}{\phi}\int_2^t \frac{u^{\phi-1}}{\log u}\left(uK'(u) - \frac{K(u)}{\log u}\right)du,\end{aligned}$$

and thus in order to obtain (2.24) it is clearly sufficient to show that

$$I := \int_2^t \frac{u^{\phi-1}}{\log u}\left(|uK'(u)| + \frac{|K(u)|}{\log u}\right)du \le \phi^2 c^{-1}\frac{s}{\log^2 s}(C_0+M_m)\left(1+\frac{C_0}{\log s}\right) \tag{2.25}$$

for some absolute constant C_0. Now there are absolute constants A and A_1 with $|K(u)| \le A$ $(u \ge 2)$ and, by Lemma 3, with

$$|uK'(u)| \le \frac{A_1}{\log u} + |K(u) + K(cu^{\phi-1})| \quad (u \ge 2).$$

Thus on appealing to the induction hypothesis we have

$$I \le (A_1+A+M_m)\int_2^t \frac{u^{\phi-1}}{\log^2 u}\,du \le (A_1+A+M_m)\left(\phi^2 c^{-1}\frac{s}{\log^2 s} + O\left(\frac{s}{\log^3 s}\right)\right),$$

whence (2.25) and (2.24) follow. Now we use (2.12) and (2.24) in (2.23) and obtain

$$\frac{s^{\phi-1}}{\log s}K(s) = -\frac{s^{\phi-1}}{\log s}K(t) + \vartheta(s)\frac{s^{\phi-1}}{\log^2 s}(C_2+M_m)\left(1+\frac{C_2}{\log s}\right), \tag{2.26}$$

where C_2 is an absolute constant and ϑ some real function with $|\vartheta(s)| \le 1$. Thus

$$|K(s)+K(cs^{\phi-1})| \le \frac{(C_2+M_m)\left(1+\frac{C_2}{\log s}\right)}{\log s} \le \frac{(C_2+M_m)\left(1+\frac{C_2}{\log N_m}\right)}{\log s} \le \frac{M_{m+1}}{\log s}$$

provided C has been chosen with $C \ge C_2$, whence the proof of the lemma will be complete if we ensure that a sequence of numbers satisfying (2.21) also satisfies

$$M_m = O(m) \quad (m \ge 1). \tag{2.27}$$

Indeed, by Lemma 4 we have $t \geq 3^{\phi^{m-1}}$ if $N_{m-1} \leq t \leq N_m$ ($m \geq 1$), so that $m \leq \log\log t/\log\phi + O(1)$, and thus with (2.27) we will have

$$|K(t) + K(ct^{\phi-1})| \leq \frac{M_m}{\log t} = O\left(\frac{\log\log t}{\log t}\right).$$

In fact we prove that

$$M_m \leq (m+1)C_3 \prod_{0\leq i<m} (1 + C\phi^{-i}) \quad (m \geq 0), \tag{2.28}$$

where C_3 denotes $\max(C, M_0)$. This is sufficient to ensure that (2.27) holds: the product in (2.28) is bounded by an infinite product that converges, since $\phi > 1$. We prove (2.28) by induction on m. It holds for $m = 0$. Assume it is satisfied for some $m \geq 0$. Then by Lemma 4 we have

$$\begin{aligned} M_{m+1} &= (C + M_m)\left(1 + \frac{C}{\log N_m}\right) \leq (C + M_m)\left(1 + \frac{C}{\phi^m}\right) \\ &\leq (1 + m + 1)C_3 \prod_{0\leq i<m} (1 + C\phi^{-i})(1 + C\phi^{-m}). \end{aligned}$$

The proof of Lemma 5 is now complete. □

Now we define the function k by

$$k\left(\frac{\log\log t}{\log\phi}\right) = K(t)$$

and we prove:

LEMMA 6. *For $x \geq 1$ we have*

$$k(x) + k(x-1) = O(x\phi^{-x}). \tag{2.29}$$

Proof. First note that

$$\frac{\log\log(ct^{\phi-1})}{\log\phi} = \frac{\log\log t}{\log\phi} - 1 + O\left(\frac{1}{\log t}\right).$$

Then note that by Lemmas 3 and 5 we have

$$K'(t) = O\left(\frac{\log\log t}{t\log t}\right),$$

which with

$$K'(t) = \frac{k'\left(\frac{\log\log t}{\log\phi}\right)}{t\log t\log\phi}$$

implies that

$$k'\left(\frac{\log\log t}{\log\phi}\right) = O(\log\log t), \tag{2.30}$$

whence, in particular,

$$k\left(\frac{\log\log t}{\log\phi} - 1 + O\left(\frac{1}{\log t}\right)\right) = k\left(\frac{\log\log t}{\log\phi} - 1\right) + O\left(\frac{\log\log t}{\log t}\right).$$

Thus

$$K(t) + K(ct^{\phi-1}) = k\left(\frac{\log\log t}{\log\phi}\right) + k\left(\frac{\log\log t}{\log\phi} - 1\right) + O\left(\frac{\log\log t}{\log t}\right).$$

An appeal to Lemma 5, and the change of variable $x = \log\log t/\log\phi$, conclude the proof. $\square$

Let now $x_0 \geq 0$ be fixed, and consider the sequences

$$x_i := x_0 + 2i \quad \text{and} \quad y_i := k(x_i) \quad (i \geq 0). \tag{2.31}$$

With the help of Lemma 6, by adding and subtracting terms of the form $(-1)^{j+1}k(x_0+j)$ $(j \geq 1)$, it is not difficult to see that $\{y_i\}$ is a Cauchy sequence and thus converges to some real number, which we call $h(x_0)$. Similarly, for $x_0 \geq 1$, y_i' converges to $h(x_0 - 1)$, where $y_i' := k(x_i - 1)$ $(i \geq 0)$. And again by Lemma 6 we see that

$$h(x_0) + h(x_0 - 1) = \lim_{i\to\infty}(k(x_i) + k(x_i - 1)) = 0. \tag{2.32}$$

We need a precise bound for the quantity $|h(x) - k(x)|$.

LEMMA 7. *We have*

$$k\left(\frac{\log\log t}{\log\phi}\right) = h\left(\frac{\log\log t}{\log\phi}\right) + O\left(\frac{\log\log t}{\log t}\right). \tag{2.33}$$

Proof. Put $x = x_0 = \log\log t/\log\phi$. Then with the notation (2.31) we have

$$|h(x) - k(x)| = \lim_{i\to\infty}|y_i - y_0|.$$

Now with Lemma 6 we have

$$\begin{aligned} |y_i - y_0| &\leq |y_1 - y_0| + |y_2 - y_1| + \cdots + |y_i - y_{i-1}| \\ &= O\left(\sum_{i\geq 0}(x+2i)e^{-(x+2i)\log\phi}\right) \\ &= O(xe^{-x\log\phi}), \end{aligned}$$

and the lemma is proved. $\square$

In order to conclude the proof of Theorem 1 it thus remains to prove:

LEMMA 8. *The function h is continuous.*

Proof. Suppose on the contrary that h is not continuous at some x. Then there is a sequence $\epsilon_i \to 0$ with

$$|h(x) - h(x + \epsilon_i)| \geq C > 0,$$

where C is some positive constant. By (2.31), for every positive integer j we have

$$|h(x + 2j) - h(x + 2j + \epsilon_i)| \geq C.$$

Now if we write $x + 2j = \log\log t_j / \log\phi$, then by Lemma 7 we have

$$h(x + 2j) - h(x + 2j + \epsilon_i) = k(x + 2j) - k(x + 2j + \epsilon_i) + A(t_j, i)\frac{\log\log t_j}{\log t_j},$$

where $|A(t_j, i)| \leq A$ for some absolute constant A. But we can choose $j = j_0$ such that for $t = t_{j_0}$ we have

$$A\frac{\log\log t}{\log t} \leq \frac{C}{2},$$

whence

$$|k(x + 2j_0) - k(x + 2j_0 + \epsilon_i)| \geq \frac{C}{2}$$

for every i, which contradicts the continuity of k.

The proof of Theorem 1 is now complete. □

3. Remarks

Remark 2. As we mentioned it in the introduction (Remark 1), Vardi's assertion in his Conjecture 2 that $|h(x)| > 0$ for $x \in (0, 1)$ is not correct, and this follows from $h(0) \neq 0$. We very briefly indicate how this latter fact can be proved. If we write

$$E(t) = \frac{t^{\phi-1}}{\log t}k_1\left(\frac{\log\log t}{\log\phi}\right),$$

then Rémy proved in [Ré] that

$$|k_1(u)| \geq a \quad (u \in [m + \alpha, m + \beta];\ m \geq m_0),$$

where a, α and β are explicit real constants with $\alpha < \beta$ and $a > 0$. By refining the computations needed to achieve that, it is possible to choose the constants a, α and

β in such a way that the additional condition $\alpha < 0 < \beta$ be satisfied. (We obtain $\alpha = -0.0018$ and $\beta = 0.2948$, with $a = 0.0007486$.) Thus $|k_1(m)| \geq a$ for $m \geq m_0$, whence

$$|h(0)| = |h(m)| = \lim_{m\to\infty} |h(m)| = \lim_{m\to\infty} |k_1(m)| \geq a > 0.$$

Remark 3. It should be noted that in Section 2 we prove more than just Theorem 1. In fact we gave an almost complete proof of the following result.

THEOREM 2. *Let E_1 be an integrable real function defined on* $[2, \infty)$ *which is a solution of some approximate functional equation of the type*

$$E_1(t) = -c^{-\phi} t^{\phi-2} \int_2^{ct^{\phi-1}} E_1(u)\, du + O\left(\frac{t^{\phi-1}}{\log^2 t}\right). \tag{3.1}$$

Then the conclusions of Theorem 1 *hold for E_1 instead of E.*

Proof. We have to ensure that the proof of Theorem 1 is valid from equation (2.6) on, if we replace E by E_1. From this point on the only property of E we use (other than equation (2.5)) is Vardi's result (1.2) (which is used in the derivations of equations (2.17) and (2.24)). So we only need to show the following. □

LEMMA 9. *If E_1 is a solution of an equation of type* (3.1), *then*

$$E_1(t) = O\left(\frac{t^{\phi-1}}{\log t}\right). \tag{3.2}$$

Proof of the lemma. Let $N_0, N_1, \ldots$ be a sequence of real numbers as in (2.18), with N_0 large enough to ensure the validity of Lemma 4. We prove by induction on m that if $\ell \in [2, N_m]$, then

$$|E_1(\ell)| \leq A_m \frac{\ell^{\phi-1}}{\log \ell} \tag{3.3}$$

for some increasing sequence of positive real numbers exceeding 1, $A_0, A_1, \ldots$ with

$$A_{m+1} = A_m(1 + C\phi^{-m})$$

for an absolute constant C. This will imply that there is an absolute constant A with $A_m < A$ for every m, whence the lemma. For A_0 large enough, (3.3) holds for $m = 0$ and $m = 1$. Suppose (3.3) holds for some $m \geq 1$, and let $L \in [N_m, N_{m+1}]$. Then $L = c\phi^{-1}\ell^{\phi}$ and $\ell = cL^{\phi-1}$ for some $\ell \in [N_{m-1}, N_m]$. Note that from Lemma 4 we can infer $\log \ell \geq \phi^{m-1}$. For some absolute constants B, B_1, B_2 and B_3 we have

$$|E_1(L)| \leq \frac{1}{c\ell^{\phi-1}} \int_2^{\ell} |E(u)|\, du + B\frac{L^{\phi-1}}{\log^2 L}$$

$$\begin{aligned}
&\leq \frac{A_m}{c\phi\ell^{\phi-1}}\left(1+\frac{B_1}{\log\ell}\right)\frac{\ell^\phi}{\log\ell}+B\frac{L^{\phi-1}}{\log^2 L}\\
&\leq \frac{A_m L^{\phi-1}}{\log L}\left(1+\frac{B_2}{\log\ell}\right)\leq \frac{A_m L^{\phi-1}}{\log L}(1+B_3\phi^{-m})\leq A_{m+1}\frac{L^{\phi-1}}{\log L},
\end{aligned}$$

provided C has been chosen with $C \geq B_3$.

Remark 4. If a function E_1 of the positive integers satisfies a functional equation of the type

$$E_1(n) = -c^{-\phi}n^{\phi-2}\sum_{k\leq cn^{\phi-1}} E_1(k) + O\left(\frac{n^{\phi-1}}{\log^2 n}\right), \tag{3.4}$$

then it is not difficult to show, by an argument similar to that of Lemma 9, that

$$E_1(n) = O\left(\frac{n^{\phi-1}}{\log n}\right).$$

It then easily follows that the extended function $E_1(t) = E_1([t])$ satisfies a functional equation of type (2.5), and thus that Theorem 2 applies to E_1.

It should be noted at this point that if one replaces the error term of (3.4) by $O(n^{\phi-1}/\log^{1+\epsilon} n)$ where $0<\epsilon<1$, then Theorem 2 can still be proved for E_1, the only difference in its conclusions being that the error term in the expression of E_1 in terms of h is now $O(n^{\phi-1}\log\log n/\log^{1+\epsilon} n)$.

In the remark on page 3 of [Va] a relation less precise than (3.4) (without error term and with the symbol $\approx$ instead of $=$) is displayed, followed by an assertion, the most natural interpretation of which appears to be that the function $n^{\phi-1}h(\log\log n/\log\phi)/\log n =: E_2(n)$ is a solution to (3.4) (possibly with a larger error term) when $h(x+1) = -h(x)$. But we just saw that for E_2, with $h(x+1) = -h(x)$, to be an asymptotic solution of (3.4), it is at least necessary that h be continuous.

And in fact there are continuous functions h with $h(x+1) = -h(x)$ such that E_2 is not a solution of (3.4). In order to verify the latter, first note that a necessary condition for a function $E_2(n) = O(n^{\phi-1}/\log n)$ to be a solution of (3.4) is $E_2(n+1)-E_2(n) = O(n^{\phi-1}/\log^2 n)$. Thus, if

$$E_2(n) = \frac{n^{\phi-1}}{\log n}h\left(\frac{\log\log n}{\log\phi}\right),$$

a necessary condition is

$$h\left(\frac{\log\log(n+1)}{\log\phi}\right) - h\left(\frac{\log\log n}{\log\phi}\right) = O\left(\frac{1}{\log n}\right).$$

We can find a sequence of positive integers $n_i \to \infty$ $(i\to\infty)$ such that the integral parts $[\log\log n_i/\log\phi]$ are all of the same parity, and such that the fractional parts

$\delta_i := \{\log\log n_i/\log\phi\}$ decrease to 0 as $i \to \infty$. By making the sequence sparse enough we may also assume that $\epsilon_{i+1} < \delta_i < \epsilon_i$, where $\epsilon_i := \{\log\log(n_i+1)/\log\phi\}$. Thus if we put $h(\delta_i) = 1/n_i$ and $h(\epsilon_i) = 1/n_i + 1/\sqrt{\log n_i}$, we can complete the construction of a continuous function h with $h(x+1) = -h(x)$ and such that

$$h\left(\frac{\log\log(n+1)}{\log\phi}\right) - h\left(\frac{\log\log n}{\log\phi}\right) = \Omega\left(\frac{1}{\sqrt{\log n}}\right).$$

Note that it is even possible to ensure that h be indefinitely differentiable (with the restriction $h^{(k)}(0) = \infty$ for $k = 1, 2, \ldots$).

Remark 5. The error term estimate $O(t^{\phi-1}\log\log t/\log t)$ we obtain in our Theorem 1 is not as good as the error term asserted bound $O(t^{\phi-1}/\log t)$ of Conjecture 2. Here we give an example of a (non trivial) functional equation of type (3.1) for which the better error term bound holds. In this case we can exhibit an explicit expression for the function h (here: g), which in addition permits a rather easy (compared to the argument in [Ré]) derivation of an Ω-estimate as strong as that of Conjecture 1.

THEOREM 3. *There is a unique solution for the exact functional-differential equation*

$$d(t) = -t^{\phi-2}\int_0^{t^{\phi-1}} d(u)\,du + 1. \tag{3.5}$$

This solution satisfies an equation of type (3.1). *Moreover, for $u \geq e$, we have*

$$d(u) = \frac{u^{\phi-1}}{\log u}\,g\left(\frac{\log\log u}{\log\phi}\right) + O\left(\frac{u^{\phi-1}}{\log^2 u}\right), \tag{3.6}$$

where

$$g(x) = -g(x+1) \quad (x \geq 0) \tag{3.7}$$

and

$$g \quad \text{is continuous}, \tag{3.8}$$

and where in addition

$$g \quad \text{is not identically zero}. \tag{3.9}$$

Proof. We first establish the existence of a solution. The argument is similar to that in [Yo, Chapitre 4, p. 151–153]. We are looking for a solution of (3.5) of the form

$$d(t) = \sum_{n\geq 0} e_n(t), \tag{3.10}$$

where for the time being we assume that the sum converges absolutely and uniformly in each bounded interval $[0, t]$. Then by using (3.10) in (3.5) we obtain

$$\sum_{n\geq 0} e_n(t) = 1 - \sum_{n\geq 0} t^{\phi-2} \int_0^{t^{\phi-1}} e_n(u)\,du.$$

We seek a solution satisfying

$$e_0(t) = 1$$

and

$$e_n(t) = -t^{\phi-2} \int_0^{t^{\phi-1}} e_{n-1}(u)\,du \quad (n \geq 1), \tag{3.11}$$

with $e_n(t) = \alpha_n t^{\beta_n}$, and where α_n and β_n are real numbers. An elementary computation yields

$$\begin{aligned}
\alpha_0 &= 1, \quad \beta_0 = 0,\\
\beta_n &= (\beta_{n-1} + 1)(\phi - 1) + (\phi - 2) = (\phi - 1)\beta_{n-1} + (\phi - 1)^3,\\
\alpha_n &= -\frac{\alpha_{n-1}}{\beta_{n-1} + 1}.
\end{aligned}$$

The sequence $\beta_0, \beta_1, \ldots$ satisfies a first degree recurrence relation whose solution is

$$\beta_k = -(\phi - 1)^{k+1} + (\phi - 1) = (\phi - 1)\left(1 - \frac{1}{\phi^k}\right), \tag{3.12}$$

whence

$$\alpha_n = \frac{(-1)^n}{\phi^n} \prod_{k=1}^{n} \frac{1}{\left(1 - \frac{1}{\phi^{k+1}}\right)}. \tag{3.13}$$

Note that as $n \to \infty$ the product in (3.13) remains bounded. Also note that $0 \leq \beta_n \leq \phi - 1$. Now there remains to ensure that with these values of α_n and β_n the sum in (3.10) does indeed converges absolutely and uniformly in each bounded interval $[0, t]$. This follows from

$$\alpha_n t^{\beta_n} = O\left(\frac{1 + t^{\phi-1}}{\phi^n}\right). \tag{3.14}$$

Now we show that the solution of (3.5) is unique. Let d_1 and d_2 be two solutions of (3.5) and consider their difference $\varepsilon := d_1 - d_2$. Let $t_0 \geq 0$ and M be the sup $|\varepsilon(u)|$ for $u \in [0, t_0]$. Then if $t \leq t_0$ we have

$$|\varepsilon(t)| \leq t^{\phi-2} \int_0^{t^{\phi-1}} |\varepsilon(u)|\,du \leq Mt^{\phi-2} \int_0^{t^{\phi-1}} du = Mt^{2\phi-3} = Mt^{(\phi-1)^3}.$$

Now let

$$\phi_k = \beta_{k+1} \quad \text{and} \qquad M_k = M|\alpha_{k+1}| \quad (k \geq 0) \tag{3.15}$$

Then it is easy to see by induction on k, with (3.14) and (3.15), that if $t \leq t_0$ then

$$|\varepsilon(t)| \leq M_k t^{\phi_k} = M|\alpha_{k+1}| t^{\beta_{k+1}} \leq \frac{C}{\phi^{k+1}}$$

for every positive integer k, where C is a positive constant depending only on t_0. Hence $\varepsilon(t) = 0$ if $t \leq t_0$ and $d_1 \equiv d_2 \equiv d$.

Now we write

$$d(t) = \frac{t^{\phi-1}}{\log t} J(t) = \frac{t^{\phi-1}}{\log t} j\left(\frac{\log\log t}{\log \phi}\right) \tag{3.16}$$

and we show that

$$J(t) + J(t^{\phi-1}) = O\left(\frac{1}{\log t}\right). \tag{3.17}$$

Using (3.14) in (3.16) we may write

$$J(t) = \sum_{n \geq -1} \alpha_n t^{-\phi^{-n-1}} \log t, \tag{3.18}$$

where, by convention, $\alpha_{-1} = 0$. Hence we have

$$J(t) + J(t^{\phi-1}) = \sum_{n\geq 0} \left(\alpha_n + \frac{\alpha_{n-1}}{\phi}\right) t^{\phi^{n-1}} \log t = \log t \sum_{n \geq 0} (-1)^n \gamma_n, \tag{3.19}$$

where on appealing to (3.15) we see that

$$\gamma_0 = t^{-\phi^{-1}},$$

and for $n \geq 1$,

$$\begin{aligned} \gamma_n &= \frac{1}{\phi^n} \prod_{k=1}^{n} \left(1 - \frac{1}{\phi^{k+1}}\right)^{-1} \left(1 - \left(1 - \frac{1}{\phi^{n+1}}\right)\right) t^{-\phi^{-n-1}} \\ &= \frac{1}{\phi^{2n+1}} \prod_{k=1}^{n} \left(1 - \frac{1}{\phi^{k+1}}\right)^{-1} t^{-\phi^{-n-1}} \end{aligned} \tag{3.20}$$

When $t > (\phi^2)^{\phi^3}$, the sequence $\{\gamma_n\}$ in (3.17) is unimodal; i.e., there is an integer n_0 such that $\gamma_{n+1} > \gamma_n$ for $n < n_0$ and $\gamma_{n+1} \leq \gamma_n$ for $n \geq n_0$. Indeed $\gamma_0 < \gamma_1$, and for $n \geq 1$,

$$\frac{\gamma_{n+1}}{\gamma_n} = \frac{1}{\phi^2} \frac{x^u}{1-u} =: v(u)$$

where $u = \phi^{-n-2}$ and $x = t^{\phi-1}$; and for x fixed the function $v(u)$ is strictly increasing from ϕ^{-2} to $+\infty$ for $u \in [0, 1)$. It follows that there is a unique $u_0 > 0$ with $v(u_0) = 1$. And thus n_0 is the smallest positive integer satisfying $\phi^{-n_0-2} \leq u_0$. Now with the unimodality of $\{\gamma_n\}$ and (3.19) we see that

$$|J(t) + J(t^{\phi-1})| \leq \gamma_{n_0} \log t. \tag{3.21}$$

The quantities n_0 and u_0 depend on t. As t increases to $+\infty$, n_0 also increases to $+\infty$, and u_0 decreases to 0. Thus from the definition of u_0, i.e. $v(u_0) = 1$, we see that t^{u_0} remains bounded as $t \to \infty$, whence, since $u_0 > 0$, $u_0 = O(1/\log t)$. With (3.20) it follows that

$$\gamma_{n_0} \ll \phi^{-2n_0} t^{-\phi^{-n_0-1}} \ll u_0^2 \ll \frac{1}{\log^2 t},$$

and this with (3.21) implies (3.17).

Now we show that d satisfies an approximate functional-differential equation of type (3.1). As in Lemma 3 we obtain

$$tJ'(t) + J(t) + J(t^{\phi-1}) = O\left(\frac{1}{\log t}\right), \tag{3.22}$$

and with (3.17) this implies

$$J'(t) = O\left(\frac{1}{t \log t}\right), \quad \text{that is,} \quad j'(u) = O(1). \tag{3.23}$$

With (3.5) we have

$$-c^{-\phi} t^{\phi-2} \int_2^{ct^{\phi-1}} d(u)\, du = \frac{d(c^\phi t)}{c} - 1 + O(t^{\phi-2}) \tag{3.24}$$

and, with (3.23) and (3.16),

$$\begin{aligned} \frac{d(c^\phi t)}{c} &= \frac{t^{\phi-1}}{\log t} j\left(\frac{\log\log t}{\log\phi} + O\left(\frac{1}{\log t}\right)\right) + O\left(\frac{t^{\phi-1}}{\log^2 t}\right) \\ &= d(t) + O\left(\frac{t^{\phi-1}}{\log^2 t}\right). \end{aligned} \tag{3.25}$$

Thus Theorem 2 applies to e. But we must ensure that the better estimate (3.6) holds, with (3.7) and (3.8). To that purpose we replace Lemma 5 by

$$J(t) + J(ct^{\phi-1}) = O\left(\frac{1}{\log t}\right), \tag{3.26}$$

which follows from (3.17) and (3.23). Then with (3.23) instead of (2.30) we replace Lemma 6 by

$$j(x) + j(x-1) = O(e^{-x\log\phi}) \tag{3.27}$$

and (with g instead of h) Lemma 7 by

$$j\left(\frac{\log\log t}{\log\phi}\right)=g\left(\frac{\log\log t}{\log\phi}\right)+O\left(\frac{1}{\log t}\right). \tag{3.28}$$

The continuity of g is obtained exactly as in Lemma 8.

There remains to prove (3.9). In fact we prove that

$$g\left(2+\frac{\log\log\phi}{\log\phi}\right)=g(0.47998\cdots)\neq 0. \tag{3.29}$$

We derive a closed formula for $g(2+\log\log\phi/\log\phi)$. As in the proof of (3.17), we show that the series (3.18) for $J(t)$ is an alternating series whose terms in modulus constitute a unimodal sequence. The maximal value of the moduli $|\alpha_n t^{-\phi^{-n-1}}\log t|$ of the terms in (3.18) is reached when n_0 is the smallest integer such that

$$t_{n_0}\geq t,\quad \text{where}\quad t_n=\left(\phi\left(1-\frac{1}{\phi^{n+2}}\right)\right)^{\phi^{n+3}}$$

Now if we write $u_n:=\log\log t_n/\log\phi$ we clearly have

$$u_n=n+3+\frac{\log\log\phi}{\log\phi}+o(1), \tag{3.30}$$

and we may write

$$\begin{aligned} j(u_n)=J(t_n) &= \sum_{\ell=0}^{\infty}\alpha_\ell t_n^{-\phi^{-\ell-1}}\log t_n \\ &= \sum_{\ell=-n}^{\infty}\frac{\alpha_{\ell+n}}{\alpha_n}t_n^{-\phi^{-\ell-n-1}}\alpha_n\log t_n =: \sum_{\ell=-\infty}^{\infty}(-1)^{n+\ell}a_{\ell,n} \end{aligned} \tag{3.31}$$

(where $a_{\ell,n}=0$ if $\ell<-n$). It is not difficult to see that

$$(-1)^\ell\frac{\alpha_{\ell+n}}{\alpha_n}=\frac{1}{\phi^\ell}\lambda_{\ell,n},\qquad t_n^{-\phi^{-\ell-n-1}}=\phi^{-\phi^{2-\ell}}\mu_{\ell,n},$$

and

$$(-1)^n\alpha_n\log t_n=\Pi\phi^3\log\phi\,\nu_n,\quad \text{where}\quad \Pi:=\prod\left(1-\phi^{-k-1}\right)^{-1},$$

and where $\lambda_{\ell,n}$, $\mu_{\ell,n}$ and ν_n converge, uniformly in ℓ, to 1 as $n\to\infty$. It follows that the terms $a_{\ell,n}$ of (3.31) converge uniformly in ℓ to $\Pi\phi^{3-\ell-\phi^{2-\ell}}\log\phi$, whence

$$\lim_{n\to\infty}(-1)^n j(u_n)=\Pi\phi^3\log\phi\sum_{\ell=-\infty}^{\infty}(-1)^\ell\phi^{-\ell-\phi^{2-\ell}}=0.001289257\cdots.$$

Now by (3.28) we have $j(u_n) = g(u_n) + O(1/\log t_n)$, and thus, with (3.30),

$$0.001289257\cdots = \lim_{n\to\infty}(-1)^n j(u_n) = \lim_{n\to\infty}(-1)^n g(u_n) = -g\left(2 + \frac{\log\log\phi}{\log\phi}\right),$$

and (3.29) is proved. □

Remark 6. It is easy to see that if λ is any real number the exact functional-differential equation

$$d_\lambda(t) = -t^{\phi-2}\int_0^{t^{\phi-1}} d_\lambda(u)\,du + \lambda \tag{3.32}$$

has exactly one solution, which is given by

$$d_\lambda(t) = \lambda d_1(t), \tag{3.33}$$

where $d_1 = d$ is the solution of (3.5). Returning to the error term E of (1.1), it is tempting to hope that E behaves similarly as d_λ for some λ, and in particular that the functions h and g have the same zeros. But experimental data, although supporting the conjecture that there is some β with $0 < \beta < 1$ such that $|g(x)| > 0$ for $x \in (\beta, 1+\beta)$, also strongly indicates that probably β is distinct from the α of Conjecture 3. In the accompanying figure the dotted curve represents the function k_1 of Remark 2 (approximating the function h of Theorem 1), and the continuous line represents the function $d_{\lambda_0}(t)$ of (3.33), where $\lambda_0 = 1.054559132\cdots$ is chosen in such a way that the amplitudes of both functions are equal. Note that it appears unlikely that the limit function h will be a translation of d_{λ_0}. In fact the following result shows that many very different functions E_1 can satisfy an approximate functional equation of type (3.1).

THEOREM 4. *If f is a real or complex-valued function defined on the real numbers, satisfying*

$$f(x+1) = -f(x) \quad (x \ge 0),$$

and if f has a Fourier series representation

$$f(x) = \sum_{k\ \text{odd}} a_k e^{\pi i k x}$$

with the property that $\sum_{k\ \text{odd}} k|a_k| < \infty$, then the function

$$E_1(t) = \frac{t^{\phi-1}}{\log t} f\left(\frac{\log\log t}{\log\phi}\right)$$

satisfies an approximate functional equation of type (3.1).

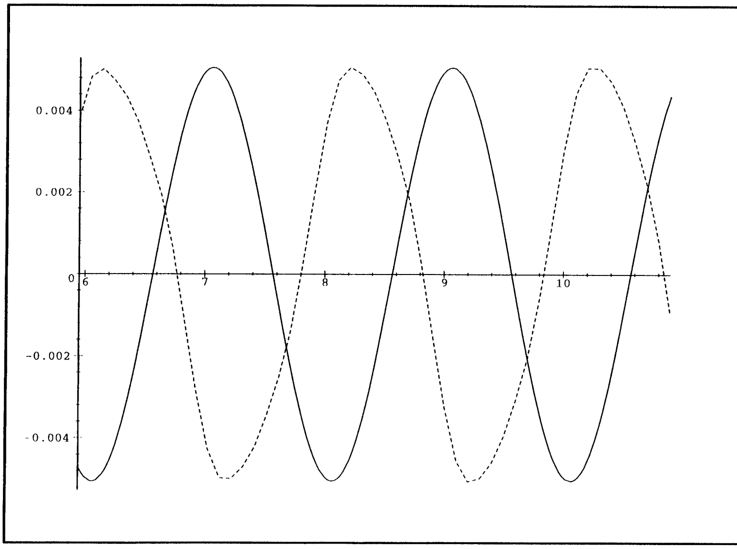

Figure 1

Proof. Let E_1 be as in the theorem. Then

$$I := -c^{-\phi}t^{\phi-2}\int_2^{ct^{\phi-1}} E_1(u)\,du = -c^{-\phi}t^{\phi-2}\sum_{k\text{ odd}} a_k \int_2^{ct^{\phi-1}} u^{\phi-1}(\log u)^{\frac{\pi ik}{\log\phi}-1}\,du.$$

Now the last integral is

$$\begin{aligned}
&\frac{u^\phi}{\phi}(\log u)^{\frac{\pi ik}{\log\phi}-1}\Big|_2^{ct^{\phi-1}} - \frac{1}{\phi}\int_2^{ct^{\phi-1}} u^{\phi-1}\left(\frac{\pi ik}{\log\phi}-1\right)(\log u)^{\frac{\pi ik}{\log\phi}-2}\,du\\
&\quad= \frac{c^\phi t}{\phi}(\log(ct^{\phi-1}))^{\frac{\pi ik}{\log\phi}-1} + O\left(\frac{tk}{\log^2 t}\right)\\
&\quad= c^\phi\frac{t}{\log t}\left(\frac{\log t}{\phi}\right)^{\frac{\pi ik}{\log\phi}} + O\left(\frac{tk}{\log^2 t}\right)\\
&\quad= c^\phi\frac{t}{\log t}e^{\pi ik\left(\frac{\log\log t}{\log\phi}-1\right)} + O\left(\frac{tk}{\log^2 t}\right).
\end{aligned}$$

Thus

$$I = -\frac{t^{\phi-1}}{\log t}\sum_{k\text{ odd}} a_k e^{\pi ik\left(\frac{\log\log t}{\log\phi}-1\right)} + O\left(\frac{t^{\phi-1}}{\log^2 t}\right)$$

$$= \frac{t^{\phi-1}}{\log t} \sum_{k \text{ odd}} a_k e^{\pi i k\left(\frac{\log\log t}{\log \phi}\right)} + O\left(\frac{t^{\phi-1}}{\log^2 t}\right),$$

and the theorem is proved. □

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Y.-F. S. Pétermann, Université de Genève, Section de Mathématique 2-4, rue du Lièvre, C.P. 240, 1211 Genève 24, Suisse
peterman@ibm.unige.ch

Jean-Luc Rémy, Centre de Recherche en Informatique de Nancy (CNRS) et INRIA Lorraine, B.P. 239, 54506 Vandœuvre-lès-Nancy Cedex, France