ILLINOIS JOURNAL OF MATHEMATICS
Volume 42, Number 3, Fall 1998

AN INVARIANT MEAN VALUE PROPERTY IN THE POLYDISC

JAESUNG LEE

I. Introduction

In [AFR], the authors showed that:

If f is a bounded function on an n-dimensional unit ball $B_n \subset \mathbf{C}^n$ satisfying

$$\int_{B_n} f \circ \psi \, dm = f(\psi(0)) \quad \text{for every } \psi \in \operatorname{Aut}(B_n) \tag{1}$$

(where m is the normalized Lebesgue measure) then f is M-harmonic.

And if $f \in L^1(B_n, m)$ satisfies (1), then f is M-harmonic if and only if $n \leq 11$.

In this paper, we answer the question of whether the similar phenomenon happens in the n-dimensional polydisc D^n.

Following Definition 2.1.1 from [Ru1], we say that $f \in \mathbf{C}^2(D^n)$ is n-harmonic if

$$\Delta_1 f = \Delta_2 f = \cdots = \Delta_n f = 0.$$

We can see that if $f \in \mathbf{C}^2(D^n)$ is n-harmonic then f satisfies the invariant volume mean value property, i.e.,

$$\int_D \cdots \int_D f \circ \psi \, dm \cdots dm = f(\psi(0, \ldots, 0)), \quad \forall\, \psi \in \operatorname{Aut}(D^n) \tag{2}$$

since $f \circ \psi$ is n-harmonic and thus satisfies the ordinary volume mean value property.

This paper is about the converse of the above statement , asking if $f \in L^p(D^n)$ satisfies (2), is f is n-harmonic?

Furstenberg [Fur] has already given a positive answer in the space which includes the unit ball and the polydisc of all dimension when $p = \infty$, using the methods of symmetric spaces. But his proof is not very widely known to analysts. Even in recent years, many papers with related results such as [AC], [Eng], [AFR] were written without noticing or mentioning the results of Funstenberg.

In this paper, we get the proof in the case when $p = \infty$ by using the results of [AFR], giving a completely independent analytic proof of Furstenberg's result in the case of the polydisc (Theorem 3.1).

Received April 23, 1997.
1991 Mathematics Subject Classification. Primary 31B05; Secondary 30C05, 47B38.
Research partially supported by TGRC-KOSEF.

When $1 \leq p < \infty$, we show that (2) does not imply that f is n-harmonic even when $n = 2$ (Theorem 2.1). Indeed, when $1 \leq p < \infty$ we show there are uncountably many joint eigenfunctions of invariant Laplacians in $L^p(D^2, m \times m)$ which satisfy the invariant volume mean value property.

For $n \geq 2$, we introduce the linear operator defined by

$$(Bf)(z_1, \ldots, z_n) = \int_D \cdots \int_D f(\varphi_{z_1}(x_1), \ldots, \varphi_{z_n}(x_n))\, dm(x_1) \cdots dm(x_n)$$

for $f \in L^1(D^n, m \times \cdots \times m)$, where $\varphi_a \in \mathrm{Aut}(D)$ is defined by

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}.$$

Then from the structure of the automorphisms of the polydisc (p. 167 of [Ru1]) and the rotation invariance of m, it follows that $f \in L^1(D^n)$ satisfies (2) if and only if $Bf = f$.

II. $Bf = f$ for $f \in L^p(D^2, m \times m)$ when $1 \leq p < \infty$

In [AFR], the authors show that functions in $L^1(B_n)$ which satisfy the invariant mean value property are M-harmonic iff $n \leq 11$. But in the bidisc $D \times D$, the analogue is not true. The next theorem states this.

THEOREM 2.1. *For $1 \leq p < \infty$, there exists $f \in L^p(D^2, m \times m)$ such that $Bf = f$ and f is not 2-harmonic.*

Before proving this, we need some preliminaries. Throughout this paper we use the following notations as definitions. These notations agree with those in [AFR], [Ru2].

Definition 2.2. We define $\tilde{\Delta}_1, \tilde{\Delta}_2$ as the invariant Laplacians with respect to the first and second variable respectively; i.e.,

$$(\tilde{\Delta}_1 f)(z, w) = (1 - |z|^2)^2 (\Delta_1 f)(z, w) \text{ for } f \in \mathbf{C}^2(D^2).$$

For $\lambda, \mu \in \mathbf{C}$, we let $\alpha, \beta \in \mathbf{C}$ be such that $\lambda = -4\alpha(1 - \alpha)$, $\mu = -4\beta(1 - \beta)$ and define

$$X_{\lambda,\mu} = \{f \in \mathbf{C}^2(D^2) \mid \tilde{\Delta}_1 f = \lambda f \text{ and } \tilde{\Delta}_2 f = \mu f\}.$$

We also define g_α, the radial function on D, by

$$g_\alpha(r) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1 - r^2}{|1 - re^{i\theta}|^2} \right)^\alpha d\theta$$

and define

$$\sum\nolimits_p = \left\{\alpha \in \mathbf{C} \mid -\frac{1}{p} < \operatorname{Re}\alpha < 1 + \frac{1}{p}\right\}, \text{ for } 1 \le p < \infty$$
$$\sum\nolimits_\infty = \{\alpha \in \mathbf{C} \mid 0 \le \operatorname{Re}\alpha \le 1\}.$$

LEMMA 2.3. *For g_α as defined above, we get*

$$\int_D g_\alpha \, dm = \frac{\pi\alpha(1-\alpha)}{\sin(\pi\alpha)}.$$

Proof. We use the formula

$$(1-z)^{-\alpha} = \sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha)}{k!\,\Gamma(\alpha)} z^k$$

and polar coordinates to obtain

$$\begin{aligned}
\int_D g_\alpha \, dm &= \int_0^1 2r(1-r^2)^\alpha \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1-re^{i\theta}|^{2\alpha}} \, d\theta \, dr \\
&= \int_0^1 2r(1-r^2)^\alpha \frac{1}{2\pi} \int_0^{2\pi} (1-re^{i\theta})^{-\alpha}(1-re^{-i\theta})^{-\alpha} \, d\theta \, dr \\
&= \sum_{k=0}^{\infty} \frac{\Gamma^2(k+\alpha)}{(k!)^2\Gamma^2(\alpha)} \int_0^1 (1-r^2)^\alpha r^{2k} 2r \, dr \\
&= \sum_{k=0}^{\infty} \frac{\Gamma^2(k+\alpha)}{(k!)^2\Gamma^2(\alpha)} \frac{\Gamma(\alpha+1)k!}{\Gamma(k+\alpha+1)} \\
&= \frac{1}{\alpha+1} F(\alpha, \alpha; \alpha+2; 1) \qquad (F \text{ is the hypergeometric function}) \\
&= \alpha\Gamma(\alpha)\Gamma(2-\alpha) \\
&\quad \text{(by the formula} \\
&\quad F(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \, dt) \\
&= \alpha(1-\alpha)\Gamma(\alpha)\Gamma(1-\alpha) \\
&= \alpha(1-\alpha)\frac{\pi}{\sin(\pi\alpha)}.
\end{aligned}$$

This completes the proof. □

LEMMA 2.4. *For $1 \le p \le \infty$,*

$$L^p(D^2, m \times m) \cap X_{\lambda,\mu} \ne \{0\} \quad \textit{iff} \quad \alpha \in \sum\nolimits_p \ \textit{and}\ \beta \in \sum\nolimits_p.$$

Proof. Let $f \in X_{\lambda,\mu}$. Then the radialization

$$(Rf)(z,w) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} f(ze^{i\theta}, we^{i\xi})\,d\theta\,d\xi$$

belongs to $X_{\lambda,\mu}$ and by 4.2.3 of [Ru 2], Rf is a constant multiple of $g_\alpha(z)g_\beta(w)$. Hence we conclude that

$$X_{\lambda,\mu} \cap L^p \neq \{0\} \quad \text{if and only if} \quad g_\alpha \in L^p \text{ and } g_\beta \in L^p.$$

By 1.4.10 of [Ru2], if Re $\alpha \le \frac{1}{2}$ and $\alpha \neq \frac{1}{2}$ then

$$\int_0^{2\pi} \frac{1}{|1 - ze^{i\theta}|^{2\alpha}}\,d\theta$$

is bounded in D while

$$\int_0^{2\pi} \frac{1}{|1 - ze^{i\theta}|}\,d\theta \approx \log\frac{1}{1-|z|^2}.$$

Since $g_\alpha = g_{1-\alpha}$, it follows that $g_\alpha \in L^p(D)$ if and only if $\alpha \in \Sigma_\infty$ and the proof is complete. □

LEMMA 2.5. For $1 \le p < \infty$, the equation

$$\frac{\pi\alpha(1-\alpha)}{\sin(\pi\alpha)} \cdot \frac{\pi\beta(1-\beta)}{\sin(\pi\beta)} = 1$$

has infinitely many pairs of solutions (α, β) in $\sum_p \times \sum_p$.

Proof. Define h on $\sum_p$ by

$$h(z) = \frac{\pi z(1-z)}{\sin(\pi z)} \qquad \text{for} \quad z \in \sum\nolimits_p. \tag{1}$$

Then it is easy to check that

(i) h is holomorphic in $\sum_p$ and
(ii) $h(1) = 1$.

Thus by the open mapping theorem for a holomorphic function, we can choose an open ball $B(1,\epsilon)$ with $B(1,\epsilon) \subset \sum_p$, and $h(B(1,\epsilon)) \subset \sum_p$. And since $h(B(1,\epsilon))$ is an open neighborhood of the point $z = 1$, it contains an arc of the unit circle around $z = 1$, namely

$$L_\delta = \{e^{i\theta} \mid -\delta < \theta < \delta\}$$

which consists of uncountably many pairs of $(\alpha, \beta) \in \sum_p \times \sum_p$ satisfying $h(\alpha) = 1/h(\beta)$. In other words,

$$\frac{\pi\alpha(1-\alpha)}{\sin(\pi\alpha)} \cdot \frac{\pi\beta(1-\beta)}{\sin(\pi\beta)} = 1.$$

This ends the proof. □

Now we are ready to prove Theorem 2.1.

2.6. *Proof of Theorem* 2.1 If $f \in X_{\lambda,\mu}$, then by 4.2.4 of [Ru2] we have

$$\int_T f(\varphi_z(r\eta),\ \varphi_w(t\xi))\, d\sigma(\eta) = g_\alpha(r) f(z, \varphi_w(t\xi))$$

where T is the unit circle. Thus by repeating the previous step we get

$$\iint_{T^2} f(\varphi_z(r\eta),\ \varphi_w(t\xi)) d\sigma(\xi)\, d\sigma(\eta) = g_\alpha(r)\, g_\beta(t)\, f(z, w). \tag{1}$$

Using polar coordinates, we get

$$\begin{aligned}
&\int\int_{D^2} f(\varphi_z(x), \varphi_w(y))\, dm(x)\, dm(y)\\
&= \int_0^1 \int_0^1 2r\ 2t \int\int_{T^2} f(\varphi_z(r\eta), \varphi_w(t\xi))\, d\sigma(\eta)\, d\sigma(\xi)\, dr\, dt\\
&= \int_0^1 2r\ g_\alpha(r)\, dr \int_0^1 2t\ g_\beta(t)\, dt \cdot f(z, w) \qquad \text{by (1)}\\
&= \int_D g_\alpha\, dm \int_D g_\beta\, dm \cdot f(z, w).
\end{aligned}$$

In other words, for $f \in X_{\lambda,\mu} \bigcap \mathrm{L}^p(D^2, m \times m)$ we have

$$(Bf)(z, w) = \int_D g_\alpha\, dm \int_D g_\beta\, dm \cdot f(z, w). \tag{2}$$

Hence by Lemma 2.3, we get

$$(Bf)(z, w) = h(\alpha)h(\beta)f(z, w) \tag{3}$$

where

$$h(\alpha) = \frac{\pi\alpha(1-\alpha)}{\sin(\pi\alpha)}.$$

But by Lemma 2.4 and Lemma 2.5 there are infinitely many $(\lambda, \mu)'s$ satisfying $h(\alpha)h(\beta) = 1$ while every $f \in X_{\lambda,\mu} \bigcap L^p$ satisfies $Bf = f$. This completes the proof of Theorem 1. □

Remark 2.7. The analogue of Theorem 2.1 for $f \in L^\infty(D^2)$ is not true. The reason is that there is no bounded joint eigenfunction of $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ which satisfies $Bf = f$ (other than the 2-harmonic one.). By Lemma 2.4, it is enough to show that

$$(h(\alpha)\, h(\beta) =)\quad \frac{\pi\alpha(1-\alpha)}{\sin(\pi\alpha)} \cdot \frac{\pi\beta(1-\beta)}{\sin(\pi\beta)} = 1$$

has no solution (α, β) in $\sum_\infty \times \sum_\infty$, except if both α, β are either 0 or 1.

To prove that assertion: For $0 \le \operatorname{Re} z \le 1$,

$$\begin{aligned} \frac{\sin(\pi z)}{\pi z(1-z)} &= \frac{1}{1-z}\frac{\sin(\pi z)}{\pi z} = (h(z))^{-1} \\ &= \frac{1}{1-z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)\left(1-\frac{z}{n}\right) \\ &= \prod_{n=1}^{\infty}\left(1+\frac{z(1-z)}{n(n+1)}\right). \end{aligned} \tag{1}$$

Thus when $z = x$ is real, $0 \le x \le 1$, then

$$\frac{\sin(\pi x)}{\pi x(1-x)} > 1 \qquad \text{except when} \quad x = 0 \text{ or } x = 1.$$

When $z = x + iy$, from (1) we have

$$\frac{\sin(\pi z)}{\pi z(1-z)} = \prod_{n=1}^{\infty}\left(1 + \frac{x(1-x)+y^2}{n(n+1)} + i\frac{y(1-2x)}{n(n+1)}\right).$$

Thus for fixed x,

$$\left|\frac{1}{h}(x+iy)\right| \quad \nearrow \quad \infty \quad \text{as} \quad |y| \to \infty \tag{2}$$

Thus from (1) and (2) we get $|h(z)| < 1$ on $\sum_\infty$, except when $z = 0$ or $z = 1$. This ends the proof. □

III. $Bf = f$ for $f \in L^\infty(D^n)$

THEOREM 3.1. *If $f \in L^\infty(D^n)$ satisfies $Bf = f$ then f is n-harmonic.*

We will give a proof using the result of [AFR], which states *if $u \in L^\infty(D)$ satisfies $Bu = u$, then u is harmonic*, together with the main theorem of [KT].

To prove the above result, we need the following.

Definition 3.2. For $u \in L^1(D, m)$, $z \in D$ we define

$$(Tu)(z) = \int_D u(\varphi_z(x))\, dm(x) \tag{1}$$

by replacing x by $\varphi_z(x)$, we obtain

$$(Tu)(z) = \int_D u(x) K_1(z, x)\, dm(x) \tag{2}$$

where

$$K_1(z, x) = \frac{(1 - |z|^2)^2}{|1 - \bar{z}x|^4}. \tag{3}$$

Here we can see that

$$\int_D K_1(z, x)\, dm(x) = 1, \ \forall z \in D. \tag{4}$$

Now let T^n be the iteration of T, n times; then by induction we can write

$$(T^n u)(z) = \int_D u(x) K_n(z, x)\, dm(x) \tag{5}$$

where

$$\int_D K_n(z, x)\, dm(x) = 1, \ \forall z \in D, n \geq 1. \tag{6}$$

Let μ be a measure on D defined by

$$d\mu(z) = (1 - |z|^2)^{-2}\, dm(z).$$

Then by 2.2.6 of [Ru2],

$$\int_D u\, d\mu = \int_D u \circ \psi\, d\mu$$

for $u \in L^1(D, \mu)$ and $\psi \in \mathrm{Aut}(D)$. The advantage of using the invariant measure μ is that even though μ is not a finite measure on D, the space $L^\infty(D, \mu)$ is the same as $L^\infty(D, m)$ (i.e., μ is a measure equivalent to m on D). Thus we consider $L^\infty(D, m)$ as the dual space of $L^1(D, d\mu)$ on which the operator T has a nice behavior. (see Lemma 3.3)

Finally, we denote the space $L^p_R(D^n)$ as the subspace of $L^p(D^n)$ which consists of all radial functions, i.e.,

$$L^p_R(D^n) = \{f \in L^p(D^n) \mid f(z_1, \ldots, z_n) = f(|z_1|, \ldots, |z_n|), \text{ for any } (z_1, \ldots, z_n) \in D^n\}.$$

LEMMA 3.3. *Let* $1 \le p \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ (*if* $p = 1$ *then* $q = \infty$ *and vice versa*).

(a) T *is a bounded operator on* $L^p(D, \mu)$ *with* $\|T\|_p \le 1$.
(b) *For* $u \in L^p(D, \mu)$, $v \in L^q(D, \mu)$ *we have*

$$\int_D Tu \cdot v \, d\mu = \int_D u \cdot Tv \, d\mu.$$

Proof. (a) Let $u \in L^1(D, \mu)$. Then

$$\begin{aligned} \|Tu\|_1 &= \int_D |Tu(z)| \, d\mu(z) \\ &\le \int_D \int_D |u(x)| K_1(z, x) \, dm(x) \, d\mu(z) \\ &= \int_D |u(x)| \int_D K_1(x, z) \, dm(z) \, d\mu(x) \quad \text{by Fubini} \\ &= \|u\|_1 \end{aligned}$$

(since $\int_D K_1(x, z) \, dm(z) \equiv 1$.) Hence

$$\|T\|_1 \le 1. \tag{1}$$

Let $v \in L^\infty(D)$. Then

$$\begin{aligned} \|Tv\|_\infty &= \sup_{z \in D} \left| \int_D v(x) K_1(z, x) \, dm(x) \right| \\ &\le \|v\|_\infty \sup_{z \in D} \left| \int_D K_1(z, x) \, dm(x) \right| = \|v\|_\infty. \end{aligned}$$

Thus

$$\|T\|_\infty \le 1. \tag{2}$$

By (1), (2) and the Riesz-Thorin Interpolation Theorem we get (a).
(b)

$$\begin{aligned} \int_D T|u||v| \, d\mu &\le \|T|u|\|_p \|v\|_q \\ &\le \|u\|_p \|v\|_q < \infty \quad \text{by (a)} \end{aligned}$$

Thus we can use Fubini's Theorem:

$$\begin{aligned} \int_D u(z)(Tv)(z) \, d\mu(z) &= \int_D u(z) \int_D v(x) K_1(z, x) \, dm(x) \, d\mu(z) \\ &= \int_D v(x) \int_D u(z) K_1(x, z) \, dm(z) \, d\mu(x) \\ &= \int_D v(x)(Tu)(x) \, d\mu(x) \end{aligned}$$

This proves (b) . □

Note. In (a), actually $\|T\|_1 = \|T\|_\infty = 1$ since

$$\int_D u\,d\mu = \int_D Tu\,d\mu$$

when $0 \le u \in L^1(D, \mu)$ and $Tu = u$ when u is bounded and harmonic.

LEMMA 3.4. *Let $f \in L^1(D^n, m \times \cdots \times m)$. Then for any $\psi \in \mathrm{Aut}(D^n)$ we have*

$$B(f \circ \psi) = (Bf) \circ \psi.$$

Proof. It is enough to prove the lemma when

$$\psi(z_1, \ldots, z_n) = (\psi_1(z_1), \ldots, \psi_n(z_n))$$

for some $\psi_1, \ldots, \psi_n \in \mathrm{Aut}(D)$.

For $z_1, \ldots, z_n \in D$ there exist $\theta_1, \ldots, \theta_n \in [0, 2\pi)$ such that

$$\varphi_{\psi_k(z_k)} \circ \psi_k \circ \varphi_{z_k} = e^{i\theta_k}, \quad k = 1, 2, \ldots, n$$

since these antomorphisms take 0 to 0. Thus

$$\begin{aligned}
&B(f \circ \psi)(z_1, \ldots, z_n) \\
&= \int_D \cdots \int_D (f \circ \psi)(\varphi_{z_1}(x_1), \ldots, \varphi_{z_n}(x_n))\, dm(x_1) \cdots dm(x_n) \\
&= \int_D \cdots \int_D f(\varphi_{\psi_1(z_1)}(e^{i\theta_1}x_1), \ldots, \varphi_{\psi_n(z_n)}(e^{i\theta_n}x_n))\, dm(x_1) \cdots dm(x_n) \\
&= \int_D \cdots \int_D f(\varphi_{\psi_1(z_1)}(x_1), \ldots, \varphi_{\psi_n(z_n)}(x_n))\, dm(x_1) \cdots dm(x_n) \\
&= [(Bf) \circ \psi](z_1, \ldots, z_n)
\end{aligned}$$

This ends the proof. □

LEMMA 3.5. *Let $u \in L^1_R(D, \mu)$. Then*

$$\lim_{n\to\infty} \int_D |T^n u(z)|\, d\mu(z) = 0 \quad \textit{if and only if} \quad \int_D u\, d\mu = 0.$$

Proof. The "only if" part is obvious from the fact that

$$\int_D T^n u\, d\mu = \int_D u\, d\mu \quad \text{for all} \quad n \ge 1.$$

On the other hand, under the convolution

$$(u * v)(z) = \int_D u(\varphi_z(x)) v(x)\, d\mu(x), \quad u, v \in L^1_R(\mu),$$

$L^1_R(\mu)$ is a commutative Banach algebra with the maximal ideal space

$$\Sigma_\infty = \{0 \le \operatorname{Re}\alpha \le 1\},$$

whose Gelfand transform is defined by

$$\hat{u}(\alpha) = \int_D u(z) g_\alpha(z)\, d\mu(z) \quad \text{for } u \in L^1_R(\mu) \text{ and } \alpha \in \Sigma_\infty.$$

Since

$$Tu = u * q \quad \text{where } q(z) = (1 - |z|^2)^2 \in L^1_R(\mu)$$

the spectrum of T on $L^1_R(\mu)$ is

$$\hat{q}(\Sigma_\infty) = \left\{ \int_p g_\alpha\, dm \mid \alpha \in \Sigma_\infty \right\} = h(\Sigma_\infty)$$

by Lemma 2.3, where

$$h(z) = \frac{\pi z(1-z)}{\sin(\pi z)}.$$

Thus in view of Remark 2.7, we get

$$|h(\Sigma_\infty)| < 1 \quad \text{on } \Sigma_\infty \backslash \{0, 1\}$$

while $h(0) = h(1) = 1$. Hence we showed that T *is a linear contraction on* $L^1_R(\mu)$ *whose spectrum intersects the unit circle only at one point* $z = 1$. Now we apply Theorem 1 of [KT] to the operator T on $L^1_R(\mu)$ to get

$$\lim_{n\to\infty} \| T^n(I - T) \| = 0 \quad \text{on } L^1_R(\mu)$$

which implies that

$$\lim_{n\to\infty} \int_D |T^n u|\, d\mu = 0 \quad \text{for all} \quad u \in (I - T)L^1_R(\mu) \tag{1}$$

Now let X be the subspace of $L^1_R(\mu)$ defined by

$$X = \left\{ u \in L^1_R(\mu) \mid \int_D u\, d\mu = 0 \right\}.$$

Then obviously

$$(I - T)L^1_R \subset X.$$

Hence from (1), the proof is complete when we show that $(I - T)L^1_R$ is dense in X. Now let $w \in L^\infty_R(D)$ satisfy

$$\int_D (v - Tv) \cdot w\, d\mu = 0 \quad \text{for every } v \in L^1_R(D, \mu).$$

Then by 3.3 (b) we get

$$\int_D v \cdot (w - Tw)\, d\mu = 0, \quad \forall v \in L^1_R(D, \mu).$$

Thus $w = Tw$ and, by [AFR], w is radial harmonic, hence a constant. Therefore we get

$$\int_D u \cdot w\, d\mu = 0, \quad \text{for every } u \in X.$$

By the Hahn-Banach Theorem, this implies that $(I - T)L^1_R$ is dense in X. This proves the lemma. □

3.6. *Proof of Theorem* 3.1 The proof is divided by two parts; the radial case and the general case.

Step (1). The radial case.

We will prove the radial case of Theorem 3.1 using induction on n where n is the dimension of the polydisc.

When $n = 1$, if $u \in L^\infty_R(D)$ satisfies $Tu = u$, then by [AFR], u is a constant.

Now assume $Bf = f$ for $f \in L^\infty_R(D^n)$ implies that f is constant. Choose $g \in L^\infty_R(D^{n+1})$ such that $Bg = g$. Fix $w = (w_1, \ldots, w_n) \in D^n$ and for $m \geq 1$ define $(g_w)_m \in L^\infty_R(D)$ by

$$(g_w)_m(z) = \int_D \cdots \int_D g(z, y_1, \ldots, y_n) K_m(w_1, y_1) \cdots K_m(w_n, y_n)\, dm(y_1) \cdots dm(y_n) \tag{1}$$

(K_m is defined in 3.2 (5).) Then for any $m \geq 1$, $\|(g_w)_m\|_\infty \leq \|g\|_\infty$.

$$\begin{aligned}[T^m (g_w)_m](z) &= \int_D \cdots \int_D g(z, y_1, \ldots, y_n) K_m(w_1, y_1) \\ &\quad \cdots K_m(w_n, y_n)\, dm(y_1) \cdots dm(y_n) \\ &= (B^m g)(z, w_1, \ldots, w_n). \end{aligned} \tag{2}$$

Now pick $u \in L^1_R(D, \mu)$ satisfying

$$\int_D u\, d\mu = 0.$$

Then we have

$$\begin{aligned}\int_D u(z) g(z, w)\, d\mu(z) &= \int_D u(z)(B^m g)(z, w)\, d\mu(z) \qquad (w \text{ is fixed}) \\ &= \int_D u \cdot (T^m (g_w)_m)\, d\mu \qquad \text{by (2)} \\ &= \int_D T^m u \cdot (g_w)_m\, d\mu \qquad \text{by 3.3 (b)}\end{aligned}$$

Hence

$$\left|\int_D u(z)g(z,w)\,d\mu(z)\right| \le \|G\|_\infty \left|\int_D T^m u\,d\mu\right| \qquad \text{for any } m \ge 1.$$

But by Lemma 3.5,

$$\lim_{m\to\infty}\left|\int_D T^m u\,d\mu\right| = 0.$$

Hence

$$\int_D u(z)g(z,w)\,d\mu(z) = 0$$

for every $u \in L^1_R(D,\mu)$ with

$$\int_D u\,d\mu = 0.$$

This means for every fixed $w \in D^n$, $g(z,w)$ is a constant. Hence there exists $f \in L^\infty_R(D^n)$ such that $g(z,w) = f(w)$ and $Bg = g$ implies that $Bf = f$. Now, by the assumption, f is a constant. Therefore g is a constant.

Step (2). The general case.

Let $f \in L^\infty(D^n)$ satisfy $Bf = f$. Consider Rf, the radialization of f, defined by

$$(Rf)(z_1,\dots,z_n) = \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} f(z_1e^{i\theta_1},\dots,z_ne^{i\theta_n})\,d\theta_1\cdots d\theta_n.$$

Since both R and B are contraction on $L^\infty(D^n)$ we can use Fubini to get

$$B(Rf) = R(Bf) = Rf.$$

Thus by step (1), Rf is a constant. This means

$$f(0,\dots,0) = \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} f(x_1e^{i\theta_1},\dots,x_ne^{i\theta_n})\,d\theta_1\cdots d\theta_n \tag{3}$$

Now pick $z = (z_1,\dots,z_n) \in D^n$ and let $\psi \in \mathrm{Aut}(D^n)$ be defined by

$$\psi(x_1,\dots,x_n) = (\varphi_{z_1}(x_1),\dots,\varphi_{z_n}(x_n)).$$

Then since $B(f\circ\psi) = (Bf)\circ\psi = f\circ\psi$ (Lemma 3.4), (3) remains true when we replace f by $f\circ\psi$; i.e.,

$$f(z_1,\dots,z_n) = \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} f(\varphi_{z_1}(xe^{i\theta_1}),\dots,\varphi_{z_n}(xe^{i\theta_n}))d\theta_1\cdots d\theta_n \tag{4}$$

for any $x_1, \ldots, x_n \in D$. Now put $x_2 = \cdots = x_n = 0$ in (4). Then since $Bf = f$ implies that $f \in \mathbf{C}^\infty(D^n)$, by 4.2.4 of [Ru2] we get $\Delta_1 f = 0$. Similarly we can see that

$$\Delta_2 f = \cdots = \Delta_n f = 0.$$

This completes the proof of Theorem 3.1. □

Now we have some corollaries.

COROLLARY 3.7. *If $f \in L^1(D^n, m \times \cdots \times m)$ satisfies $Bf = f$ and $R(f \circ \varphi) \in L^\infty(D^n)$ for every $\varphi \in \mathrm{Aut}(D^n)$, then f is n-harmonic.*

Proof. Let $f \in L^1(D^n, m \times \cdots \times m)$ satisfy the above conditions. Then since R, B are contractions on $L^\infty(D^n)$, for every $\varphi \in \mathrm{Aut}(D^n)$ we have

$$\begin{aligned} B(R(f \circ \varphi)) &= R(B(f \circ \varphi)) \\ &= R(Bf \circ \varphi) \qquad \text{by 3.4} \\ &= R(f \circ \varphi) \end{aligned}$$

Thus by Theorem 3.1, $R(f \circ \varphi)$ is a constant. This implies (4) in step (2) of 3.6. Hence by the method of 3.6 we get the result.

Corollary 3.8. *For $1 \le p < \infty$, if $u \in L^p(D, \mu)$ satisfies $Tu = u$ then $u \equiv 0$. Similarly if $f \in L^p(D^n, \mu \times \cdots \times \mu)$ satisfies $Bf = f$ then $f \equiv 0$.*

Proof. Since the only harmonic function on D which belongs $L^p(D, \mu)$ is the constant 0, by Theorem 3.1 it is enough to show that u is bounded. When $u \in L^p(D, \mu)$ since $1 < p < \infty$ we have

$$\begin{aligned} u(z) = (Tu)(z) &= \int_D u(x) \frac{(1 - |z|^2)^2}{|1 - \bar{z}x|^4}\, dm(x) \\ &= (1 - |z|^2)^2 \int_D u(x) \frac{(1 - |x|^2)^2}{|1 - \bar{z}x|^4}\, d\mu(x) \end{aligned}$$

Thus

$$\begin{aligned} |u(z)| &\le (1 - |z|^2)^2 \|u\|_p \left(\int_D \frac{(1 - |x|^2)^{2q-2}}{|1 - \bar{z}x|^{4q}}\, dm(x) \right)^{\frac{1}{q}} \qquad \left(\frac{1}{p} + \frac{1}{q} = 1 \right) \\ &\le (1 - |z|^2)^2 \|u\|_p c (1 - |z|^2)^{-2} \text{ for some } c > 0 \quad \text{(by 1.4.10 of [Ru2])} \\ &= c\|u\|_p. \qquad (1) \end{aligned}$$

When $u \in L^1(D, d\mu)$

$$\begin{aligned} |u(z)| &= |Tu(z)| \leq \sup_{z \in D} \frac{(1-|z|^2)^2(1-|x|^2)^2}{|1-z\bar{x}|^4} \int_D |u(x)|\, d\mu(x) \\ &= \|u\|_1 \end{aligned} \tag{2}$$

From (1) and (2), we complete the proof for u.

In the same way we can show that such f is bounded and Theorem 3.1 forces f to be n-harmonic. And constant zero is the only n-harmonic function which belongs to $L^p(D^n, \mu \times \cdots \times \mu)$. □

REFERENCES

[AC] S. Axler and Z. Cuckovic, *Commuting Toeplitz operator with harmonic symbols*, Integral Equations and Operator Theory **14** (1991), 1–12.

[AFR] P. Ahern, M. Flores and W. Rudin, *An invariant volume mean value property*, J. Funct. Anal. **111** (1993), 380–397.

[Eng] M. Englis, *Functions invariant under the Berezian transform*, Preprint, 1992.

[Fur] H. Furstenberg, "Boundaries of Riemannian symmetric spaces" in *Symmetric spaces* (W. M. Boothby and G. L. Weiss, Editors), Marcel Dekker, New York, 1972, pp. 359–377.

[KT] Y. Katznelson and L. Tzafriri, *On power bounded operators*, J. Funct. Anal. **68** (1986), 313–328.

[Ru1] W. Rudin, *Function theory in polydiscs*, Benjamin, New York, 1969.

[Ru2] ————, *Function theory in the unit Ball of* $\mathbf{C}^n$, Springer-Verlag, 1980.

Topology and Geometry Research Center, Kyungpook National University, Taegu 702-701, Korea

jalee@math.wisc.edu