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STRUCTURE OF FOLIATIONS ON 2-MANIFOLDS

HABIB MARZOUGUI

Introduction

In this paper, we intend to study qualitative properties of foliations with finitely many singularities on closed 2-manifolds. Considering such a foliation as a regular foliation on the punctured 2-manifold obtained from a closed 2-manifold by removing the singular points, we will give an analogy of a structure's theorem (in Salhi [10], Theorem 1) on codimension one regular foliations on closed manifolds. Singular foliations on 2-manifolds have been investigated by many authors from a geometric point of view (for example, see [2], [5], [6]). We are interested more precisely in the following questions:

1. Describe the foliation near a leaf.
2. Establish a structure's theorem.

We mention that the results given here are known for foliations with singularities saddles and/or thorns.

In Section 1, we give some preliminaries (definitions and notations of the general theory of singular foliations on 2-manifolds, and some topological results which will be needed later.). In Section 2, we give a description of foliations near a leaf, especially near an exceptional leaf, by establishing analogues of Sacksteder's Theorem [9] for singular foliations on 2-manifold (Theorems 2.1 and 2.2). Some consequences as in [10], [11] are given.

1. Preliminary

(A) Basic definitions.

This section is devoted to the basic facts of the general theory of singular foliations on 2-manifolds. Let $\mathfrak{J}$ be a C° singular foliation with a finite number of singularities on a compact orientable 2-manifold S of genus g. We let sing $\mathfrak{J}$ be the set of singularities of $\mathfrak{J}$, $\mathfrak{J}/U$ the restriction of $\mathfrak{J}$ to an invariant open set U of S, $\mathfrak{J}^*$ the restriction of $\mathfrak{J}$ to $S^* = S - \operatorname{sing}\mathfrak{J}$, and let U_1 be the complement in S^* of the union of closed leaves of $\mathfrak{J}^*$. By [3], Theorem p. 386, U_1 is an open invariant set of S.

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A leaf L of $\mathfrak{I}$ is said to be proper if $\bar{L} - L$ is closed in S, locally dense if $\bar{L}$ has non-empty interior, and exceptional if L is non proper and nowhere dense. L is said to be totally proper if $\bar{L}$ consists of singularities and proper leaves. A subset M of S is called invariant (or $\mathfrak{I}$-saturated) if it is an union of leaves and singularities. M is called a minimal set of $\mathfrak{I}$ if it is a closed non-empty and invariant set which is minimal (in the sense of inclusion) for these properties. We call the class (resp. higher structure) of a leaf L of $\mathfrak{I}$ the union $\mathrm{cl}(L)$ (resp. $\mathrm{SS}(L)$) of leaves G of $\mathfrak{I}$ such that $\bar{G} = \bar{L}$ (resp. $L \subset \bar{G}$ with $\bar{G} \neq \bar{L}$) (cf. [11]). If L is proper, $\mathrm{cl}(L) = L$. A quasiminimal set K of $\mathfrak{I}$ is the closure of a non-proper leaf. It is showed in [7] that if $\mathfrak{I}$ is orientable, the closure of any non-proper leaf is a quasiminimal set of $\mathfrak{I}$ and every totally proper leaf of $\mathfrak{I}$ is closed in S^* or closed in U_1.

Let L be a non-closed leaf of $\mathfrak{I}$. A point $x \in L$ divides L into two half-leaves $L^{(-)}$ and $L^{(+)}$. Denote the limit set of the half-leaf $L^{(\cdot)}$ (resp. of the leaf L) by $\lim L^{(\cdot)}$ (resp. $\lim L$). The set $\lim L^{(\cdot)}$ is closed, invariant and non-empty. For a non-closed leaf, $\lim L = \overline{\bar{L} - L}$. We have $\lim L = \bar{L} - L$ if L is proper and non-closed, and $\lim L = \bar{L}$ otherwise.

In the case where $\mathfrak{I}$ is orientable, $\mathfrak{I}$ can be defined by a flow ϕ: $\mathbb{R} \times S \to S$. For every leaf L of $\mathfrak{I}$ and $x \in L$, the half-leaf $L^{(+)}$ (resp. $L^{(-)}$) is denoted by $L_x^+ = \{\phi(t,x)/t \in \mathbb{R}_+\}$ (resp. $L_x^- = \{\phi(t,x)/t \in \mathbb{R}_-\}$ and called the positive (resp. negative) half-leaf of origin x. The set $\lim L_x^+$ (resp. $\lim L_x^-$) is denoted by $\Omega_L = \{y \in S : \exists (t_n)_{n\in\mathbb{N}} \to +\infty, y = \lim \phi(t_n, x)\}$ (resp. $A_L = \{y \in S : \exists (t_n)_{n\in\mathbb{N}} \to -\infty$, $y = \lim \phi(t_n, x)\}$ and called the ω-limit (resp. α-limit) set of L. The limit set $\lim L$ of L is $\Omega_L \cup A_L$.

(B) Some results.

The following theorem is a consequence of the theorem obtained in [7] classifying the limit sets.

THEOREM 0.1. *Let $\mathfrak{I}$ be an orientable singular foliation with finite singularities on a compact orientable 2-manifold S. For every leaf L of $\mathfrak{I}$, each of its limit sets Ω_L (resp. A_L) is one of the following type*:

(i) a singular point

(ii) a compact leaf

(iii) a union of singularities and non-compact leaves which are closed in S^*

(iv) a quasiminimal set.

Below, we give some topological results which we need in the sequel.

PROPOSITION 0.2 [8]. *Let M be a non-compact connected orientable 2-manifold of finite genus k. Then its end point compactification $\hat{M}$ is a compact connected orientable 2-manifold of finite genus k where the space* $\mathrm{Bt}(M) = \hat{M} - M$ *of ends of M is a totally disconnected compact set.*

PROPOSITION 0.3 [12, Lemma 4.3, p. 259]. *Let S be a compact connected orientable 2-manifold of genus g, and let X be a compact subset of S having finitely many connected components. If W is a connected component of $S - X$ then W is a connected 2-manifold with genus $\leq g$ and finitely many ends.*

2. Foliation near a leaf

In all the proofs below, the foliation $\mathfrak{I}$ is assumed to be orientable. If $\mathfrak{I}$ is non orientable, these proofs are then straightforward by passing to a double branched covering of $\mathfrak{I}$.

THEOREM 2.1. *Let G be a proper non-closed leaf of $\mathfrak{I}$ and let O be a leaf such that $O \subset \lim G^{(\cdot)}$. Then there exists an open connected invariant set W in S, containing G such that for every leaf γ of $\mathfrak{I}/W$, γ is proper and $O \subset \lim \gamma$.*

The following result is analogous to Sacksteder's Theorem [9].

THEOREM 2.2. *Let $\mathfrak{I}$ be a singular foliation with a finite number of singularities on a compact orientable 2-manifold S. Let L be an exceptional leaf of $\mathfrak{I}$. Then:*

(i) The union $V = \mathrm{SS}(L) \cup \mathrm{cl}(L)$ is open and connected in S.

(ii) For every leaf G of $\mathfrak{I}/V$, $\lim G \subset \bar{L} \cup \mathrm{Fr}(V)$ with $\lim G^{(+)} = \bar{L}$ or $\lim G^{(-)} = \bar{L}$ ($\mathrm{Fr}(V)$ denotes the frontier of V.)

(A) Proofs of Theorem 2.1 and 2.2.

Let $K_1, K_2, \ldots, K_p$ be the quasiminimal sets of $\mathfrak{I}$ (we know [7] that $p \leq g$, where g is the genus of S) and let L be an exceptional leaf of $\mathfrak{I}$. We let $K_p = \bar{L}$ and let U be the connected component of $U_1 - (K_1 \cup K_2 \ldots \cup K_{p-1})$ containing L.

LEMMA 2.1. *Let $(G_n)_{n\in\mathbb{N}}$ be an infinite sequence of leaves of $\mathfrak{I}/U$. Then the sequence $(\Omega_{G_n})_{n\in\mathbb{N}}$ (resp. $(A_{G_n})_{n\in\mathbb{N}}$) has one of the following properties:*

(i) $(\Omega_{G_n})_{n\in\mathbb{N}}$ (resp. $(A_{G_n})_{n\in\mathbb{N}}$) is a union of singularities and closed leaves of $\mathfrak{I}^*$, and there exists a singular point $s_o \in \Omega_{G_n}$ (resp. A_{G_n}) for infinitely many integers n.

(ii) There exists a compact leaf γ such that $\Omega_{G_n} = \gamma$ (resp. $A_{G_n} = \gamma$) for infinitely many integers n.

(iii) There exists a quasiminimal set K_r ($r \in [1, p]$) such that $\Omega_{G_n} = K_r$ (resp. $A_{G_n} = K_r$) for infinitely many integers n.

Proof. Let $(G_n)_{n\in\mathbb{N}}$ be an infinite sequence of leaves of $\mathfrak{I}/U$. If we have neither (i) nor (iii), then for every $s \in \operatorname{sing}\mathfrak{I}$ (resp. every quasiminimal set $K_i, i = 1, 2, \ldots, p$), there is a finite number of integers n such that $s \in \Omega_{G_n}$ (resp. $\Omega_{G_n} = K_i$). The set $\operatorname{sing}\mathfrak{I}$ is finite so for n large enough, Ω_{G_n} is reduced to a compact leaf γ_n (Theorem 0.1). Now let us show that for infinitely many integers n, all γ_n coincide with the same leaf: To the contrary, if the γ_n are pairwise distinct for infinitely many integers

n, then [4, Appendix] there exist three integers p, q and r such that every pair of leaves γ_p, γ_q and γ_r bound an annulus. One supposes for example that γ_q is in the interior of the annulus (γ_p, γ_r). It follows that the leaves G_p, G_q and G_r are not contained in the same connected component U, a contradiction. □

LEMMA 2.2. *Let $(G_n)_{n\in\mathbb{N}}$ be an infinite sequence of leaves of $\mathfrak{I}/U$. Then there exists an infinite subsequence $(G_{n_k})_{k\in\mathbb{N}^*}$ of $(G_n)_{n\in\mathbb{N}}$ such that $(\bigcup_{k\in\mathbb{N}^*}\Omega_{G_{n_k}})$ (resp. $(\bigcup_{k\in\mathbb{N}^*}A_{G_{n_k}})$) is connected.*

Proof. Each of properties (i), (ii) and (iii) of Lemma 2.1 implies the existence of an infinite subsequence $(G_{n_k})_{k\in\mathbb{N}^*}$ of $(G_n)_{n\in\mathbb{N}}$ such that $(\bigcap_{k\in\mathbb{N}^*}\Omega_{G_{n_k}})$ (resp. $(\bigcap_{k\in\mathbb{N}^*}A_{G_{n_k}})$) is non-empty. Since for every $k\in\mathbb{N}^*$, $\Omega_{G_{n_k}}$ (resp. $A_{G_{n_k}}$) is connected, it follows that $(\bigcup_{k\in\mathbb{N}^*}\Omega_{G_{n_k}})$ (resp. $(\bigcup_{k\in\mathbb{N}^*}A_{G_{n_k}})$ is connected. □

PROPOSITION 2.1. *If L is an exceptional leaf of $\mathfrak{I}$ and $(G_n)_{n\in\mathbb{N}}$ is a sequence of leaves which converges to a leaf L, then for n large enough, we have $L\subset\overline{G_n}$.*

Proof. Let U be the connected component of $U_1-(K_1\cup K_2\dots K_{p-1})$ containing L and let $(G_n)_{n\in\mathbb{N}}$ be an infinite sequence of leaves of $\mathfrak{I}$ which converge to L. For n large enough, we have $G_n\subset U$. If the proposition is not true then for infinitely many indices n, G_n is a closed leaf in U. We may assume, passing to a subsequence if necessary, that for each $n\in\mathbb{N}$, G_n is a closed leaf of $\mathfrak{I}/U$ and (Ω_{G_n}) (resp. $(A_{G_n})_{n\in\mathbb{N}}$) has one of the properties (i), (ii), (iii) of Lemma 2.1. In the case (iii), since G_n is closed in U, Ω_{G_n} (resp. A_{G_n}) $\neq K_p$. Therefore we have

$$L\subset S-\overline{\bigcup_{n\in\mathbb{N}}(\Omega_{G_n}\cup A_{G_n})},$$

because otherwise we would have $L\subset U\cap\overline{\bigcup_{n\in\mathbb{N}}(\Omega_{G_n}\cup A_{G_n})}\subset K_1\cup K_2\dots\cup K_{p-1}$, which is impossible. Now by Lemma 2.2, $\bigcup_{n\in\mathbb{N}}\Omega_{G_n}$ (resp. $\bigcup_{n\in\mathbb{N}}A_{G_n}$) is connected. Then the set $\overline{\bigcup_{n\in\mathbb{N}}(\Omega_{G_n}\cup A_{G_n})}$ is a compact subset of S having at most two connected components. Denote by W the connected component of $S-\overline{\bigcup_{n\in\mathbb{N}}(\Omega_{G_n}\cup A_{G_n})}$ containing L. Then W is a connected orientable 2-manifold with finite genus and its space of ends $\mathrm{Bt}(W)$ is finite (Proposition 0.3). Since for n large enough, G_n is closed in W, the leaf L will be closed in W [3, Theorem p. 386], which is impossible because L is non-proper. □

Completing the Proof of Theorem 2.1. We will prove precisely:

THEOREM 2.1′. *Let G be a proper non-compact leaf of $\mathfrak{I}$ and let O be a leaf such that $O\subset\Omega_G$ (resp. A_G). Then there exists an open connected invariant set W in S containing G such that for every leaf γ of $\mathfrak{I}/W$, γ is proper and $O\subset\Omega_\gamma$ (resp. $O\subset A_\gamma$).*

It suffices to show the theorem for Ω_G, the proof being similar for A_G. Under the hypotheses of the theorem, let U be the connected component of $U_1 - (K_1 \cup K_2 \ldots \cup K_p)$ containing G. Suppose the theorem is false; then there exists an infinite sequence of proper leaves $(G_n)_{n\in\mathbb{N}}$ of $\mathfrak{I}/U$ which converges to G and, for every $n \in \mathbb{N}$, $O \not\subset \Omega_{G_n}$. One can assume, passing to a subsequence if necessary, that for every $n \in \mathbb{N}$, $(\Omega_{G_n})_{n\in\mathbb{N}}$ (resp. $(A_{G_n})_{n\in\mathbb{N}}$) has one of the properties (i), (ii), (iii) of Lemma 2.1. Since

$$G \subset S - \overline{(\cup_{n\in\mathbb{N}}\Omega_{G_n})}$$

(because otherwise we would have $G \subset U \cap \overline{(\cup_{n\in\mathbb{N}}\Omega_{G_n})} \subset K_1 \cup K_2 \ldots \cup K_P$, which is impossible), let W be the connected component of $S - \overline{(\cup_{n\in\mathbb{N}}\Omega_{G_n})}$ containing G. Since $\overline{(\cup_{n\in\mathbb{N}}\Omega_{G_n})}$ is connected (Lemma 2.2), $\overline{(\cup_{n\in\mathbb{N}}\Omega_{G_n})}$ is a compact connected subset of S. Therefore W is an open connected orientable 2- manifold with finite genus and finitely many ends (Proposition 0.3). The endpoint compactification $\hat{W}$ of W is a compact connected 2-manifold and the foliation $\hat{\mathfrak{I}}$ of $\hat{W}$ extends the foliation $\mathfrak{I}/W$ where each point of $\mathrm{Bt}(W)$ is a singular point of $\hat{\mathfrak{I}}$. Since G is proper, let $x \in G$ and let T be an open transverse arc such that $T \subset U$ and $T \cap G = \{x\}$. For each $n \in \mathbb{N}$, choose a point $x_n \in G_n \cap T$ with $(x_n)_{n\in\mathbb{N}}$ converging to x. If we denote by $\hat{\Omega}_{G_n}$ the ω-limit set of G_n in $\hat{W}$, then $\hat{\Omega}_{G_n} = \{s_n\}$ where $s_n \in \operatorname{sing}\hat{\mathfrak{I}}$. Since $\operatorname{sing}\hat{\mathfrak{I}}$ is finite, one can suppose, passing to a subsequence if necessary, that for every $n \in \mathbb{N}$, $\hat{\Omega}_{G_n} = \{p\}$ where $p \in \operatorname{sing}\hat{\mathfrak{I}}$. Denote by $[x_1, x_n]$ the transverse segment contained in T and let

$$\theta_n = G^+_{x_n} \cup G^+_{x_1} \cup \{p\} \cup [x_1, x_n] \quad \text{for } n \geq 2$$

where $G^+_{x_n} = \{\phi(t, x_n)/t \in \mathbb{R}_+\}$. Now we will use now an argument which is originally due to Thurston: Each θ_n induces a class $[\theta_n]$ in $H_1(\hat{W}; Z)$. Since the subgroup H of $H_1(\hat{W}; Z)$ generated by the $([\theta_n])_{n\geq 2}$ is of finite type, let k the integer such that $[\theta_2], [\theta_3], \ldots, [\theta_k]$ generate H. Since $O \not\subset \Omega_{G_n}$ for every $n \in \mathbb{N}$, G^+_x will be cut by a closed transversal curve which is disjoint from $\theta_2, \theta_3, \ldots, \theta_k$ (for example, see [1], page 18). Hence, the closed transversal curve τ has a zero intersection number with each generator H; this contradicts the fact that G^+_x is adherent to the union of θ_n, $n \in \mathbb{N}^*$. □

Completing the proof of Theorem 2.2.

Assertion (*i*). Let L be an exceptional leaf of $\mathfrak{I}$. It follows from Theorem 2.1 that $\mathrm{SS}(L)$ is open in S. Now, if V is not open there exists an infinite sequence of leaves $(G_n)_{n\in\mathbb{N}}$ not contained in V which converge to L. This is impossible by Proposition 2.1. The connectedness of the open V is clear.

Assertion (*ii*). Let G be a leaf of $\mathfrak{I}/V$. If G is non-proper, then $\lim G = \Omega_G \cup A_G = \bar{G}$ and $\Omega_G = \bar{G}$ or $A_G = \bar{G}$. Since $\bar{G}$ is a quasiminimal set and $L \subset \bar{G}$, we obtain $\bar{G} = \bar{L}$. The assertion is then verified. If G is proper then $L \subset \lim G =$

$\bar{G} - G = \Omega_G \cup A_G$, and $\Omega_G = \bar{L}$ or $A_G = \bar{L}$ (Theorem 0.1). One supposes for example that $\Omega_G = \bar{L}$. We have $\lim G = \bar{L} \cup A_G$. It follows that if A_G meets V then $A_G = \bar{L}$ and $\lim G = \bar{L}$. Otherwise, $\lim G \subset \bar{L} \cup \mathrm{Fr}(V)$.

Remark 2.1. If L is a locally dense leaf, the set $V = \mathrm{SS}(L) \cup \mathrm{cl}(L) = \mathrm{cl}(L)$ is the connected component of U_1 containing L. Every leaf of $\mathfrak{J}/V$ is dense in V.

Remark 2.2. If L is an exceptional leaf of $\mathfrak{J}$ and G is a proper leaf such that $\lim G^{(\cdot)} = \bar{L}$, then there exists an open connected invariant set W in S, containing G, such that for every leaf γ of $\mathfrak{J}/W$, γ is proper and $\lim \gamma^{(+)} = \bar{L}$ or $\lim \gamma^{(-)} = \bar{L}$. In the case where $\mathfrak{J}$ is orientable, we have precisely, by Theorem 0.1: If $\bar{L} = \Omega_G$ (resp. A_G) for every leaf γ of $\mathfrak{J}/W$, γ is proper and $\bar{L} = \Omega_\gamma$ (resp. A_γ).

(B) Corollaries.

COROLLARY 2.1. *The higher structure* SS(L) *of every leaf L of $\mathfrak{J}$ is open in S.*

Proof. We remark first that if L is locally dense, SS(L) is empty. We suppose then L is either exceptional or proper. Let G be a leaf contained in SS(L). Then G is non-compact. If L is exceptional, G is proper (Theorem 0.1), and we have $\bar{L} = \Omega_G$ or $\bar{L} = A_G$. The corollary is deduced from Remark 2.2. If L is proper, the corollary is deduced from Theorem 2.2 if G is exceptional, from Remark 2.2 if G is proper, and from Remark 2.1 if G is locally dense. □

COROLLARY 2.2. *If W is an open invariant non empty set contained in U_1, then the union of closed leaves of $\mathfrak{J}/W$ is closed in W.*

Proof. Suppose the proposition is not true. Then there exists an infinite sequence of closed leaves $(L_n)_{n\in\mathbb{N}}$ of $\mathfrak{J}/W$ which converges to a non-closed leaf L of $\mathfrak{J}/W$. By [7, Corollary 3.2], there exists a minimal set E of $\mathfrak{J}/W$ contained in $\bar{L}$. The set E is either a closed leaf of $\mathfrak{J}/W$ or equal to $\bar{G} \cap W$ where G is a non-proper leaf of $\mathfrak{J}/W$. Consider the first case. Since $W \subset U_1$, E is a proper and non-closed leaf in S^* contained in $\bar{L}$; this is impossible by Theorem 0.1. In the second case, if G is locally dense, the leaf L and the leaves L_n are also locally dense for n large enough; this contradicts the fact that L_n is closed in W. If G exceptional, for n large enough we have $G \subset \overline{L_n}$ (Proposition 2.1); this is impossible because $G \subset W$ and L_n is closed in W. □

It follows from Corollary 2.2 and Theorem 0.1 that if we take $W = U_1$, then *the union* TP$(\mathfrak{J})$ *of totally proper leaves of $\mathfrak{J}$ is a closed set in S^*.*

COROLLARY 2.3. (STRUCTURE'S THEOREM). *Let $\mathfrak{J}$ be a singular foliation with a finite number h of singularities on a compact orientable 2-manifold S of genus g.*

Then:

(1) $\Im$ *has a finite number n of quasiminimal sets* $\overline{L_1} = K_1, \overline{L_2} = K_2, \ldots, \overline{L_n} = K_n$ *of* $\Im$, *where* $n \leq g$ *if* $\Im$ *is orientable, and* $n \leq [2g - 1 + \frac{h}{2}]$ *if* $\Im$ *is non orientable, and* $L_1, L_2, \ldots, L_n$ *are non-proper leaves of* $\Im$.

(2) *The subsets* $V_i = \mathrm{SS}(L_i) \cup \mathrm{cl}(L_i)$ $(1 \leq i \leq n)$ *are open and connected in S and their union R has at most n connected components, each of which is a union of some* V_i.

(3) *The complementary* $\mathrm{TP}(\Im)$ *in S of the union R is a compact invariant subset consisting of the union of singularities, closed leaves of* $\Im^*$, *and closed leaves of* $\Im^*/U_1$.

Proof. Assertion 1 is known [7]. Let us prove assertion 2. If C is a connected component of R, then C will contains at least a non-proper class $\mathrm{cl}(L)$, where L is a non-proper leaf. Since there exist n such classes $(n \leq g)$ [7, Theorem 4.1], then C has at most n connected components. Denote by $\mathrm{cl}(L_1), \mathrm{cl}(L_2), \ldots, \mathrm{cl}(L_p)$ the non-proper classes contained in C. We have $C = V_1 \cup V_2 \cup \cdots \cup V_p$. Assertion 3 follows from Theorem 0.1 because a totally proper leaf L of $\Im$ is closed in S^* or closed in U_1. □

Remark 2.3. The structure's theorem above is close to the structure's theorem for C°-regular foliations of codimension one on compact manifold given in [10], Theorem 1.

Remark 2.4. We can apply the results above to transverse invariant measures for orientable foliations. By the same methods as in [6] we obtain (for arbitrary singularities) the results given there for foliations with saddles.

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References

[1] C. Godbillon, *Dynamical system on surfaces*, Springer-Verlag, 1983.

[2] C. Gutiérrez, *Foliations on surfaces having exceptional leaves*, Lect. Notes in Math., no. 1331, Springer-Verlag, 1988, pp. 73–85.

[3] A. Haefliger, *Variétés feuilletées*, Ann. Sci. Norm. Sup. Pisa **16** (1962), 367–397.

[4] C. L. Hagopian, *Homogeneous continua in 2-manifolds*, Topology Appl. **19** (1985), 157–163.

[5] G. Levitt, *La décomposition dynamique et la différentiabilité des feuilletages des surfaces*, Ann. Inst. Fourier **37** (1987), 85–116.

[6] ———, *Feuilletages des surfaces*, Ann. Inst. Fourier. Grenoble **32** (1982), 179–217.

[7] H. Marzougui, *Structure des feuilles sur les surfaces ouvertes*, C. R. Acad. Sci. Paris (Sér.1) **323** (1996), 185–188.

[8] I. Richard, *On the classification of non-compact surfaces*, Trans. Amer. Math. Soc. **106** (1963), 259–269.

[9] R. Sacksteder, *Foliations and pseudogroups*, Amer. J. Math. **87** (1965), 79–102.

[10] E. Salhi, *Sur un théorème de structure des feuilletages de codimension* 1, C. R. Acad. Sci. Paris (Sér. 1) **300** (1984), 635–638.
[11] ———, *Niveau des feuilles*, C. R. Acad. Sci. Paris (Sér. I) **301** (1985), 219–222.
[12] A. J. Schwartz and E. S. Thomas, *The depth of the center of 2-manifolds*, *Proc. Symp. Pure Math.*, no. 14, Amer. Math. Soc. Providence, R.I, 1970, pp. 253–264.

Département de Mathématiques, Faculté des Sciences de Bizerte, 7021, Zarzouna. Tunisie