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HECKE MODULAR FORMS AND q-HERMITE POLYNOMIALS

BY

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1. Introduction

In this paper we shall use a technique of L.J. Rogers, expansion in terms of q-Hermite polynomials,

$$A_n(\cos\theta|q) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \cos(n-2i)\theta, \tag{1.1}$$

where

$$\begin{bmatrix} n \\ i \end{bmatrix} = \prod_{j=1}^{i} \frac{(1-q^{n-i+j})}{(1-q^j)}$$

is the Gaussian polynomial, to derive a number of identities which express a summation of the form

$$\sum_{(n,m)\in D} (-1)^{f(n,m)} q^{Q(n,m)+L(n,m)} \tag{1.2}$$

as a rational product of η-functions, where Q is a quadratic form, L is a linear form and $D \subseteq \{(n,m) \in \mathbf{Z} \times \mathbf{Z} | Q(n,m) \geq 0\}$.

The most famous identity of this type is due to Jacobi [7, Theorem 357]:

$$\begin{aligned} \prod_{n\geq 1}(1-q^n)^3 &= \sum_{m=-\infty}^{\infty} \sum_{n\geq |m|} (-1)^n q^{(n^2+n)/2} \\ &= \sum_{n\geq 0} (-1)^n (2n+1) q^{(n^2+n)/2}. \end{aligned} \tag{1.3}$$

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E. Hecke [8] described the effect of the transformation $\tau \to -1/\tau$ on a class of sums of the type given in (1.2) with $q = \exp(2\pi i \tau)$ and used this to prove the following identity:

$$\prod_{n\geq 1}(1-q^n)^2 = \sum_{m=-\infty}^{\infty}\sum_{n\geq 2|m|}(-1)^{n+m}q^{(n^2-3m^2+n+m)/2}. \tag{1.4}$$

More recently, V.G. Kac and D.H. Peterson [9] showed how to prove (1.4) and similar results using affine Lie algebras. This in turn led G.E. Andrews [1] to demonstrate how these identities can be derived by computing the constant term in the series expansion of a certain infinite product by two very different methods.

What should surprise no one familiar with the history of theta function identities is that L.J. Rogers had stated and proved equation (1.4) back in 1894 [12, p. 323]. In fact, the derivation of (1.4) is almost a warm-up exercise before the tackling of the more difficult derivation of the Rogers-Ramanujan identities. Rogers' technique is philosophically close to that of Andrews as it also involves computing a constant term by two very different methods. But his starting point is different and the constant term he seeks is with respect to the q-Hermite polynomials which Rogers was the first to study [10], [11], [12], [13] and which have since received attention from G. Szegö [14], L. Carlitz [5], [6], R. Askey and M. Ismail [2] and others.

We shall explain and exploit Rogers' approach to equation (1.4) to obtain the following general identities which involve formal sums in a completely arbitrary sequence $\{f(m)\}$. For convenience, we shall use the rising q-factorial notation ($|q| < 1$):

$$(a;q)_\infty = \prod(1-aq^i), \quad i\geq 0, \qquad (a;q)_x = \frac{(a;q)_\infty}{(aq^x;q)_\infty}, \quad x\in \mathbf{R}.$$

We shall prove the following identities:

(1.5)

$$(q;q)_\infty^2\sum_{m\geq 0}q^m f(m)$$
$$= \sum_{m=-\infty}^{\infty}\sum_{n\geq 2|m|}(-1)^{n+m}q^{(n^2-3m^2+n+m)/2}\sum_{k=0}^{|m|}(q^{-m};q)_k(q^m;q)_k q^k f(k),$$

(1.6)
$$(q;q)_\infty (q^2;q^2)_\infty \sum_{m\ge 0} (q;q^2)_m q^m f(m)$$
$$= \sum_{m=-\infty}^{\infty} \sum_{n\ge |m|} (-1)^n q^{(2n^2-m^2+2n+m)/2} \sum_{k=0}^{|m|} (q^{-m};q)_k (q^m;q)_k q^k f(k),$$

(1.7)
$$(q;q)_\infty (q;q^2)_\infty \sum_{m\ge 0} (-q;q)_{2m} q^m f(m)$$
$$= \sum_{m=-\infty} \sum_{n\ge 2|m|} (-1)^{n+m} q^{(n^2-2m^2+n+2m)/2}$$
$$\times \sum_{k=0}^{|m|} (q^{-2m};q^2)_k (q^{2m};q^2)_k q^{2k} f(k).$$

If in (1.5) we set $f(0) = 1$, $f(m) = 0$ for $m \ge 1$, we get equation (1.4). Jacobi's identity (1.3) will be shown in §3 to be equation (1.6) with $f(m)$ set equal to $(q^2;q^2)_m^{-1}$.

Other corollaries of these identities will also be proved, all using Heine's summation [15; Corollary 2.4],

(1.8)
$$\sum_{k\ge 0} \frac{(a;q)_k (b;q)_k}{(q;q)_k (c;q)_k} \left(\frac{c}{ab}\right)^k = \frac{(c/a;q)_\infty (c/b;q)_\infty}{(c;q)_\infty (c/ab;q)_\infty},$$

its two finite corollaries in the form

(1.9)
$$\sum_{k=0}^{m} \frac{(q^{-m};q)_k (q^m;q)_k}{(q;q)_k (c;q)_k} c^k = \frac{(cq^{-m};q)_m}{(c;q)_m}$$
$$= (-c)^m q^{-m^2/2-m/2} \frac{(c^{-1}q;q)_m}{(c;q)_m},$$

(1.10)
$$\sum_{k=0}^{m} \frac{(q^{-m};q)_k (q^m;q)_k}{(q;q)_k (c;q)_k} q^k = \frac{(cq^{-m};q)_m}{(c;q)_m} q^{m^2}$$
$$= (-c)^m q^{m^2/2-m/2} \frac{(c^{-1}q;q)_m}{(c;q)_m}$$

(equation (1.10) is a restatement of (1.9) with q replaced by q^{-1}, c by c^{-1}), the

q-binomial theorem [15, (2.2.1)],

$$\sum_{k\geq 0} \frac{(a;q)_k}{(q;q)_k} x^k = \frac{(ax;q)_\infty}{(x;q)_\infty}, \tag{1.11}$$

and one of its corollaries,

$$2\sum_{k\geq 0} \frac{(a;q)_{2k}}{(q;q)_{2k}} x^k = (x;q^2)_\infty^{-1}\{(x^{1/2};q)_\infty(-ax^{1/2};q)_\infty + (-x^{1/2};q)_\infty(ax^{1/2};q)_\infty\}. \tag{1.12}$$

2. Proof of (1.5)

Rogers' q-Hermite polynomials, $A_n = A_n(\cos\theta|q)$, can be defined by their generating function:

$$\frac{1}{(re^{i\theta};q)_\infty(re^{-i\theta};q)_\infty} = \sum_{n\geq 0} \frac{A_n r^n}{(q;q)_n}. \tag{2.1}$$

The representation given in (1.1) follows from (2.1) when each infinite product of the left side is expanded as a power series in r. It is clear from the representation given in (1.1) that A_n is a polynomial of exact degree n in $\cos\theta$ and thus two expansions in terms of the A_n are equal only if the corresponding coefficients are equal. For this reason, if we can show two expansions in terms of A_n to be equal, then we can replace the A_n on each side by the same arbitrary sequence without losing equality.

Rogers begins his study by establishing the expansion of $\cos n\theta$ in terms of $\{A_m\}$ (see [12, p. 319] or [4, §2]):

$$2\cos n\theta = \sum \frac{(-1)^i q^{C(i,2)}(1-q^n)(q;q)_{n-1-i}}{(q;q)_i(q;q)_{n-2i}} A_{n-2i}, \quad 0\leq i\leq n/2,\ n\geq 1, \tag{2.2}$$

where $C(i,2) = i(i-1)/2$. He now considers the following classical theta function identity, derived by partial fraction decomposition (see [1, Lemma 1]).

(2.3)
$$\frac{(q;q)_\infty^2}{(e^{i\theta}q^{1/2};q)_\infty(e^{-i\theta}q^{1/2};q)_\infty} = \sum_{m=-\infty}^{\infty} \sum_{n\geq|m|} \cos m\theta(-1)^{n+m} q^{C(n+1,2)-m^2/2}.$$

It follows from equation (2.1) that we also have

$$\frac{(q;q)_\infty^2}{(e^{i\theta}q^{1/2};q)_\infty(e^{-i\theta}q^{1/2};q)_\infty} = (q;q)_\infty^2 \sum_{n\ge 0}\frac{A_n q^{n/2}}{(q;q)_n}. \tag{2.4}$$

We now use equation (2.2) to expand the right hand side of (2.3) in terms of $\{A_m\}$:

$$\begin{aligned}(2.5)\quad &\sum_m \sum_{n\ge|m|} \cos m\theta(-1)^{n+m} q^{C(n+1,2)-m^2/2}\\ &= \sum_{n\ge 0}(-1)^n q^{C(n+1,2)} + \sum_{m\ge 1}\sum_{n\ge m}(-1)^{n+m} q^{C(n+1,2)-m^2/2}\\ &\quad\times \sum_{i=0}^{[m/2]}\frac{(-1)^i q^{C(i,2)}(1-q^m)(q;q)_{m-1-i}}{(q;q)_i(q;q)_{m-2i}}A_{m-2i}.\end{aligned}$$

Equation (1.4) can now be obtained directly be equating the constant terms with respect to $\{A_m\}$ in the right hand side of (2.4) and (2.5). To obtain (1.5), we keep all of the A_m with even subscripts:

(2.6)

$$\begin{aligned}(q;q)_\infty^2 \sum_{m\ge 0}\frac{A_{2m}q^m}{(q;q)_{2m}} &= \sum_{n\ge 0}(-1)^n q^{C(n+1,2)}\\ &\quad+\sum_{m\ge 1}\sum_{n\ge 2m}(-1)^n q^{C(n+1,2)-2m^2}(1-q^{2m})\\ &\quad\times\sum_{i=0}^{m}(-1)^i q^{C(i,2)}(q^{i+1};q)_{2m-1-2i}\frac{A_{2m-2i}}{(q;q)_{2m-2i}}\\ &= \sum_{n\ge 0}(-1)^n q^{C(n+1,2)}\\ &\quad+\sum_{m\ge 1}\sum_{n\ge 2m}(-1)^{n+m} q^{C(n+1,2)-3m^2/2-m/2}(1-q^{2m})\\ &\quad\times\sum_{t=0}^{m}(-1)^t q^{C(t+1,2)-mt}\frac{(q^{m-t+1};q)_t(q^m;q)_t}{(1-q^m)}\frac{A_{2t}}{(q;q)_{2t}}\\ &= \sum_{m=-\infty}^{\infty}\sum_{n\ge 2|m|}(-1)^{n+m} q^{C(n+1,2)-3m^2/2+m/2}\\ &\quad\times\sum_{t=0}^{|m|}(q^{-m};q)_t(q^m;q)_t q^t\frac{A_{2t}}{(q;q)_{2t}}.\end{aligned}$$

Equation (1.5) now follows when A_{2m} is replaced by $(q;q)_{2m}f(m)$.

We list some simple corollaries of (1.5).
if $f(m) = \delta_{0,m}$,

$$(2.7)\qquad (q;q)_\infty^2 = \sum_{m=-\infty}^{\infty} \sum_{n\geq 2|m|} (-1)^{n+m} q^{(n^2-3m^2+n+m)/2};$$

if $f(m) = (q;q)_m^{-1}$,

$$(2.8)\qquad (q;q)_\infty = \sum_{m=-\infty}^{\infty} \sum_{n\geq 2|m|} (-1)^{n+m} q^{(n^2-m^2+n+m)/2};$$

if $f(m) = (q^{1/2};q^{1/2})_{2m}^{-1} = (q;q)_m^{-1}(q^{1/2};q)_m^{-1}$,

$$(2.9)\qquad (q;q)_\infty\left\{(q^{1/2};q^{1/2})_\infty + (-q^{1/2};q^{1/2})_\infty\right\} = 2\sum_{m=-\infty}^{\infty} \sum_{n\geq 2|m|} (-1)^{n} q^{(n^2-2m^2+n+m)/2};$$

if $f(m) = q^{-m/2}(q^{1/2};q^{1/2})_{2m}^{-1} = q^{-m/2}(q;q)_m^{-1}(q^{1/2};q)_m^{-1}$,

$$(2.10)\qquad \frac{(q;q)_\infty^2}{(q^{1/2};q)_\infty}\left\{(q^{1/4};q^{1/2})_\infty + (-q^{1/4};q^{1/2})_\infty\right\} = 2\sum_{m=-\infty}^{\infty} \sum_{n\geq 2|m|} (-1)^{n} q^{(n^2-4m^2+n+m)/2}.$$

To get the next two corollaries, it is convenient to leave the right side of (1.5) in the form

$$\sum_{n\geq 0} (-1)^n q^{(n^2+n)/2} + \sum_{m\geq 1} \sum_{n\geq 2m} (-1)^{n+m} q^{(n^2-3m^2+n-m)/2}(1+q^m) \times \sum_{t=0}^{m} (q^{-m};q)_t (q^m;q)_t q^t f(t).$$

We now make the following choices for $f(m)$:
if $f(m) = (q^2;q^2)_m^{-1} = (q;q)_m^{-1}(-q;q)_m^{-1}$,

$$(2.11)\qquad (q;q)_\infty(q^2;q^2)_\infty = \sum_{m=-\infty}^{\infty} \sum_{n\geq 2|m|} (-1)^{n+m} q^{(n^2-2m^2+n)/2};$$

if $f(m) = (-1)^m(q^2;q^2)_m^{-1} = (-1)^m(q;q)_m^{-1}(-q;q)_m^{-1}$,

$$\frac{(q;q)_\infty^3(q^4;q^4)_\infty}{(q^2;q^2)_\infty^2} = \sum_{m=-\infty}^{\infty}\sum_{n\geq 2|m|}(-1)^{n+m}q^{(n^2-4m^2+n)/2}. \tag{2.12}$$

Equation (2.11) can also be found in [1, equ. (3.17)].

3. Proofs of (1.6) and (1.7)

The starting point is the following identity discovered by Rogers (see [11], p. 343; also see [3]):

(3.1)

$$\frac{(\mu\nu;q)_\infty}{(\mu e^{i\theta};q)_\infty(\mu e^{-i\theta};q)_\infty(\nu e^{i\theta};q)_\infty(\nu e^{-i\theta};q)_\infty} = \sum_{n\geq 0} H_n(\mu,\nu|q)A_n(\cos\theta|q),$$

where $H_n(\mu,\nu|q)$ satisfies

$$\frac{1}{(\mu x;q)_\infty(\nu x;q)_\infty} = \sum_{n\geq 0} H_n(\mu,\nu|q)x^n. \tag{3.2}$$

We shall use two special cases of (3.1). First let $\mu = r^{1/2}$, $\nu = -r^{1/2}$. Then

$$H_{2n+1}(r^{1/2},-r^{1/2}|q) = 0 \quad\text{and}\quad H_{2n}(r^{1/2},-r^{1/2}|q) = \frac{r^n}{(q^2;q^2)_n}.$$

We thus get that

$$\frac{(-r;q)_\infty}{(re^{2i\theta};q^2)_\infty(re^{-2i\theta};q^2)_\infty} = \sum_{n\geq 0}\frac{A_{2n}(\cos\theta|q)r^n}{(q^2;q^2)_n}. \tag{3.3}$$

If we set $\mu = r$, $\nu = rq^{1/2}$, then

$$H_n(r,rq^{1/2}|q) = \frac{r^n}{(q^{1/2};q^{1/2})_n}$$

and so (3.1) becomes

$$\frac{(r^2q^{1/2};q)_\infty}{(re^{i\theta};q^{1/2})_\infty(re^{-i\theta};q^{1/2})_\infty} = \sum_{n\geq 0}\frac{A_n(\cos\theta|q)r^n}{(q^{1/2};q^{1/2})_n}. \tag{3.4}$$

We now proceed exactly as in §2 with equation (2.1) replaced by either (3.3) or (3.4). In equation (3.3) we set $r = q$ and then expand the left hand side using (2.3) with q replaced by q^2, θ by 2θ:

$$(3.5)\qquad \frac{(-q;q)_\infty}{(qe^{2i\theta};q^2)_\infty(qe^{-2i\theta};q^2)_\infty} = \frac{(-q;q)_\infty}{(q^2;q^2)_\infty^2}\sum_{m=-\infty}^{\infty}\sum_{n\ge|m|}\cos 2m\theta(-1)^{n+m}q^{n^2+n-m^2}.$$

We use equation (2.2) to obtain an expansion in terms of $\{A_m\}$:

$$(3.6)\qquad \frac{(-q;q)_\infty}{(qe^{2i\theta};q^2)_\infty(qe^{-2i\theta};q^2)_\infty}$$

$$= \frac{(-q;q)_\infty}{(q^2;q^2)_\infty^2}\left\{\sum_{n\ge0}(-1)^n q^{n^2+n} + \sum_{m\ge1}\sum_{n\ge m}(-1)^{n+m}q^{n^2+n-m^2}\right.$$
$$\left.\times\sum_{i=0}^{m}\frac{(-1)^i q^{C(i,2)}(1-q^{2m})(q;q)_{2m-1-i}}{(q;q)_i(q;q)_{2m-2i}}A_{2m-2i}\right\}$$

$$= \frac{(-q;q)_\infty}{(q^2;q^2)_\infty^2}\left\{\sum_{n\ge0}(-1)^n q^{n^2+n} + \sum_{m\ge1}\sum_{n\ge m}(-1)^{n+m}q^{n^2+n-m^2}(1-q^{2m})\right.$$
$$\left.\times\sum_{t=0}^{m}(-1)^{m+t}q^{m^2/2-m/2+t^2/2+t/2-mt}(q^{m-t+1};q)_{2t-1}\frac{A_{2t}}{(q;q)_{2t}}\right\}$$

$$= \frac{(-q;q)_\infty}{(q^2;q^2)_\infty^2}\left\{\sum_{n\ge0}(-1)^n q^{n^2+n} + \sum_{m\ge1}\sum_{n\ge m}(-1)^n q^{n^2+n-m^2/2-m/2}(1+q^m)\right.$$
$$\left.\times\sum_{t=0}^{m}(-1)^t q^{t^2/2+t/2-mt}(q^{m-t+1};q)_t(q^m;q)_t\frac{A_{2t}}{(q;q)_{2t}}\right\}$$

$$= \frac{(-q;q)_\infty}{(q^2;q^2)_\infty^2}\sum_{m=-\infty}^{\infty}\sum_{n\ge|m|}(-1)^n q^{n^2+n-m^2/2+m/2}$$
$$\times\sum_{t=0}^{|m|}(q^{-m};q)_t(q^m;q)_t\frac{A_{2t}q^t}{(q;q)_{2t}}.$$

On the other hand, from equation (3.3) with $r = q$ we see that

$$(3.7)\qquad \frac{(-q;q)_\infty}{(qe^{2i\theta};q^2)_\infty(qe^{-2i\theta};q^2)_\infty} = \sum_{n\ge0}\frac{A_{2n}q^n}{(q^2;q^2)_n}.$$

Thus we obtain equation (1.6) by equating the right hand side of (3.6) and (3.7), replacing A_{2n} by $(q;q)_{2n}f(n)$ and multiplying each side by

$$(-q;q)^{-1}(q^2;q^2)^2 = (q;q)(q^2;q^2).$$

We consider several choices for $f(m)$:
if $f(m) = \delta_{0,m}$,

$$(3.8)\qquad (q;q)_\infty(q^2;q^2)_\infty = \sum_{m=-\infty}^{\infty}\sum_{n\ge|m|}(-1)^n q^{(2n^2-m^2+2n+m)/2},$$

if $f(m) = (q;q)_m^{-1}(q^{1/2};q)_m^{-1}$ (multiply each side by $1+q^{1/2}$),

$$(3.9)\qquad (-q^{1/2};q)_\infty(q^2;q^2)_\infty = \sum_{m=-\infty}^{\infty}\sum_{n\ge|m|}(-1)^{n+m}q^{n^2+n+m/2}(1+q^{1/2})$$
$$= \sum_{n\ge0} q^{n^2+n/2}(1+q^{n+1/2}) = \sum_{n=-\infty}^{\infty} q^{n^2+n/2};$$

if $f(m) = q^{-m/2}(q;q)_m^{-1}(q^{1/2};q)_m^{-1}$,

$$(3.10)\qquad \frac{(q^2;q^2)_\infty^2}{(q^{1/2};q)_\infty} = \sum_{m=-\infty}^{\infty}\sum_{n\ge|m|}(-1)^{n+m}q^{n^2-m^2+n+m/2}.$$

For the following corollaries, we use (1.6) with the right side in the form

$$\sum_{n\ge0}(-1)^n q^{n^2+n} + \sum_{m\ge1}\sum_{n\ge m}(-1)^n q^{(2n^2-m^2+2n-m)/2}(1+q^m)$$
$$\times\sum_{t=0}^{m}(q^{-m};q)_t(q^m;q)_t q^t f(t).$$

We now make the following choices for $f(m)$:
if $f(m) = (q^2;q^2)_m^{-1} = (q;q)_m^{-1}(-q;q)_m^{-1}$,

$$(3.11)\qquad (q^2;q^2)_\infty^3 = \sum_{m=-\infty}^{\infty}\sum_{n\ge|m|}(-1)^n q^{n^2+n} = \sum_{n\ge0}(-1)^n(2n+1)q^{n^2+n};$$

if $f(m) = (-1)^m(q^2;q^2)_m^{-1} = (-1)^m(q;q)_m^{-1}(-q;q)_m^{-1}$,

$$(3.12)\qquad \frac{(q;q)_\infty^2(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^2} = \sum_{m=-\infty}^{\infty}\sum_{n\ge|m|}(-1)^n q^{n^2+n-m^2}.$$

Equation (3.9) is also a consequence of the Jacobi triple product identity [15, (2.2.10)]. Equation (3.11) is the Jacobi identity (1.3) with q replaced by q^2. Equation (3.8) is interesting in view of (2.11).

Now, starting with equation (3.4) we derive (1.7). We set $r = q^{1/4}$ and expand the left hand side using (2.3):

$$(3.13)\quad \frac{(q;q)_\infty}{(q^{1/4}e^{i\theta};q^{1/2})_\infty(q^{1/4}e^{-i\theta};q^{1/2})_\infty} = \frac{(q;q)_\infty}{(q^{1/2};q^{1/2})^2_\infty}\sum_{m=-\infty}^{\infty}\sum_{n\geq|m|}\cos m\theta(-1)^{n+m}q^{(n^2+n-m^2)/4}.$$

From equation (3.4) with $r = q^{1/4}$ we also have

$$(3.14)\quad \frac{(q;q)_\infty}{(q^{1/4}e^{i\theta};q^{1/2})_\infty(q^{1/4}e^{-i\theta};q^{1/2})_\infty} = \sum_{n\geq 0}\frac{A_n q^{n/4}}{(q^{1/2};q^{1/2})_n}.$$

Exactly as before, we expand $\cos m\theta$ in terms of $\{A_n\}$ using equation (2.3) and then compare the coefficients of the A_n with n even to obtain

$$(3.15)\quad \frac{(q^{1/2};q^{1/2})_\infty}{(-q^{1/2};q^{1/2})_\infty}\sum_{n\geq 0}\frac{A_{2n}q^{n/2}}{(q^{1/2};q^{1/2})_{2n}} = \sum_{m=-\infty}^{\infty}\sum_{n\geq 2|m|}(-1)^{n+m}q^{(n^2+n)/4-m^2/2+m/2} \times\sum_{t=0}^{|m|}(q^{-m};q)_t(q^m;q)_t\frac{A_{2t}q^t}{(q;q)_{2t}}$$

Equation (1.7) is obtained when A_{2t} is replaced by $(q;q)_{2t}f(t)$ and then q is replaced by q^2 throughout. We give corollaries of equation (1.7):
if $f(m) = \delta_{0,m}$,

$$(3.16)\quad (q;q)_\infty(q;q^2)_\infty = \sum_{m=-\infty}^{\infty}\sum_{n\geq 2|m|}(-1)^{n+m}q^{(n^2-2m^2+n+2m)/2};$$

if $f(m) = (q;q)_{2m}^{-1} = (q^2;q^2)_m^{-1}(q;q^2)_m^{-1}$ (multiply each side by $1-q$),

$$(3.17)\quad (q;q)_\infty\left\{(q^{1/2};q)^2_\infty(1+q^{1/2}) + (-q^{1/2};q)^2_\infty(1-q^{1/2})\right\} = 2\sum_{m=-\infty}^{\infty}\sum_{n\geq 2|m|}(-1)^n q^{(n^2+n+2m)/2}(1-q) = 2\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2/2+n/2-[n/2]}.$$

For the following corollaries, we use (1.7) with the right side in the form

$$\sum_{n\geq 0}(-1)^n q^{(n^2+n)/2} + \sum_{m\geq 1}\sum_{n\geq 2m}(-1)^{n+m} q^{(n^2-2m^2+n-2m)/2}(1+q^{2m})$$
$$\times \sum_{t=0}^{m}(q^{-2m};q^2)_t(q^{2m};q^2)_t q^{2t} f(t).$$

We now make the following choices for $f(m)$:
if $f(m) = (q^2;q^2)_m^{-1}(-q^2;q^2)_m^{-1}$,

$$(3.18)\qquad \frac{(q;q)_\infty(q^4;q^4)_\infty}{(q^2;q^2)_\infty} = \sum_{m=-\infty}^{\infty}\sum_{n\geq 2|m|}(-1)^{n+m}q^{(n^2+n)/2}$$
$$= \sum_{n\geq 0}(-1)^{n+[n/2]}q^{(n^2+n)/2};$$

if $f(m) = (-1)^m(q^2;q^2)_m^{-1}(-q^2;q^2)_m^{-1}$,

$$(3.19)\qquad \frac{(q;q)_\infty^3(q^4;q^4)_\infty}{(q^2;q^2)_\infty^2} = \sum_{m=-\infty}^{\infty}\sum_{n\geq 2|m|}(-1)^{n+m}q^{(n^2-4m^2+n)/2}.$$

Equations (3.17) and (3.18) are also consequences of the Jacobi triple product identity [15, (2.2.10)]. Equation (3.16) makes an interesting companion to (2.11), and equation (3.19) is, of course, the same as equation (2.12).

References

1. G.E. Andrews, *Hecke modular forms and the Kac-Peterson identities*, Trans. Amer. Math. Soc., vol. 283 (1984), pp. 451–458.
2. R. Askey and M. Ismail, *A generalization of ultraspherical polynomials*, Studies in Pure Mathematics, ed. P. Erdös, Akad. Kaido, Budapest 1983, pp. 55–78.
3. D. Bressoud, *A simple proof of Mehler's formula for q-Hermite polynomials*, Indiana Univ. Math. J., vol. 29 (1980), pp. 577–580.
4. ______, *On partitions, orthogonal polynomials and the expansion of certain infinite products*, Proc. London Math. Soc. (3), vol. 42 (1981), pp. 478–500.
5. L. Carlitz, *Some polynomials related to theta functions*, Ann. Math. Pura Appl. Ser. 4, vol. 41 (1955), pp. 359–373.
6. ______, *Some polynomials related to theta functions*, Duke Math. J., vol. 24 (1957), pp. 521–527.
7. G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers*, 4th ed., Oxford University Press, Oxford, 1960.
8. E. Hecke, *Über einen Zusammenhang zwischen elliptischen Modulfunktionen und indefiniten quadratischen Formen*, Mathematische Werke, Vandenhoeck and Ruprecht, Göttingen, 1959, pp. 418–427.
9. V.G. Kac and D.H. Peterson, *Infinite dimensional Lie algebras, theta functions and modular forms*, Adv. in Math., vol. 53 (1984), pp. 125–264.

10. L.J. Rogers, *On a three-fold symmetry in the elements of Heine's series*, Proc. London Math. Soc., vol. 24 (1893), pp. 171–179.
11. ______, *On the expansion of some infinite products*, Proc. London Math. Soc., vol. 24 (1893), pp. 337–352.
12. ______, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc., vol. 25 (1894), pp. 318–343.
13. ______, *Third memoir on the expansion of certain infinite products*, Proc. London Math. Soc., vol. 26 (1895), pp. 15–32.
14. G. Szego, *Ein Beitrag zur Theorie der Thetafunktionen*, Sitz, Preuss. Akad. Wiss. Phys. Math., vol. 19 (1926), pp. 242–252.
15. G.E. Andrews, *The theory of partitions*, Addison-Wesley, Reading, Mass., 1976.

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