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INTEGRAL REPRESENTATIONS FOR POSITIVE SOLUTIONS OF THE HEAT EQUATION ON SOME UNBOUNDED DOMAINS

BY

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0. Introduction

The main result (Theorem 3.4) in this paper extends the integral representation theorem of Widder [7, Theorem 5.2, p. 143] for positive solutions of the heat equation on $\mathbf{R}_+ \times (0, T)$ to positive solutions on $\mathbf{R}^{n-1} \times \mathbf{R}_+ \times (0, T)$ for arbitrary n. (This result may be part of the mathematical folk-lore but no proof exists in the literature.) Then in Section 4, the Appell transform is used to obtain similar results on $\mathbf{R}^{n-1} \times \mathbf{R}_+ \times (-\infty, 0)$ and $\mathbf{R}^{n-1} \times \mathbf{R}_+ \times \mathbf{R}$, thus generalizing the result in [4, Theorem A] which was obtained by different methods. The integral representation obtained for positive solutions on $\mathbf{R}^{n-1} \times \mathbf{R}_+ \times \mathbf{R}$ verifies an assumption needed in [6, Remarks (2)] to compare fine limits with parabolic limits at points on $\mathbf{R}^{n-1} \times \{0\} \times \mathbf{R}$.

1. Preliminaries

Let $0 < T \leq \infty$,

$$X = \mathbf{R}^{n-1} \times \mathbf{R}_+ \times (0, T) = \{(x', x_n, t) : x' \in \mathbf{R}^{n-1}, x_n > 0, 0 < t < T\},$$

$$H = \mathbf{R}^{n-1} \times \mathbf{R}_+ \times \{0\} \text{ (the horizontal boundary of } X),$$

$$V = \mathbf{R}^{n-1} \times \{0\} \times [0, T) \text{ (the vertical boundary of } X),$$

$$B = H \cup V, \text{ and } V_+ = V \setminus (\mathbf{R}^{n-1} \times \{0\} \times \{0\}).$$

The fundamental solution of the heat equation $\Delta_x u = \partial u / \partial t$ on $\mathbf{R}^n \times \mathbf{R}$ is given by

$$W(x, t; y, s) = \begin{cases} [4\pi(t-s)]^{-n/2} \exp\left(-\dfrac{\|x-y\|^2}{4(t-s)}\right) & \text{if } t > s \\ 0 & \text{if } t \leq s. \end{cases}$$

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The potential theory for the heat equation on X (cf. [1], [2]) will be used freely in this paper. A solution of the heat equation on an open subset U of X will be said to be *harmonic* on U.

For each $(x, t) = (x', x_n, t) \in \mathbf{R}^{n-1} \times \mathbf{R}_+ \times \mathbf{R}$, define

$$K_b(x,t) = \begin{cases} W(x,t;(b',b_n),0) - W(x,t;(b',-b_n),0) \\ \qquad \text{if } b = (b', b_n, 0),\ b_n > 0 \\ \dfrac{\partial}{\partial b_n}\left[W(x,t;(b',b_n),s) - W(x,t;(b',-b_n),s)\right]\Big|_{b_n=0} \\ \qquad \text{if } b = (b',0,s). \end{cases}$$

In the case of $n = 1$, denote K_b defined above by $K_b^{(1)}$.

For each Borel set $E \subset B$ and Borel measure μ on E, the function $K\mu : X \to [0, \infty]$ is defined by

$$K\mu(x,t) = \int_E K_b(x,t)\, d\mu(b).$$

If $d\mu(b) = f(b)\, db$ we denote $K\mu$ by Kf.

The following factorisation will be useful in deducing results for arbitrary n from those for $n = 1$.

Let W' denote the fundamental solution of the heat equation on $\mathbf{R}^{n-1} \times \mathbf{R}$. Then,

$$(*) \qquad K_b(x,t) = \begin{cases} W'(x',t;b',0)K^{(1)}_{(b_n,0)}(x_n,t) & \text{if } b = (b', b_n, 0) \\ W'(x',t;b',s)K^{(1)}_{(0,s)}(x_n,t) & \text{if } b = (b',0,s). \end{cases}$$

By using $K^{(1)}1 = 1$ it follows that $K1 = 1$.

Theorem 8.1 in [7, p. 40] implies that

$$\int_0^\infty K^{(1)}_{(0,t)}(\lambda, s) K^{(1)}_{(0,s)}(\tau, r)\, ds = K^{(1)}_{(0,t)}(\lambda + \tau, r)$$

for all strictly positive λ, τ, r. Hence, by using $(*)$ and the semi-group property of W it is seen that for any (x, t) and $(z, r) \in X$,

$$\int_0^\infty \int_{\mathbf{R}^{n-1}} K_{(z,0)}(b,r) K_{(b,0)}(x,t)\, db = K_{(z,0)}(x, r+t)$$

and

$$\int_0^\infty \int_{\mathbf{R}^{n-1}} K_{(x',0,t)}(b', x_n, s) K_{(b',0,s)}(z,r)\, db'\, ds = K_{(x',0,t)}(z', x_n + z_n, r).$$

An analogue of the Schwarz reflection principle for analytic functions was proved (in the case $n = 1$) in [7, Theorem 7, p. 115]. The proof of this result can be simplified by using the local criterion for harmonicity in terms of averages over regular neighbourhoods of a point. Observe that if D is a rectangle in $\mathbf{R}^{n+1}$ which is symmetric with respect to the plane $x_n = 0$, then the Green function for D satisfies

$$G((x',0),t;(y',y_n),s) = G((x',0),t;(y',-y_n),s)$$

(cf. [3, p. 85]). Hence the following reflection principle holds for arbitrary n.

If u is a continuous function on $X \cup V_+$ which is harmonic on X and $u = 0$ on V_+, then the function

$$\tilde{u}(x,t) = \begin{cases} u(x,t) & \text{if } (x,t) \in X \cup V_+ \\ -u((x',-x_n),t) & \text{if } (x',-x_n,t) \in X \end{cases}$$

is harmonic on $\mathbf{R}^n \times \mathbf{R}_+$.

By using methods in [7, p. 61] it can be shown that if $f: B \to \mathbf{R}$ is a bounded Borel function which is continuous at $y \in B$ then $Kf(x,t) \to f(y)$ as $(x,t) \to y$.

The first main departure from methods in [7] will now be made by showing that $K_b\, db$ is the harmonic measure on X.

THEOREM 1.1. *For each bounded, Borel $f: B \to \mathbf{R}$, Kf is the solution of the Dirichlet problem (in the sense of Perron-Wiener-Brelot) on X corresponding to f.*

Proof. It suffices to assume f is continuous with compact support. Then Kf is bounded, harmonic on X and approaches f continuously on B. Hence $Kf \geq H_f^X$ (the solution of the Dirichlet problem on X corresponding to f). By the reflection principle, $Kf - H_f^X$ can be extended to a bounded harmonic function on $\mathbf{R}^n \times \mathbf{R}_+$ having continuous boundary value 0 on $\mathbf{R}^n \times \{0\}$. Hence $Kf = H_f^X$ (cf. [3, p. 29]).

COROLLARY 1.2. *If u is a positive continuous function on $X \cup B$ which is harmonic on X, then $u \geq Ku$.*

Proof. For each $m = 1, 2, 3, \ldots$, define $u_m = \min(u, m)$. Then $u_m \geq H_{u_m}^X = Ku_m$. The fact that $u = K^{(1)}u$ for u as above is an important step in proving the integral representation theorem in [7]. The proof of this result in [7] depends, among other things, on the similar result for bounded functions u and on estimating certain integrals on sides of rectangles. A simpler, less geometric proof of this result for arbitrary n is presented in the next section.

2. To show $u = Ku$

In the remainder of this paper, C denotes a general strictly positive constant (not necessarily the same at different occurrences).

A uniqueness theorem for X will now be proved by using Corollary 1.2 and a uniqueness theorem for $\mathbf{R}^n \times \mathbf{R}_+$ (cf. [3, Theorem 16, p. 29]).

THEOREM 2.1. *If u is a positive continuous function on $X \cup B$ which is harmonic on X and $u = 0$ on B, then $u \equiv 0$.*

Proof. Fix $x' = 0$, $x_n = 1$ and $0 < t < T$. For any $0 < s < t$, Corollary 1.2 applied to $\mathbf{R}^{n-1} \times \mathbf{R}_+ \times (s, T)$ gives

$$\int_0^\infty \int_{\mathbf{R}^{n-1}} \exp\left(- \frac{\|b'\|^2 + 1 + b_n^2}{4(t-s)} \right) \sinh\left(\frac{b_n}{2(t-s)} \right) u(b, s)\, db'\, db_n$$
$$\leq [4\pi(t-s)]^{n/2} u(x, t).$$

Let $0 < r < t$. Then for $0 < s < t - r$,

$$\int_{1/2}^\infty \int_{\mathbf{R}^{n-1}} e^{-\|b\|^2/4r} u(b, s)\, db'\, db_n \leq C,$$

which implies that $\int_0^{t-r} \int_{1/2}^\infty \int_{\mathbf{R}^{n-1}} e^{-\|b\|^2/4r} u(b, s)\, db'\, db_n\, ds$ is finite. Now, for any $0 < b_n < 1/2$, Corollary 1.2 applied to $\mathbf{R}^{n-1} \times (b_n, \infty) \times (0, T)$ gives

$$\int_0^t \int_{\mathbf{R}^{n-1}} (t-s)^{-(n+2)/2} (1 - b_n) \exp\left(- \frac{\|b'\|^2 + (1 - b_n)^2}{4(t-s)} \right) u(b', b_n, s)\, db'\, ds$$
$$\leq (4\pi)^{n/2} u(x, t).$$

Hence $\int_0^{t-r} \int_0^{1/2} \int_{\mathbf{R}^{n-1}} e^{-\|b\|^2/4r} u(b, s)\, db'\, db_n\, ds$ is finite for every $0 < t < T$ and $0 < r < t$. Now, let $\tilde{u}$ be the extension of u to $\mathbf{R}^n \times [0, T)$ obtained by the reflection principle. Then $\tilde{u}(x, 0) = 0$ for all $x \in \mathbf{R}^n$ and

$$\int_0^{t-r} \int_{\mathbf{R}^n} e^{-\|b\|^2/4r} |\tilde{u}(b, s)|\, db\, ds$$

is finite for every $0 < t < T$ and $0 < r < t$. Hence it follows from [3, p. 29] that $\tilde{u} \equiv 0$.

The main result in this section is now easily deduced.

THEOREM 2.2. *If u is a positive continuous function on $X \cup B$ which is harmonic on X then $u = Ku$ on X.*

Proof. There exists an increasing sequence of bounded continuous functions on B which converge pointwise to u on B and $u \geq Ku$, hence $Ku(x, t) \to u(b)$ as $(x, t) \to b$. The proof is completed by applying Theorem 2.1 to $u - Ku$.

3. Integral representation theorem on X

In this section, the Lebesgue Dominated Convergence Theorem is abbreviated by LDCT.

LEMMA 3.1. *Let τ be a Borel measure on $\mathbf{R}_+$ and $0 < r < T$ such that*

$$g(s, t) = \int \left\{ \exp\left[-\frac{(s-q)^2}{4t} \right] - \exp\left[-\frac{(s+q)^2}{4t} \right] \right\} d\tau(q)$$

is finite for all $s > 0$ and $r < t < T$. Then $g(s, t) \to 0$ as $(s, t) \to (0, p)$ if $r < p < T$.

Proof.

$$g(s, t) = 2e^{-s^2/4t} \int \sinh\left(\frac{sq}{2t} \right) e^{-q^2/4t} \, d\tau(q).$$

Fix p such that $r < p < T$. Then there exists $M > 0$, $0 < \delta < 1$ such that $r < \delta M < p < M < T$. Now, $\delta M < t < M$ implies

$$\begin{aligned} g(s, t) &\leq 2e^{-s^2/4t} \int \sinh\left(\frac{sq}{2\delta M} \right) e^{-q^2/4M} \, d\tau(q) \\ &= \exp\left(\frac{s^2}{4\delta^2 M} - \frac{s^2}{4t} \right) g(\delta^{-1}s, M), \end{aligned}$$

which is finite for all $s > 0$. Also, the integrand is an increasing function of s, hence the result follows by applying the LDCT.

Observe that

$$K_{(b', b_n, 0)}(x', x_n, t) = K_{(x', x_n, 0)}(b', b_n, t)$$

but

$$K_{(b', 0, s)}(x', x_n, t) \neq K_{(x', 0, t)}(b', x_n, s).$$

However, the following easily proved result compensates for this lack of symmetry.

LEMMA 3.2. *Let $f\colon V \to \mathbf{R}$ be bounded, Borel and continuous at $b \in V$. Then*

$$\int K_b(x', \lambda, t) f(x', 0, t)\, dx'\, dt \to f(b) \text{ as } \lambda \to 0+ .$$

From now on, μ_E denotes the restriction of a measure μ to a measurable set E.

PROPOSITION 3.3. *Let μ be a Borel measure on B such that $K\mu$ is finite on X. Then*

(i) $K\mu_H(b', b_n, t)\, db'\, db_n \to d\mu_H(b', b_n, 0)$ *weakly as* $t \to 0+$,

(ii) $K\mu_V(b', \lambda, s)\, db'\, ds \to d\mu_V(b', 0, s)$ *weakly as* $\lambda \to 0+$.

Proof. Fix $(z, r) \in X$ and for each $0 < t < T - r$ define the measure μ_t on H by

$$d\mu_t(b) = K\mu_H(b', b_n, t) K_b(z, r)\, db.$$

Then

$$\mu_t(H) = \int_H \int_H K_y(b', b_n, t) K_b(z, r)\, d\mu(y)\, db = K\mu_H(z, t + r)$$

by the semi-group property. Hence each μ_t is a finite Borel measure on H. Now, for each continuous $f\colon H \to \mathbf{R}$ having compact support,

$$\lim_{t \to 0} \int f(b)\, d\mu_t(b) = \lim_{t \to 0} \int_H \int_H K_y(b', b_n, t) K_b(z, r) f(b)\, db\, d\mu(y)$$

$$= \int_H K_y(z, r) f(y)\, d\mu(y) \quad \text{(by the LDCT)}.$$

Hence $K\mu_H(b', b_n, t)\, db'\, db_n \to d\mu_H(b', b_n, 0)$ weakly as $t \to 0+$.

The result in (i) follows by observing that $K\mu_V(x, t) \to 0$.

Fix (z, r) as before and for each $\lambda > 0$ define the measure ν_λ on V by

$$d\nu_\lambda(b', 0, s) = K\mu_V(b', \lambda, s) K_{(b', 0, s)}(z, r)\, db'\, ds.$$

Then, as in the proof of (i), the semi-group property of the kernels and the LDCT can be used to obtain the result in (ii).

The main result in this paper will now be proved.

THEOREM 3.4. *For every positive harmonic function on X, there is a unique Borel measure μ on B such that $u = K\mu$. (μ is called the representing measure for u).*

Conversely, if μ is a Borel measure on B such that $K\mu$ is finite on X, then $K\mu$ is harmonic on X.

Proof. The uniqueness of the representing measure follows from Proposition 3.3.

To prove the existence of the representing measure, fix $z \in \mathbf{R}^{n-1} \times \mathbf{R}_+$. For each t such that $0 < t < T$ there exists $m_t \in \mathbf{N}$ such that $t + 1/m < T$ for all $m \geq m_t$. By applying Theorem 2.2 to $\mathbf{R}^{n-1} \times (1/m, \infty) \times (1/m, T)$,

$$u\left(x', x_n + \frac{1}{m}, t + \frac{1}{m}\right) = \int_B K_b(x, t) u\left(b \oplus \frac{1}{m}\right) db \quad \text{for } m \geq m_t,$$

where

$$b \oplus \frac{1}{m} = \begin{cases} \left(b', b_n + \frac{1}{m}, \frac{1}{m}\right) & \text{if } b = (b', b_n, 0),\ b_n > 0 \\ \left(b', \frac{1}{m}, s + \frac{1}{m}\right) & \text{if } b = (b', 0, s). \end{cases}$$

Now, for each r such that $0 < r < T$ and $m = 1, 2, 3, \ldots$, define the measure μ_m^r on

$$B_r = \mathbf{R}^{n-1} \times \mathbf{R}_+ \times (0, r)$$

by

$$d\mu_m^r(b) = K_b(z, r) u\left(b \oplus \frac{1}{m}\right) db.$$

Then for each $(x, t) \in \mathbf{R}^{n-1} \times \mathbf{R}_+ \times (0, r)$,

$$u\left(x', x_n + \frac{1}{m}, t + \frac{1}{m}\right) = \int \frac{K_b(x, t)}{K_b(z, r)} d\mu_m^r(b)$$

for sufficiently large m. Now,

$$\mu_m^r(B_r) = u\left(z', z_n + \frac{1}{m}, r + \frac{1}{m}\right)$$

which is bounded as a function of m, hence $\{\mu_m^r\}$ has a weak limit point μ^r. It is easy to prove that

$$b \to \frac{K_b(x, t)}{K_b(z, r)}$$

is continuous and vanishes at ∞ on $\mathbf{R}^{n-1} \times \mathbf{R}_+ \times (0, r)$, and it follows that for

$0 < t < r$,

$$u(x,t) = \int K_b(x,t)\,d\nu_r(b) \quad \text{where } d\nu_r(b) = [K_b(z,r)]^{-1}\,d\mu^r(b).$$

The existence then follows from the uniqueness.

To prove the second part, let F be a compact subset of H or V. Then $\mu(F) < \infty$ and the LDCT implies that $K\mu_F$ is a continuous on X. Fubini's theorem implies that $K\mu_F$ satisfies the mean-value equality for harmonic functions and so is harmonic on X. The proof is completed by using increasing sequences of compact subsets of H and V and the Doob convergence property.

4. Integral representations on a quarter-space and a right half-space

In this section, let $T = \infty$, so X is the "upper quarter-space" $\mathbf{R}^{n-1} \times \mathbf{R}_+ \times \mathbf{R}_+$. Let Y be the "lower quarter-space" $\mathbf{R}^{n-1} \times \mathbf{R}_+ \times (-\infty, 0)$ and Z be the right half-space $\mathbf{R}^{n-1} \times \mathbf{R}_+ \times \mathbf{R}$.

In the case of $n = 1$, Kaufman and Wu [4] obtained an integral representation theorem on Z by using Widder's result for a semi-infinite strip, and the uniqueness of the representing measure was proved by using analytic functions.

In this section, the Appell transform and elementary measure theory are used to obtain integral representation theorems on Y and Z from Theorem 3.4.

The Appell transform will now be defined. For each $(x,t) \in \mathbf{R}^n \times \mathbf{R}_+$, define

$$k(x,t) = W(x,t;0,0) = (4\pi t)^{-n/2}\exp\left(-\frac{\|x\|^2}{4t}\right).$$

For each $v : Y \to \mathbf{R}$, define Av: $X \to \mathbf{R}$ by

$$Av(x,t) = k(x,t)\,v(t^{-1}x, -t^{-1}).$$

Av is called the Appell transform of v. Then A is a bijection from the set of harmonic functions on Y to the set of harmonic functions on X whose inverse A^* is given by

$$A^*u(x,t) = [k(-t^{-1}x, -t^{-1})]^{-1}u(-t^{-1}x, -t^{-1})$$

(cf. [7, p. 14]). Also,

$$A^*K_b(x,t) = \begin{cases} 2\exp\left(\frac{1}{4}t\|b\|^2 + \frac{1}{2}\langle x', b'\rangle\right)\sinh\left(\frac{1}{2}x_n b_n\right) & \text{if } b = (b', b_n, 0) \\ x_n\exp\left(\frac{1}{4}t\|b'\|^2 + \frac{1}{2}\langle x', b'\rangle\right) & \text{if } b = (b', 0, 0) \\ s^{-(n+2)/2}\exp\left(\frac{\|b'\|^2}{4s^2}\right)K_{(s^{-1}b', 0, -s^{-1})}(x,t) & \text{if } b = (b', 0, s),\ s \neq 0, \end{cases}$$

where $\langle x', b'\rangle = \sum_{i=1}^{n-1} x_i b_i$. Set

$$B_Y = \left(\mathbf{R}^{n-1} \times \mathbf{R}_+ \times \{-\infty\}\right) \cup \left(\mathbf{R}^{n-1} \times \{0\} \times [-\infty, 0)\right)$$

and

$$B_Z = \left(\mathbf{R}^{n-1} \times \mathbf{R}_+ \times \{-\infty\}\right) \cup \left(\mathbf{R}^{n-1} \times \{0\} \times [-\infty, \infty)\right).$$

For each $(x, t) \in Z$ and $b \in B_Z$ define

$$\tilde{K}_b(x,t) = \begin{cases} \exp\left(t\|(b', b_n)\|^2 + \langle x', b'\rangle\right) \sinh(x_n b_n) & \text{if } b = (b', b_n, -\infty), b_n > 0 \\ x_n \exp\left(t\|b'\|^2 + \langle x', b'\rangle\right) & \text{if } b = (b', 0, -\infty) \\ K_b(x,t) & \text{if } b = (b', 0, s), s \in \mathbf{R}. \end{cases}$$

Now, define the map $\psi : B_Y \to B$ by

$$\psi(b) = \begin{cases} (2b', 2b_n, 0) & \text{if } b = (b', b_n, -\infty), b_n > 0 \\ (2b', 0, 0) & \text{if } b = (b', 0, -\infty) \\ (-s^{-1}b', 0, -s^{-1}) & \text{if } b = (b', 0, s), s \in (-\infty, 0). \end{cases}$$

Then, for any Borel measure ν on B_Y and $(x, t) \in Y$,

$$\begin{aligned} \int \tilde{K}_b(x,t)\, d\nu(b) &= \int_H A^*K_b(x,t)\, d(1/2\nu \circ \psi^{-1})(b) \\ &\quad + \int_{\mathbf{R}^{n-1} \times \{0\} \times \{0\}} A^*K_b(x,t)\, d(\nu \circ \psi^{-1})(b) \\ &\quad + \int_{V_+} A^*K_b(x,t)\, d(\nu_1 \circ \psi^{-1})(b) \end{aligned}$$

where

$$d\nu_1(b', 0, s) = (-s)^{-(n+2)/2} \exp\left(-\frac{\|b'\|^2}{4}\right) d\nu(b', 0, s).$$

The following integral representation theorem for harmonic functions on Y is now easily deduced from Theorem 3.4 by applying the Appell transform.

THEOREM 4.1. *For each positive harmonic function u on Y there exists a unique Borel measure μ on B_Y such that $u(x,t)) = \int \tilde{K}_b(x, t\, d\mu(b)$ for all $(x,t) \in Y$.*

Conversely, if μ is a Borel measure on B_Y such that $u(x,t) = \int \tilde{K}_b(x,t)\, d\mu(b)$ is finite on Y, then u is harmonic on Y.

Now, let $u \geq 0$ be harmonic on Z. For each $p = 1, 2, 3, \ldots,$ let

$$Z_p = \{(x, t) \in Z : t < p\}$$

and

$$B_p = (\mathbf{R}^{n-1} \times \mathbf{R}_+ \times \{-\infty\}) \cup (\mathbf{R}^{n-1} \times \{0\} \times [-\infty, p).$$

Then, by Theorem 4.1, for each p, there exists a unique Borel measure ν_p on B_Y such that

$$u(x, t) = \int \tilde{K}_b(x, t - p)\, d\nu_p(b) \quad \textit{for all } (x, t) \in Z_p.$$

Observe that

$$\tilde{K}_{(b', b_n, -\infty)}(x, t - p) = \exp(-p\|(b', b_n)\|^2)\tilde{K}_{(b', b_n, -\infty)}(x, t),$$
$$\tilde{K}_{(b', 0, -\infty)}(x, t - p) = \exp(-p\|b'\|^2)\tilde{K}_{(b', 0, -\infty)}(x, t)$$

and

$$\tilde{K}_{(b', 0, s)}(x, t - p) = \tilde{K}_{(b', 0, s, +p)}(x, t) \quad \textit{if } s \in (-\infty, 0).$$

Hence $u = \int \tilde{K}_b \, d\mu_p$ on Z_p for a unique measure μ_p on B_p.

Consequently, Theorem 4.1 holds when Y is replaced by Z, thus obtaining the required integral representation theorem for positive harmonic functions on Z.

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