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GEOMETRIC AND TOPOLOGICAL PROPERTIES OF CERTAIN w^* COMPACT CONVEX SUBSETS OF DOUBLE DUALS OF BANACH SPACES, WHICH ARISE FROM THE STUDY OF INVARIANT MEANS

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INTRODUCTION

Let $\mathscr{F} = (\phi \in (l^\infty)^*;\ \phi(1) = 1 = \|\phi\|,\ \phi(\delta_n) = 0$ for all $n \geq 1\}$ where $\delta_n \in l^\infty$ are defined by $\delta_n(k) = 1$ if $k = n$ and is 0 otherwise. Then $\mathscr{F}$ contains a set $H = \beta N \sim N$ which satisfies: card $H = 2^c$ (where c is the cardinality of the reals), H is a w^* perfect set and H is isometric to a canonical l^1 basis i.e. for any linear combination of elements $h_i \in H$, $\|\Sigma_1^n c_i h_i\| = \Sigma_1^n |c_i|$.

We will call a w^* compact subset A of the dual X^* of a Banach space X "big" if there is an *onto* bounded linear map t: $X \to l^\infty$ such that $t^*(\mathscr{F}) \subset A$. Note that in this case t^* is an isomorphism *into* X^* (i.e., for some $\gamma > 0$, $\|t^*\phi\| \geq \gamma\|\phi\|$ for each $\phi \in (l^\infty)^*$) and if $H = t^*(\beta N \sim N)$ then: card $H = 2^c$, H is w^* perfect and H is *isomorphic* to a canonical l^1 basis (i.e., $\|\Sigma_1^n c_i h_i\| \geq \gamma \Sigma_1^n |c_i|$ for any linear combination of elements of H).

There are many results in the literature on invariant means which, when slightly paraphrased, express the fact that a set of invariant means is "big".

The most useful corollary of our main result of Section 1 (Theorem 1.1) is Theorem 1.4 and it incorporates some of these results in the setting of Banach spaces. It is a definitive improvement of some results we obtained in the past [8, Cor. 1.3]. It states loosely, that if X is a Banach space, $K \subset X$ convex bounded (assumed embeded in X^{**}) S a countable set of operators s_n: $X \to X$, $A = \{y \in w^*\text{cl } K;\ s_n^{**}y = 0$ for all $n\}$ and if some nonvoid w^* G_δ-section A_0 of A can be pushed outside the w^*-sequential closure of K (w^*seq cl K) then A_0 is necessarily "big". Specifically we have:

THEOREM 1.4. *Let X be a Banach space, S a sequence of operators s_n: $X \to X$, $K \subset X$ convex bounded $A = \{y \in w^*\text{cl}K;\ s_n^{**}y = 0$ for $n \geq 1\}$. If for*

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[1]This paper contains the details of part of the results contained in the lecture delivered by this author at the Conference on the Geometry of Normed Linear Spaces in honor of Mahlon M. Day. It is dedicated to Mahlon M. Day with much gratitude and respect.

some $x_n^* \in X^*$ and scalars α_n the nonvoid set

$$A_0 = \{ y \in A : \langle y, x_n^* \rangle = \alpha_n \text{ for } n \geq 1 \}$$

is such that $A_0 \cap w^*\text{seq cl} K = \varnothing$ then there exists an onto linear bounded map t: $X^* \to l^\infty$ such that

(a) *$t^* \mathscr{F} \subset A_0$, hence A_0 is big, afortiori does not have the WRNP, and*

(b) *t cl lin$\{S^*X^* \cup \{x_n^*\}_1^\infty \cup K^0\} \subset c$ where*

$$K^0 = \{x^* \in X^*; x^*(K) = 0\}, \quad c = \{a = \{a_n\} \in l^\infty; \lim a_n \text{ exists}\}$$

and $S^*X^* = \cup_1^\infty s_n^* X^*$.

It is easily seen in the remark after Theorem 1.4 that $\mathscr{F}$ is the "biggest" set which can be included in such a set A_0 i.e. isomorphically the above result is definitive. Theorem 1.4 is false if card $S = c$.

Ching Chou, in solving an open problem of Joe Rosenblatt [19] has proved the following:

THEOREM (CC). *If G is a countable group of measure preserving maps which acts ergodically on the nonatomic probability space $(X, \mathscr{S}, p)$ and if there is some $\phi_0 \in M_G$ (the set of G invariant means on $L^\infty(X)$) such that $\phi_0 \neq p$ then there exists an onto bounded linear map t: $L^\infty(X) \to l^\infty$ such that t^*: $(l^\infty)^* \to L^\infty(X)^*$ is an isometry into and such that $t^*(\mathscr{F}) \subset M_G$. If $H = t^*(\beta N \sim N)$ then* card $H = 2^c$, *H is a w^* perfect set and the elements of H have pairwise disjoint supports, i.e., H is* **isometric** *to a canonical l^1* basis.

Our Theorem 1.4, being an isomorphic result yields, except for the fact that H is only isomorphic to a canonical l^1 basis, much more than Chou's result. We get a result with the following as a particular case:

THEOREM 2.6. *Let $(X\mathscr{S}\mu)$ be any σ-finite measure space, G a countable set of bounded linear maps g_n: $L^1(X) \to L^1(X)$ and let*

$$M_G = \{ \psi \in L^\infty(X)^*; \|\psi\| = 1 = \psi(1), g_n^{**}\psi = \psi \text{ for all } n \}.$$

Assume that there is some $\psi_0 \in M_G$ which is not in $L^1(X)$ and let then $X_n \subset X$ be such that $1_{X_n} \uparrow 1$ a.e. but $\psi_0(1_{X_n}) = \gamma_n \to \gamma < 1$. Let $A = \{\psi \in M_G; \psi(1_{X_n}) = \gamma_n$ for all $n\}$. Then any nonvoid w^*G_δ-section A_0 of A (afortiori A) is "big".

Joe Rosenblatt and M. Talagrand in their study of G-invariant means which are *not* invariant with respect to some given transformation, have proved in [20] a result *which when restricted* to countable G is as follows:

THEOREM (RT). *Let G be a countable* **amenable group** *G acting on a set D and let M_G be the set of G invariant means on $l^\infty(D)$. Assume that for any $E \subset D$ with* card $E <$ card D, *$\phi(1_E) = 0$ for each $\phi \in M_G$. Let v: $D \to D$ be a one to one onto map. Then the following are equivalent.*

(a) *There exists some $\phi \in M_G$ and a set $E \subset D$ such that $\phi(1_E) \neq \phi(1_{vE})$.*

(b) *There exists $0 < \delta \leq 1$ and a set H of mutually singular elements of M_G such that* card $H = 2^c$ *and for some fixed $E \subset D$, $\phi(1_E) = 1$ and $\phi(1_{vE}) \leq 1 - \delta$ for each ϕ in H.*

Using our Theorem 1.4, we get a result which is a definite improvement of a slightly weaker version of the above. We have:

THEOREM 2.1. *Let G be a countable set of bounded linear maps*

$$g_n: l^1(D) \to l^1(D)$$

and assume that $M_G \cap l^1(D) = \varnothing$. Let v_n: $l^\infty(D) \to l^\infty(D)$ be ***arbitrary*** *maps, $f_n \in l^\infty(D)$, $\psi_0 \in M_G$ and let $\alpha_n = \psi_0(f_n)$, $\beta_n = \psi_0(v_n f_n)$. Then the set*

$$A_0 = \{\phi \in M_G; \phi(f_n) = \alpha_n, \phi(v_n f_n) = \beta_n \text{ for all } n\}$$

is "big".

Hence, if we are willing to accept the fact that the elements of H are only isomorphic to a canonical l^1 basis, then we can relax the conditions on G and v. $G[v]$ need only be a countable set of (arbitrary) bounded linear maps g_n: $l^1(D) \to l^1(D)$ (v_n: $l^\infty(D) \to l^\infty(D)$).

We further apply our Theorem 1.4 to amenable actions of second countable locally compact groups G on a locally compact space Z and obtain Theorem 2.4 (see Section 2 for details).

Our last application of Theorem 1.4 is to certain algebras $PM_p(G)$ of operators on $L^p(G)$, for second countable groups G, and obtain definitive improvements of results of Ching Chou [1] and of ours [9], [10]. In [1] and [9], C^* and W^* algebra methods play a crucial role and these work only for $p = 2$. They do not seem to work for $p \neq 2$. It is at this place, when trying to generalize results of Chou to $p \neq 2$ that we discovered that H. Rosenthal's fundamental Theorem 1 of [21] can replace the W^* algebra methods of Chou and became thus one of the main ingredients of Theorem 1.1.

We come now to the next main result of Section 1 namely Theorem 1.6 and an application.

Let s_k: $l^\infty \to l^\infty$ be given by $(s_k a)(n) = a(n + k)$ and let AC be the closed linear span of $\{C1 + \Sigma_1^n(s_k a^k - a^k\}$ where $a^k \in l^\infty$, $n \geq 1$ and C is the field of scalars (complexes or reals). This is just the space of almost convergent sequences and $c \subset AC$. (As known, $WAP(N) \subset AC$ and AC is not norm separable). We have:

THEOREM 1.6. *Let s_n: $X \to X$, $K \subset X$ convex bounded,*

$$A = \{ y \in w^*\mathrm{cl} K;\ s_n^{**}y = 0 \text{ for each } n \},$$

$$\varnothing \neq A_0 = \{ y \in A;\ y(x_n^*) = \alpha_n \text{ for each } n \}$$

be such that $A_0 \cap w^*\mathrm{seq\ cl}\ K = \varnothing$ and t: $X^* \to l^\infty$ (onto), be all as in Theorem 1.4

Let $R_1' \subset l^\infty$ be separable and R_0' be the closed linear span of $\{R_1' + AC\}$. Then $X^*/t^{-1}(R_0')$ (and afortiori X^*/W for any subspace $W \subset t^{-1}(R_0')$) has l^∞ as a continuous homomorphic image.

We apply Theorem 1.6 to amenable actions of locally compact groups G on coset spaces $Z = G/G_0$ and obtain Theorem 2.5′, a particular case of which is:

THEOREM 2.5. *Let G be second countable G_0 a closed subgroup, $Z = G/G_0$ and let ψ_0 be a topologically invariant mean on $UCB_l(Z)$. Let TIM denote the set of all such means. Let F_0 be the linear span of*

$$\{ f - \alpha \Box f;\ f \in UCB_l(Z), \alpha \in P_1(G) \}$$

where $P_1(G) = \{\alpha \in L^1(G);\ \alpha \geq 0 \int \alpha\, d\lambda = 1\}$. Let $f_n \in L^\infty(Z, \nu)$ and let

$$K \subset \left\{ \beta \in L^1(Z\nu);\ \beta \geq 0, \int \beta\, d\nu = 1 \right\}$$

be convex and such that $\psi_0 \in w^*\mathrm{cl}\ K$, where ν is a quasiinvariant measure on Z. Let R_0 be the closed linear span of

$$\{ C1 + F_0 + (f_n)_1^\infty + K^0 \}$$

where $K^0 = \{ f \in UCB_l(Z);\ f(K) = 0\}$. Then $UCB_l(Z)/W$ has l^∞ as a continuous homomorphic image, for any subspace $W \subset R_0$, if G/G_0 does not admit a finite invariant measure.

The reader may want to note that if G_0 is normal in G then $WAP(G/G_0)$ is a (somewhat small) subspace of R_0. Theorem 2.5 improves a result in [9] and is related to results of Ching Chou [1] (see Section 2 for details).

The reader will find, we hope, other results in this paper which are interesting for their own sake.

0. Definitions, notations and remarks

Let E be a normed space over the reals or complexes. E^* will denote its dual. If F is a space of linear functionals on E then $\sigma(E, F)$ will denote the weakest vector topology on E which makes all linear maps $e \to \langle f, e\rangle$, $e \in E$ continuous, for each $f \in E$. The weak (w) [weak* (w^*)] topology on E [E^*] is just $\sigma(E, E^*)$ [$\sigma(E^*, E)$]. Let Q: $E \to E^{**}$ be the canonical imbedding. If $K \subset E$ we will sometimes look upon K as being canonically imbedded in E^{**} and thus write K instead of QK. We will then write (K, w^*) or $w^*\text{cl}\, K$ instead of (QK, w^*), $w^*\text{cl}\, QK$, respectively. These denote QK with the w^* topology and the w^* closure of K in E^{**}, respectively. If $K \subset E$ is convex, ext K is the set of extreme points of K. We say that the bounded set $B \subset E$ is isomorphic (isometric) to a canonical l^1 basis if there is some $c > 0$ such that

$$\left\|\sum_1^n \alpha_i b_i\right\| \geq c\sum_1^n |\alpha_i|, \quad \left(\left\|\sum_1^n \alpha_i b_i\right\| = \sum_1^n |\alpha_i|\right)$$

for any finite subset $\{b_1, \ldots, b_n\} \subset B$ and scalars $\alpha_1, \ldots, \alpha_n$.

All the results in this paper are true for spaces over the complex (C) or real (R) numbers. The only place where this needs special attention is in the use of H. Rosenthal's fundamental theorem which is true for R or C as proved by L. Dor (see [21, p. 805]).

If X is completely regular Hausdorff, $C(X)$ will denote the bounded R (or C) valued continuous functions on X. If $a \in X$, $p_a \in C(X)^*$ is defined by $p_a f = f(a)$ for f in $C(X)$. It is readily seen that $\|\Sigma_1^n \alpha_i p_{a_i}\| = \Sigma_1^n |\alpha_i|$ for any $\{a_1, \ldots, a_n\} \subset X$ and scalars α_i. For any Banach algebra A let

$$\Delta_A = \{\psi \in A^*; \phi \neq 0 \textit{ and multiplicative}\}.$$

It is then clear that if $A = C(X)$ or $A = L^\infty(X, \mathcal{S}, \mu)$ where $(X\mathcal{S}\mu)$ is a measure space then any subset $B \subset \Delta_A$ is isometric to a canonical l^1 basis.

Any $\phi \in C(X)^*$ ($\phi \in L^\infty(X)^*$) such that $\phi f \geq 0$ if $f \geq 0$ and $\phi 1 = 1$ is called a mean. If $A \subset X$ let $1_A(x) = 1$, if $x \in A$ and 0 otherwise.

If S is a set of maps s: $X \to X$ we let $Sb = \{sb;\, s \in S\}$ and write $Sb = b$ to denote the fact that $sb = b$ for all $s \in S$. $\beta X(\beta N)$ will always denote the Stone-Cech compactification of the topological space X (N denotes the discrete positive integers).

Let τ be a locally convex vector topology on the vector space E (we write (E, τ) is an l.c.s) and $K \subset E$. We write $\tau\text{seq}\,\text{cl}\, K$ for the τ-sequential closure of K; thus $x_0 \in \tau\text{seq}\,\text{cl}\, K$ iff there is some sequence $x_n \in K$ such that $\tau \lim_{n\to\infty} x_n = x_0$.

Let $\mathcal{F} = \{\phi \in (l^\infty)^*;\ \phi(1) = 1 = \|\phi\|;\ \phi\{c_0\} = 0\}$, where $l^\infty[c_0]$ is the space of bounded real or complex sequences $a = \{a_n\}_1^\infty$ (such that $\lim_{n\to\infty} a_n$

$= 0$) with the usual norm $\|a\| = \sup|a_n|$. The set $\mathscr{F}$ will be *important* in the sequel. This set $\mathscr{F}$ is "big" in the following sense: $\mathscr{F}$ contains the set $\beta N \sim N = H$ such that (a) card $H = 2^c$ where c is the cardinality of the continuum, (b) H is a w^* perfect set (i.e., compact with no isolated points) and (c) H is isometric to a canonical l^1 basis (as any subset of Δ_{l^∞}). The set $\beta N \sim N$ can be identified with the set of free ultrafilters on the positive integers N (and $A \sim B$ will always denote the set theoretical difference of A and B).

If $A \subset E$ then $\mathrm{Co}\, A$ ($\mathrm{cl\,lin}(A)$) will denote the convex hull (closed linear span) of A.

Any other notation in this paper is consistent with Dunford-Schwartz: *Linear operators*, vol. I, Interscience, New York, 1958.

1. Results on the geometry of Banach spaces

DEFINITION. (a) Let A be a subset of the *l.c.s.* (E, τ). The set A_0 is said to be a τ-G_δ section of A if there exist τ-continuous linear functionals x_n' on X and scalars α_n, $n = 1, 2, 3\ldots$ such that

$$A_0 = \{x \in A;\ x_n'(x) = \alpha_n\}.$$

Note that $A = A_0$ if we take all $x_n' = 0$, $\alpha_n = 0$.

(b) We defined

$$\mathscr{F} = \{\phi \in (l^\infty)^*;\ 1 = \phi(1) = \|\phi\|;\ \phi\{c_0\} = 0\}$$

where

$$c_0 = \left\{f \in l^\infty;\ \lim_n f(n) = 0\right\}.$$

Note that whenever $f \in l^\infty$ and $\lim_n f(n)$ exists, $\phi(f) = \lim_n f(n)$ for any $\phi \in \mathscr{F}$.

(c) We identify any subset K of the banach space X with its canonical image QK in its second dual X^{**}.

THEOREM 1.1. *Let X, X_n be Banach spaces, s_n: $X \to X_n$ be bounded linear, $a_n \in X_n$, $n = 1, 2, \ldots$, and $K \subset X$ a bounded convex set. Let*

$$A = \{w^*\mathrm{cl}\ K\} \cap \{y \in X^{**};\ s_n^{**}y = a_n \text{ for each } n \geq 1\}$$

Let A_0 be a nonvoid w^*G_δ section of A.

If $A_0 \subset (w^*\mathrm{cl}\ K) \sim w^*\mathrm{seq\ cl}\ K$ then there exists a bounded linear **onto** *map t: $X^* \to l^\infty$ such that t^*: $(l^\infty)^* \to X^{**}$ is a norm isomorphism* **into** *and such that $t^*(\mathscr{F}) \subset A_0$.*

In particular, if $H = t^*(\beta N \sim N) \subset A_0$ *then* (a) card $H = 2^c$, (b) H *is a* w^* *perfect set and* (c) H *is* **isomorphic** *to a canonical* l^1 *basis.*

Remark. $w^*\text{seq cl } K$ can be replaced by the elements of $w^*\text{cl } K$ which are Baire $-$ 1 functions on (B^*, w^*) where $B^* = \{x^* \in X^*; \|x^*\| \le 1\}$ (see Odell-Rosenthal [17, lemma 1 and remark on p. 379]).

Proof. Choose $x_n^* \in X^*$ and scalars α_n such that

$$A_0 = \{w^*\text{cl } K\} \cap \{y \in X^{**};\ s_n^{**}y = a_n, \langle y, x_n^* \rangle = \alpha_n, n \ge 1\}.$$

Let $y_0 \in A_0$ be fixed and let $w_\alpha \in K$ be such that $w^*\lim w_\alpha = y_0$. By a slight adaptation of a technique of Namioka (see [8, p. 17]) there exists a net v_β of convex combinations of the w_α such that

$$w^*\lim v_\beta = y_0,\ \lim_\beta \|s_n v_\beta - a_n\| = 0$$

and

$$\lim_\beta |\langle x_n^*, v_\beta \rangle - \alpha_n| = 0 \quad \text{for each } n = 1, 2, \ldots .$$

For each n, choose β_n such that

$$\|s_k v_{\beta_n} - a_n\| < \frac{1}{n} \quad \text{and} \quad |\langle x_k^*, v_{\beta_n} \rangle - \alpha_k| < \frac{1}{n} \quad \textit{if } k \le n.$$

We claim that $v_{\beta_n} \in X$ does not have any weak, i.e., $\sigma(X, X^*)$, Cauchy subsequence. In fact, assume that $v'_k = v_{\beta_{n_k}}$ is one. Then clearly

$$\lim_k \|s_n v'_k - a_n\| = 0 \quad \text{and} \quad \lim_k |\langle x_n^*, v'_k \rangle - \alpha_n| = 0 \quad \text{for each } n = 1, 2, \ldots .$$

Let $z \in X^{**}$ be given by $z = w^*\lim v'_k$. Then $s_n^{**}z = w^*\lim_k s_n^{**}v'_k = a_n$ and $\langle z, x_n^* \rangle = \lim_k \langle v'_k, x_n^* \rangle = \alpha_n$, for each $n \ge 1$. Hence $z \in A_0$. But $z \in w^*\text{seq cl } K$, which cannot be.

Since v_{β_n} does not have any weak Cauchy subsequence, Rosenthal's fundamental Theorem 1 on p. 805 of [7] shows that some subsequence $u_k = v_{\beta_{n_k}}$ is isomorphic to a canonical l^1 basis. (The scalars may be real or complex here as proved by L. Dor.) This means that there is a $\delta > 0$ such that for any linear combination of the u_k's we have

$$\left\| \sum_1^n c_j u_j \right\| \ge \delta \sum_1^n |c_j|. \tag{0}$$

Clearly $\lim_j \|s_n u_j - a_n\| = 0$ and $\lim_j |\langle x_n^*, u_j\rangle - \alpha_n| = 0$ for each n.

Note that the u_j's depend on the $x_n^* \in X^*$ and on the scalars α_n *which define* A_0.

Inspired by an idea of Ching Chou (see [2, Theorem 3.3]), we define t: $X^* \to l^\infty$ by $(tx^*)(n) = \langle x^*, u_n\rangle$. It is easy to show now that t is *onto* l^∞. (We avoid the difficulties encountered by Chou [2] who heavily uses order and geometric properties of W^*-algebras and their preduals, by the use of Rosenthal's fundamental theorem.)

In fact, if $b = \{b_n\} \in l^\infty$ is arbitrary define the linear functional

$$\left\langle \hat{b}, \sum_1^n c_j u_j \right\rangle = \sum_1^n c_j b_j.$$

Then by (0) we have

$$\left| \left\langle \hat{b}, \sum_1^n c_j u_j \right\rangle \right| \le \|b\| \sum_1^n |c_j| \le \|b\| \delta^{-1} \left\| \sum_1^n c_j u_j \right\|.$$

Hence $\hat{b}$ is bounded and, by the Hahn-Banach theorem, has a norm preserving extension $b^* \in X^*$. Clearly $tb^*(n) = \langle \hat{b}, u_n\rangle = b_n$ and

$$\|b^*\| \le \delta^{-1} \|b\|_\infty. \tag{1}$$

In particular $t(X^*) = l^\infty$. Also $\|tx^*\| = \sup_n |\langle x^*, u_n| \le \|x^*\| M$; thus $\|t\| \le M$, where $M = \sup\{\|x\|; x \in K\}$. Furthermore if $\phi \in (l^\infty)^*$ and δ is given by Rosenthal's theorem then, by (1),

$$\|t^*\phi\| = \frac{1}{\delta} \sup_{\|x^*\| \le 1} |\langle \phi, t\delta x^*\rangle| \ge \frac{1}{\delta} \sup_{\|b\|_\infty \le 1} |\langle \phi, b\rangle| = \frac{1}{\delta} \|\phi\|.$$

Thus t^*: $(l^\infty)^* \to X^{**}$ is an isomorphism into (with $\|t^*\phi\| \ge 1/\delta$ where δ is determined by Rosenthal's theorem.

We show now that $t^*\mathscr{F} \subset A_0$.

Let $\phi \in (l^\infty)^*$ be such that $\phi f = 0$ whenever $f \in c_0$ and such that $\phi(1) = 1 = \|\phi\|$. Then $\langle \phi, b\rangle = \lim b_n$ whenever $\lim_n b_n$ exists. But

$$\langle s_n^{**} t^* \phi, x^*\rangle = \langle \phi, ts_n^* x^*\rangle$$

and

$$(ts_n^* x^*)(k) = \langle x^*, s_n u_k\rangle \to \langle x^*, a_n\rangle \quad \text{as } k \to \infty. \tag{2}$$

Thus $\langle \phi, ts_n^* x^*\rangle = \langle x^*, a_n\rangle$ and $s_n^{**} t^* \phi = a_n$ for all n.

Furthermore $\langle t^*\phi, x_n^*\rangle = \langle \phi, tx_n^*\rangle$ and since u_k is a subsequence of v_{β_j} we have

$$(t, x_n^*)(k) = \langle x_n^* u_k\rangle \to \alpha_n \quad \text{as } k \to \infty. \tag{3}$$

Thus $\langle \phi, tx_n^*\rangle = \alpha_n$ for all n.

Now we use the fact that $\|\phi\| = 1 = \phi(1)$ to show that $t^*\phi \in w^*\text{cl}\, K$. Let $\delta_n \in (l^\infty)^*$ be defined by $\langle \delta_n, b\rangle = b_n$ for $b = \{b_n\} \in l^\infty$. Then

$$\langle t^*\delta_n, x^*\rangle = \langle \delta_n, tx^*\rangle = \langle x^*, u_n\rangle$$

since $tx^*(k) = \langle x^*, u_k\rangle$. It follows that $t^*\delta_n = u_n \in K \subset X^{**}$. But $\mathscr{F} \subset w^*\text{cl}\,\text{Co}\{\delta_n; n \geq 1\}$. Hence $t^*\mathscr{F} \subset w^*\text{cl}\, K$. It follows that $t^*\mathscr{F} \subset A_0$ which finishes the proof of the first part. Now consider the set $\beta N \sim N \subset \Delta_{l^\infty} = \{\phi \in (l^\infty)^*; \phi \neq 0$ and multiplicative$\}$. As known, card $(\beta N \sim N) = 2^c$ (see Rudin [22, p. 411]) and $\beta N \sim N$ is a w^* perfect set [22, p. 414]). As remarked in Section 0 any subset of Δ_{l^∞} is *isometric* to a canonical l^1 basis. We have shown above that t^* is a $w^* - w^*$ continuous norm isomorphism *into*. Thus if $H = t^*(\beta N \sim N) \subset A_0$ then (a) card $H = 2^c$, (b) H is a w^* perfect set and (c) H is (only) isomorphic to a canonical l^1 basis. (Thus $\|h_1 - h_2\| \geq 2/\delta$ if $h_1 \neq h_2$ and h_1, h_2 belong to H. In particular, the least cardinality of a set norm dense in A_0 is $\geq 2^c$.) QED

K. Musial introduced in [15] the weak Radon-Nikodym property (WRNP). A subset K of a Banach space E is said to have the WRNP if for every finite measure space $(X\mathscr{S}\mu)$ and bounded linear T: $L^1(X\mathscr{S}\mu) \to E$ such that $T(\mu(A)^{-1}1_A) \in K$ whenever $\mu(A) \neq 0$, T is represented by a Pettis kernel with values in K, i.e., there is some $f: X \to K$ such that $e^*(f(\omega))$ is measurable for each $e^* \in E^*$. Moreover for each $A \in \mathscr{S}$ there is some $e_A \in K$ such that $e^*(e_A) = \int_A e^*(f(\omega))\, d\mu(\omega)$.

In contrast with the RNP, the WRNP is not a hereditary property. If however $K \subset E^*$ is a w^* compact convex WRNP set and $K_1 \subset K$ is a w^* compact convex subset then K_1 has the WRNP. This, together with the following beautiful theorem is due to E. Saab [24, p. 308].

THEOREM. *Let K be a w^* compact convex subset of the dual E^* of a Banach space E. The following are equivalent.*

(a) *K has the WRNP.*

(b) *Every bounded sequence $\{x_n\}$ in E has a subsequence $\{x_{n_k}; k \geq 1\}$ such that for each $x^* \in K$, $\lim_k x^*(x_{n_k})$ exists.*

(c) *For every w^*-compact subset M of K the restriction of every $x^{**} \in E^{**}$ to $(M, \sigma(E^*, E))$ has a point of continuity. Consequently w^* compact convex subsets of w^* compact convex WRNP sets have the WRNP.*

In what follows we need the next lemma which we could not find in the literature:

LEMMA 1.2. *Let E be a Banach space, $K \subset E^*$ a w^* compact convex set. Assume that the set $L \subset K$ is isomorphic to a canonical l^1 basis and contains a w^* perfect subset L_1. Then K does not have the WRNP.*

COROLLARY 1.3. *The set A_0 of Theorem 1, and afortiori any w^* compact convex set containing it, do not have the WRNP (hence do not have the RNP).*

This follows from Lemma 1.2 with $A_0 = K$ and $H = t^*(\beta N \sim N) = L = L_1$.

Remarks. It follows that the norm closed linear span of A_0 does not have the RNP and hence *cannot be isomorphic* to the following:

(i) $l^1(\Gamma)$, Γ any cardinality, and in fact moreover to any $(\Sigma_{\alpha\in\Gamma} \oplus E_\alpha)_p, 1 \leq p < \infty$, if each Banach space E_α has the RNP;

(ii) $L_p(\mu, E), 1 < p < \infty$, if E has the RNP;

(iii) dual subspaces of a w.c.g. space;

(iv) weakly locally uniformly convex duals, etc.

For these and more, see Diestel-Uh*l* [4, p. 218].

Clearly, any w^* closed subspace of X^{**} (of Theorem 1.1) which contains the set A_0 is a dual Banach space which does not have the WRNP.

Observation. Let X be compact Hausdorff, $S \subset X$ a countable set dense in itself and let $T = \text{cl}\, S$. Define $g\colon T \to [0,1]$ by $g(S) = 1$, $g(T \sim S) = 0$. Then g does not have any point of continuity.

Proof. g is clearly not continuous at any x in $T \sim S$. ($T \sim S \neq \varnothing$ since otherwise S would be countable and closed and would by Baire's cathegory theorem contain an isolated point in S which cannot be). Assume that g is continuous at $x \in S$. Then there exists an open (in T) set U such that $x \in U \subset S$. Let V be open in T and such that $\text{cl}_T V \subset U \subset S$. Then $\text{cl}_T V$ is compact and countable hence has some isolated (in $\text{cl}_T V$) point y. Let W be open in X satisfy $W \cap \text{cl}_T V = \{y\}$. Then $W \cap V \neq \varnothing$ and hence $W \cap V = \{y\}$ and y is isolated in V and hence in S, which cannot be.

Proof of Lemma 1.2. By Theorem 8 of W. Rudin [23, p. 204], (L_1, w^*) contains a (countable) subset S dense in itself. Then $T = \text{cl}\, S$ is a w^* perfect set and the function $g\colon T \to [0,1]$ defined by $g(S) = 1$, $g(T \sim S) = 0$ does not have any point of continuity, by the above observation. By assumption there is some $\delta > 0$ such that any linear combination of elements of T satisfies $\|\Sigma_1^n \alpha_i \phi_i\| \geq \delta \Sigma_1^n |\alpha_i|$. Define the linear functional ψ on the linear span $[T]$ of T

by

$$\psi\left(\sum_1^n \alpha_i\phi_i\right) = \sum_1^n \alpha_i g(\phi_i);$$

then

$$\left|\psi\left(\sum_1^n \alpha_i\phi_i\right)\right| \le \sum_1^n |\alpha_i| \le \delta^{-1}\left\|\sum_1^n \alpha_i\phi_i\right\|.$$

By the Hahn Banach theorem there is some bounded linear extension again denoted by ψ to all of E^*. Then $\psi \in E^{**}$ and ψ restricted to the w^* compact set (T, w^*) does not have any point of continuity. This shows by *E*. Saab's theorem above that K does not have the WRNP. QED

Recall that

$$c = \left\{a = (a_n) \in l^\infty;\ \lim_n a_n \text{ exists}\right\}$$

and

$$c_0 = \left\{a = \{a_n\} \in l^\infty;\ \lim_n a_n = 0\right\}.$$

The *most useful result* for the sequel is the following theorem which includes most of the previous results for the special case where $X_n = X$ and $a_n = 0$:

THEOREM 1.4. *Let X be a Banach space (over C or R), let s_n: $X \to X$ be bounded linear and $K \subset X$ be bounded convex. Let*

$$A = \{y \in w^*\text{cl } K;\ s_n^{**}y = 0 \text{ for } n \ge 1\} \ne \varnothing.$$

Suppose some nonvoid w^*G_δ section A_0 of A given by

$$A_0 = \{y \in A\colon\ y(x_n^*) = \alpha_n \text{ for } n \ge 1\}$$

is such that $A_0 \subset \{w^*\text{cl } K\} \sim w^*\text{seq cl } K$. Then there exists a bounded ***onto*** *linear map t: $X^* \to l^\infty$ such that*

(a)

$$t \text{ cl lin}\left\{S^*X^* \cup \{x_n^*\}_1^\infty \cup K^0\right\} \subset c$$

where $S^*X^* = \cup_1^\infty s_n^* X^*$, and $K^0 = \{x^* \in X^*;\ x^*(K) = 0\}$.

(b) *t^*: $(l^\infty)^* \to X^{**}$ is a norm isomorphism* ***into*** *(which is $w^* - w^*$ continuous) such that $t^*\mathcal{F} \subset A_0$. (Thus A_0 is "big").*

In particular if $H = t^*(\beta N \sim N) \subset A_0$ then

(c) *card $H = 2^c$, H is a w^* perfect set which is isomorphic to a canonical l^1 basis. Afortiori A_0 does not have the WRNP.*

Remark. The sequence $S = \{s_n\}$ cannot be replaced by a set S of cardinality c: Let $(X, \mathscr{B}, p)$ be an infinite probability space, $L_1(X\mathscr{B}p) = L_1$. For each f in L_∞ let $l_f: L_1 \to L_1$ be defined by $l_f(g) = (\int fg\,dp)1_X = \langle f, g\rangle 1_X \in L_1$. By w^* continuity we have

$$l_f^{**}\psi = \langle \psi, f\rangle 1_X \in L_1 \subset L_\infty^* \quad \text{for all } \psi \in L_\infty^*.$$

Fix some $\psi_0 \in L_\infty^* \sim L_1$ such that $\|\psi_0\| = 1 = \langle \psi_0, 1_X\rangle$. For each f in

$$B = \{f \in L_\infty; \|f\| \le 1\}$$

let $a_f = \langle \psi_0, f\rangle$. If $\psi \in L_\infty^*$ satisfies $\langle \psi, 1_X\rangle = 1$ then $l_{f-a_f1_X}^{**}(\psi) = 0$ for all f in B iff $\langle \psi, f - a_f 1_X\rangle 1_X = 0$ iff $\langle \psi, f\rangle = \langle \psi_0, f\rangle$ for all f in L_∞. Hence, if

$$K = \left\{g \in L_1;\ g \ge 0, \int g = 1\right\}$$

then

$$A = \{w^*\text{cl}\, K\} \cap \left\{\psi \in L_\infty^*;\ l_{f-a_f1_X}^{**}\psi = 0 \textit{ for } f \in B\right\} = \{\psi_0\}.$$

Thus $A \cap w^*\text{seq cl}\, K = \varnothing$. However A is not "big". If $X = \{1, 2, 3, \dots\}$ with $p\{n\} = 2^{-n}$ then $\text{card}\{l_f;\ f \in B\} = c$, the cardinality of the continuum.

Remark. It happens in applications that the bounded convex set K is included in some hyperplane $K \subset \{x \in X; \langle x, x_0^*\rangle = 1\}$ with $x_0^* \in X^*$. Then the w^*G_δ-section A_0 can be written as

$$A_0 = \{y \in A : \langle y, y_n^*\rangle = 0 \text{ if } n \ge 1\} \quad \text{where } y_n^* = x_n^* - \alpha_n x_0^*.$$

If we construct the map t for the y_n^*'s then

$$(ty_n^*)(k) = \langle y_n^*, u_k\rangle \to 0 \quad \text{if } k \to \infty.$$

Thus $ty_n^* \in c_0$ and $t\,\text{cl}\,\text{lin}\{S^*X^* \cup \{y_n\}_1^\infty \cup K^0\} \subset c_0$, and $tx_0^* = 1 \in c$, in this case.

Proof. We only need to prove (a). If $x^* \in X^*$ then by (2) of the proof of Theorem 1.1,

$$t(s_n^* x^*)(k) = \langle x^*, s_n u_k\rangle \to \langle x^*, a_n\rangle = 0 \quad \text{if } k \to \infty.$$

Thus $t(s_n^* X^*) \subset c_0$ for each n, i.e., $t(S^*X^*) \subset c_0$.

Furthermore, by (3) in the proof of Theorem 1, we have

$$(tx_n^*)(k) = \langle x_n^*, u_k\rangle \to \alpha_n \quad \text{if } k \to \infty.$$

Thus $t(x_n^*) \in c$ for all n. Since $(tx^*)(k) = \langle x^*, u_k\rangle$ and $u_k \in K$, we have $tx^* = 0$ if $x^* \in K^0$, and (a) follows.

Remark. Isomorphically, the above theorem is the best possible: If $X = l^1$ then let s: $l^1 \to l^1$ be the zero operator. Let $p_n \in l^\infty$ be defined by $p_n(k) = 1$ if $k = n$ and 0 otherwise. Let $K = \{\phi \in l^1;\ \phi(n) \geq 0$ and $\Sigma\phi(n) = 1\}$. Then

$$\{w^*\mathrm{cl}\, K\} \cap \{\phi \in l^{\infty *};\ \phi(p_n) = 0 \text{ for all } n\} = \mathscr{F}.$$

Hence the set $\mathscr{F}$ of Theorem 1.4 is "attained", and the result is best possible in this sense.

Theorem 1.4 above is an improvement on a result of ours [8, Cor 1.3, p. 21]. In [8], we were only able to prove that card $A_0 \geq 2^{\aleph_1}$ (see p. 61) and that A_0 is not norm separable.

Theorem 1.4 yields the following:

COROLLARY 1.4′. *Let X, K, s_n: $X \to X$, A be as in Theorem* 1.4 *and let A_0 be a w^*G_δ-section of A given by $A_0 = \{y \in A\colon \langle y, x_n^*\rangle = \alpha_n, n \geq 1\}$.*
If (i) card $A_0 < 2^c$ *or* (ii) A_0 *has the WRNP then* $A_0 \cap w^*\mathrm{seq\,cl}\, K \neq \varnothing$.

Remarks. Assume that the semigroup S acts on the set D and define t_s on $l^\infty(D)$ by $(t_s f)(d) = f(sd) - f(d)$. Then the closed subspace R_S spanned by $t_S\{l^\infty(D)\}$ is just the closure of $\{\Sigma_1^n t_{s_i} f_i,\ s_i \in S,\ f_i \in l^\infty(D),\ n \geq 1\}$. Note that the dual of $l^\infty(D)/R_S$ is just the space of S-invariant elements of $l^\infty(D)^*$, i.e.,

$$(l^\infty(D)/R_S)^* = \{\phi \in l^\infty(D)^*;\ t_s^*\phi = 0 \text{ for each } s \in S\} = I_S.$$

The fact that $l^\infty(D)/R_S$ is "big" (for example, it contains l^∞) is another way of expressing the fact that I_S is "big". There are quite a few results in the literature which express the fact that spaces analogous to $l^\infty(D)/R_S + R_1$ (with $R_1 = C1$ or separable and $R_1 \subset l^\infty(D)$) are not norm separable or contain l^∞. For example, see [9, p. 381], [1, p. 197]. A deep result of this type was obtained by Dzinotyiweyi [5, p. 226], namely that for any locally compact noncompact group G, $UC(G)/WAP(G)$ contains an isometric copy of l^∞.

We are able here to improve a slightly weaker version of a more general result. We prove that some of these spaces have l^∞ as a continuous homomorphic image. We gain the fact that the results hold for all Banach spaces.

As known, if $X_1 \subset X_2$ are closed subspaces of the Banach space X then

$$\pi\colon X/X_1 \to X/X_2$$

defined by $\pi(a + X_1) = a + X_2$ is a continuous linear map. Consequently, if X/X_2 has l^∞ as a continuous linear image, so does X/X_1.

LEMMA 1.5. *Let the nonvoid* $A_0 \subset \{w^*\text{cl } K\} \sim w^*\text{seq cl } K$ *be as in Theorem* 1.4. *Let*

$$K^0 = \{x^* \in X^*; x^*(K) = 0\}, \quad S^*X^* = \bigcup_1^\infty s_n^* X^*, \; R_S = \text{cl lin}(S^*X^*),$$

and let $R_1 \subset X^*$ *be any separable subspace. Let* $R_0 = \text{cl lin}(K^0 + R_S + R_1)$. *Then* X^*/R_0 *(and afortiori* X^*/W *for any subspace* $W \subset R_0$ *has* l^∞ *as a continuous homomorphic image.*

Proof. We have $A_0 = \{y \in w^*\text{cl } K; s_n^{**}y = 0; \langle y, x_n^* \rangle = \alpha_n \text{ for } n \geq 1\}$. We can assume that $x_n^* \in R_1$, since by proving the result for a bigger R_1 we will have a stronger result. Let $x_0^{**} \in A_0$ and let $\{r_n^*\}_1^\infty$ be dense in R_1 such that $\{x_k^*\} \subset \{r_n^*\}$. Let $\beta_n = x_0^{**}(r_n^*)$. Then the set $A_1 = \{y \in A_0; y(r_n^*) = \beta_n$ for $n \geq 1\}$ is a w^*G_δ section of A_0 such that $x_0^{**} \in A_1 \subset A_0$. Now construct the onto map $t: X^* \to l^\infty$ of Theorem 1.4 with r_n^*, β_n replacing x_n^*, α_n. Then

$$t[\text{cl lin}\{S^*X^* \cup \{r_n^*\}\}] \subset c$$

by Theorem 1.4(a). If $x^* \in X^*$ then $(tx^*)(n) = \langle x^*, u_n \rangle$ where the $u_n \in K$ are constructed with respect to r_n^*, β_n (instead of x_n^*, α_n). In any case $\langle x^*, u_n \rangle = 0$ for any $x^* \in K^0$ thus $t(K^0) = 0$. It follows that $tR_0 \subset c$. It is enough hence to show that $X^*/t^{-1}(c)$ (which is isomorphic to l^∞/c, via the map $t_1\{x^* + t^{-1}(c)\} = tx^* + c$) has l^∞ as a continuous linear image. The following remark, which is probably known, will in fact prove that l^∞/c contains an isometric copy Y_0 of l^∞. The identity map i: $Y_0 \to Y_0$ has as known a continuous linear extension to l^∞/c [3, p. 106].

Remark. Note that we have shown above that $x^*/t^{-1}(c)$ *contains* an isomorphic copy of l^∞.

Remark 1. l^∞/c contains an isometric copy of l^∞.

Proof. Write the positive integers $N = \cup_0^\infty A_i$ where each A_i, $i = 0, 1, 2, \ldots$ is *infinite* and $A_i \cap A_j = \varnothing$ if $i \neq j$. Let $Y = \{f \in l^\infty; f(A_i) = c_i, i \geq 0\}$, i.e., the functions in l^∞ which are constant on each A_i. Let $Y_0 = \{f \in Y; f(A_0) = 0\}$. Clearly Y and Y_0 are isometric copies of l^∞. Let $\phi_0 \in \beta N \sim N$ be such that $\phi_0(1_{A_0}) = 1$ (i.e., ϕ_0 lives on A_0). If $f \in l^\infty$ let $\bar{f} \in C(\beta N)$ be its unique extension and let $\tilde{f} = \bar{f}_{|\beta N \sim N}$ (the restriction of $\bar{f}$ to $\beta N \sim N$). The map $f \to \tilde{f}$ *from* Y to $C(\beta N \sim N)$ satisfies $\|f\|_{l^\infty} = \|\tilde{f}\|_{C(\beta N \sim N)}$ for each $f \in Y$ since each such f is constant on each A_i. Let $\tilde{Y} = \{\tilde{f}; f \in Y\}$, $\tilde{Y}_0 = \{\tilde{f}; f \in Y_0\}$. Then $\tilde{Y}$ and $\tilde{Y}_0$ are isometric to l^∞. Define the projection P:

$C(\beta N \sim N) \to C1$ by $Pf = (\phi_0 f)1$. Let $Q = I - P$. Then

$$C(\beta N \sim N) = C1 \oplus Q[C(\beta N \sim N)].$$

If $f \in \tilde{Y}_0$ then $Qf = f - (\phi_0 f)1 = f$ since ϕ_0 lives on A_0 and $f(A_0) = 0$ for $f \in \tilde{Y}_0$. Thus

$$\tilde{Y}_0 \subset QC(\beta N \sim N) = C(\beta N \sim N)/C1.$$

However $C(\beta N \sim N) \approx l^\infty/c_0$. Hence

$$C(\beta N \sim N)/C1 \approx (l^\infty/c_0)/(c/c_0) \approx l^\infty/c.$$

Remark 2. Let s_k: $l^\infty \to l^\infty$ be defined by $(s_k a)(n) = a(n+k)$ where $a \in l^\infty$ is given by $a = \{a(n)\}$. Then

$$\operatorname{cl}\operatorname{lin}\{Sl^\infty\} = \operatorname{cl}\operatorname{lin}\left\{\sum_{k=1}^{n} s_k a_k - a_k : a_k \in l^\infty\right\} = R_S$$

where $S = \{s_k - I;\ k \geq 1\}$. As known and easily checked,

$$R_S = \{a \in l^\infty;\ \phi(a) = 0 \text{ for each } \phi \in M_S\}$$

where

$$M_S = \{\phi \in l^\infty;\ \phi(1) = 1 = \|\phi\|,\ s_1^*\phi = \phi\}.$$

In different terminology R_S is the space of sequences which are almost convergent to 0. Denote this space by AC_0. Clearly $c_0 \subset AC_0$.

Then Lemma 1.5 implies that l^∞/W' has l^∞ as a homomorphic image for any closed subspace $W' \subset R_0' = \operatorname{cl}\operatorname{lin}\{AC + R_1'\}$ where R_1' is any separable subspace of l^∞ and, as known, $AC = C1 + AC_0$ is the space of almost convergent sequences (i.e., those $a \in l^\infty$ such that $\{\phi(a);\ \phi \in M_S\}$ consists of one scalar). As known, AC is not norm separable and contains $WAP(N)$. Moreover, there exists a family $\{A_t; 0 \leq t \leq 1\}$ of subsets A_t of the positive integers such that $A_t \subset A_s$ if $t \leq s$ and $\phi(1_{A_t}) = t$ for each $\phi \in M_S$ and $0 \leq t \leq 1$. (See our paper in Trans Amer. Math. Soc., vol. 111, 1964).

THEOREM 1.6. *Let s_n: $X \to X$, $K \subset X$ be convex bounded, $S = \{s_n\}$,*

$$A = \{y \in w^*\text{cl } K,\ S^{**}y = 0\}, \quad A_0 = \{y \in A;\ y(x_n^*) = \alpha_n \text{ for } n \geq 1\} \neq \varnothing,$$

$A_0 \subset w^*\text{cl } K \sim w^*\text{seq cl } K$ and t: $X^* \to l^\infty$ (onto) all be as in Theorem 1.4. *Let $R_1' \subset l^\infty$ be separable and $R_0' = \operatorname{cl}\operatorname{lin}\{AC + R_1'\}$. Then X^*/W has l^∞ as a homomorphic continuous image for each closed subspace $W \subset t^{-1}(R_0')$.*

Proof. $X^*/t^{-1}(R_0')$ is isomorphic to l^∞/R_0'. The above remarks show that l^∞/R_0' has l^∞ as a homomorphic image hence so does X^*/W for any subspace $W \subset t^{-1}(R_0')$.

Remark. While $t^{-1}(c)$ may be separable, $t^{-1}(R_0')$ is not.

2. Applications of Theorems 1.4 and 1.6

Part I. J. Rosenblatt and M. Talagrand in their study of G-invariant means which are *not* invariant with respect to some given transformation have proved the following in part of Theorem 2 of [20]:

(RT) Let G be an *amenable* group acting on an infinite set D such that for any $E \subset D$ with card $E <$ card D, $\phi(1_E) = 0$, for each G-invariant mean on $l^\infty(D)$. Let v: $D \to D$ be a one to one onto map. The following are equivalent.

(a) *There exists a G-invariant mean ϕ and a set $E \subset D$ such that $\phi(1_E) \neq \phi(1_{vE})$.*

(b) *There exists δ, $0 < \delta \leq 1$, and a set H of mutually singular G invariant means with card $H = 2^m$, $m = 2^{\,\text{card}\, G}$, such that for some fixed E, $\phi(1_E) = 1$ and $\phi(1_{vE}) \leq 1 - \delta$ for each $\phi \in H$.*

We have the following remarks:

(i) The fact that the means in H are mutually singular is equivalent to asserting that H is isometric to a canonical l^1 basis (i.e., that $\|\Sigma_1^n c_i h_i\| = \Sigma_1^n |c_i|$ if $h_i \in H$ and c_i are scalars).

(ii) If G is countable then card $H = 2^c$.

(iii) Let M_G be the set of G invariant means on $l^\infty(D)$. If for each $E \subset D$ with card $E <$ card D one has $\phi(1_E) = 0$ for each $\phi \in M_G$ then clearly for each finite $E \subset D$ we have $\phi(1_E) = 0$ for each $\phi \in M_G$.

(iv) $l^1(D)$ is weakly sequentially complete and in fact so is the subset

$$K_1 = \left\{\phi \in l^1(D);\ \phi \geq 0, \sum_d \phi(d) = 1\right\}.$$

Thus $w^*\text{seq cl}\, K_1 = K_1$ in $l^\infty(D)^*$. It is well known that

$$w^*cl\, K_1 = \{\phi \in l^\infty(D)^*;\ \phi(1) = 1 = \|\phi\|\}.$$

As can easily be seen [8, p. 36], the fact that G (even if it is only a semigroup of maps g: $D \to D$) does not have finite orbits in D, i.e., Gd is infinite for each $d \in D$, is equivalent to $M_G \cap l^1(D) = \varnothing$ (or to $M_G \subset w^*\text{cl}\, K_1 \sim w^*\text{seq cl}\, K_1$).

DEFINITION. A w^* compact convex set $A \subset X^*$, where X is a Banach space is said to be "big" if there exists an onto linear continuous map t:

$X \to l^\infty$ such that $t^*\mathscr{F} \subset A$. Then it follows that t^*: $(l^\infty)^* \to X^*$ is a norm isomorphism into. Any big set A contains a set $H = t^*(\beta N \sim N)$ such that card $H = 2^c$, H is w^* perfect and is isomorphic to a canonical l^1-basis. Thus A does not have the WRNP. Our first application of Theorem 1.4 is:

THEOREM 2.1. *Let D be infinite, G a* ***countable*** *set of bounded linear maps*

$$g_n: l^1(D) \to l^1(D)$$

and let $M_G = \{\phi \in l^\infty(D)^*;\ \|\phi\| = \phi(1) = 1,\ G^{**}\phi = \phi\}$.
Assume that $M_G \cap l^1(D) = \varnothing$.
Let $\psi_0 \in M_G$, v_n: $l^\infty(D) \to l^\infty(D)$ *be arbitrary maps*, $f_n \in l^\infty(D)$ *and define* $\alpha_n = \psi_0 f_n$ *and* $\beta_n = \psi_0(v_n f_n)$. *Then the set*

$$A_0 = \{\psi \in M_G;\ \psi f_n = \alpha_n,\ \psi(v_n f_n) = \beta_n \text{ for } n \geq 1\}$$

is "big".

It is enough to assume that $K \subset K_1$ is convex such that

$$\psi_0 \in w^*\text{cl}\, K \quad \text{and} \quad M_G \cap w^*\text{seq}\,\text{cl}(K) = \varnothing.$$

In this case, M_G would have to be replaced by $M_G \cap w^*\text{cl}\, K$ in the definition of A_0.

Remarks. (a) Assume that g: $D \to D$ is a map. Then g induces a map

$$t_g': l^1(D) \to l^1(D)$$

as follows: If $\phi \in l^1(D)$ then $(t_g'\phi)(E) = \phi(g^{-1}E)$ for any $E \subset D$. Then the operator t_g: $l^\infty(D) \to l^\infty(D)$ given by $(t_g f)(d) = f(gd)$ satisfies $t_g = (t_g')^*$, hence $t_g^* = (t')^{**}$. Thus $\psi \in l^\infty(D)^*$ is t_g^* invariant if and only if $(t_g - I)^{**}\psi = 0$ where I is the identity.

(b) If G is a semigroup of maps g: $D \to D$ and if G has no finite orbits in D and $M_G = \{\psi \in l^\infty(G)^*;\ \|\psi\| = \psi(1) = 1,\ t_G^*\psi = \psi\}$ then $M_G \cap l^1(D) = \varnothing$, a known result that can be readily shown [8, p. 36].

As a proof of Theorem 2.1 we just note that $A_0 \neq \varnothing$ is a w^*G_δ-section of $A = M_G \subset \{w^*\text{cl}\, K_1\} \sim w^*\text{seq}\,\text{cl}\, K_1$ and apply our Theorem 1.4.

Comparing the above theorem with the Rosenblatt-Talagrand result (RT) we have the following observations:

We have gained the fact that G need not be an amenable group of point maps g: $D \to D$; rather, *any* countable set of linear bounded g_n: $l^1(D) \to l^1(D)$ will do. The fact that the means ϕ considered are not v-invariant on the set $E(v$: $D \to D$, a one to one map) can be replaced by the (much weaker) conditions $\phi(v_n f_n) = \beta_n$, $\phi f_n = \alpha_n$, $n \geq 1$ where $f_n \in l^\infty(D)$, α_n, β_n are scalars ($\alpha_n \neq \beta_n$ is irrelevant), and v_n: $l^\infty(D) \to l^\infty(D)$ is a *countable* set of *arbitrary* maps.

As noted in (RT), the δ in (a) and (b) are the same. However the set E in (a) of (RT) may be different from the set E in (b) of (RT). In our case the same set E (identified with some f_n) will appear in (a) and (b).

In comparison with the (RT) result, we have weaker results. We need G to be *countable* and we do not get the elements of H *isometric*, rather only *isomorphic*, to a canonical l^1 basis. Furthermore, while (RT) gets a set E such that $\phi(1_E) = 1$ and $\phi(1_{vE}) \leq 1 - \delta$ for each $\phi \in H$, we do not seem to be able to get a set E such that for all $\phi \in H$, E *supports* ϕ and $\phi(vE) \leq 1 - \delta$. This is one of the places where the fact that G is an amenable group seems to be necessary (see [20, Lemma 1]).

We give two examples in which the amenability of G is not present and the (RT) result does not seem to apply, for the purpose of illustration:

Example 1. Let G be a countable nonamenable group, $G_0 \subset G$ a subgroup such that $D = G/G_0 = \{g_k G_0\}$ is an *amenable coset space*; i.e., $l^\infty(D)$ admits a G-invariant mean (where g acts on D from the left by $g(g_k G_0) = g g_k G_0$).

Let $\psi_0 \in M_G = \{\phi \in l^\infty(D)^*;\ \phi(1) = 1 = \|\phi\|, t_G^*\phi = \phi\}$, $E_n \subset D$, let v_n: $D \to D$ be arbitrary maps and define $\psi_0(1_{E_n}) = \dot{\alpha}_n$, $\psi_0(1_{v_n E_n}) = \beta_n$.

Corollary 2.2. *If D is infinite then the set*

$$A_0 = \left\{\phi \in M_G;\ \phi\left(1_{E_n}\right) = \alpha_n,\ \phi\left(1_{v_n E_n}\right) = \beta_n \text{ for } n \geq 1\right\}$$

is big.

The reader will note that the maps v_n can be assumed to be on $l^\infty(D)$, 1_{E_n} can be replaced by $f_n \in l^\infty(D)$ and G may just be a sequence of maps on $l^1(D)$.

To prove this corollary one only needs to note that G does not have finite orbits in D if D is infinite (in fact $Gd = D$ for each $d \in D$).

Example 2. Consider an "inner amenable" countable group G, i.e., a group such that if $(t_x f)(y) = f(x^{-1}yx)$ for $x, y \in G$, $f \in l^\infty(G)$ then $l^\infty(G)$ admits a mean ϕ such that $\psi(t_x f) = \phi(f)$ for all $x \in G$ and $f \in l^\infty(G)$; in addition, $\phi \neq \delta_e$ where $\delta_e(f) = f(e)$ for all $f \in l^\infty(G)$, (e being the unit of G). Any amenable group $G \neq \{e\}$ is inner amenable.

Inner amenability as well as the next example are taken from E.G. Effros, Proc Amer. Math. Soc., vol. 47 (1975), pp. 483–486.

While the group F_2, the free group on two generators, is not inner amenable, the group $G = F_2 \times A$, where A is a nontrivial abelian group, is inner amenable. For example, if $a \in A$ is not the unit of A and we define $\psi(f) = f(e, a)$ for $f \in l^\infty(F_2 \times A)$ where e is the unit of F_2, then the mean ψ is t_G^* invariant and nontrivial. Clearly $F_2 \times A$ is not an amenable group.

For any mean ϕ on $l^\infty(A)$, if $f_0(a) = f(e, a)$, then the mean $\psi f = \phi f_0$ is t_G^* invariant. The mean $\psi f = \frac{1}{2}[f(e, a) + f(e, a^{-1})]$ is t_G^* and inversion invariant (i.e., $\psi f = \psi f^*$ where $f^*(g, a) = f(g^{-1}, a^{-1})$). Now, Theorem 2.1 yields:

COROLLARY 2.3. *Let G be an inner amenable countable group. Let ψ_0 be a t_G^* invariant mean such that $\psi_0 \notin l^1(G)$ and let $E_n \subset G$ be such that $E_n \uparrow G$ but*

$$\psi_0(1_{E_n}) = \gamma_n \uparrow \gamma < 1.$$

Let $f_n \in l^\infty(G)$ and define $\psi_0 f_n = \alpha_n$, $\psi_0 f_n^* = \beta_n$. Then the set

$$A_0 = \{\psi \in M_G;\ \psi 1_{E_n} = \gamma_n,\ \psi f_n = \alpha_n,\ \psi(f_n^*) = \beta_n \text{ for } n \geq 1\}$$

is big.

Part II. Now we apply Theorem 1.4 to *jointly continuous* actions of a locally compact *second countable* group G on the locally compact Hausdorff space Z. Thus $G \times Z \to Z$ is a jointly continuous transformation group. We also assume that Z supports a *quasi invariant Radon measure* ν; this holds if Z is second countable (see Greenleaf [11, p. 297]). We use notations and some basic results from [11] in this section. If $G_0 \subset G$ is a closed subgroup then $Z = G/G_0$ admits a unique such measure ν, up to equivalence of null sets.

The following example [11, p. 304] shows the difficulties which may occur. Let G be the discrete free group on two generators, $Z = G \cup \{\infty\}$ the one point compactification. Let ν_1 be the point mass at $\{\infty\}$. Let ν_2 be the counting measure on $G \subset Z$. We let G act by left translation on $G \subset Z$, and let $G\{\infty\} = \{\infty\}$. Then ν_1 and ν_2 are quasiinvariant measures whose supports in Z are disjoint. Furthermore there is no G-invariant mean on $L^\infty(Z, \nu_2) \approx l^\infty(G)$ but there exists a G-invariant mean on $L^\infty(Z, \nu_1) \approx C$.

Thus, let ν be a fixed quasiinvariant measure on Z. The injection i: $C(Z) \to L^\infty(Z, \nu)$ may be many to one; see [11, p. 298].

The action of G on Z induces an action of $M(G)$, the finite Borel measures on G, on the function spaces $UCB_l(Z)$, $C(Z)$, $L^\infty(Z, \nu)$ by

$$\mu \square f(z) = \int f(gz)\, d\mu(g) \quad \text{for } \mu \in M(G)$$

(see [11, p. 297–298]). The usual action $M(G) \times M(Z) \to M(Z)$ denoted by $\alpha * \beta$ has the property that $L^1(Z, \nu)$ is a closed $M(G)$ submodule. Furthermore

$$\langle \mu \square f, \phi \rangle = \langle f, \mu * \phi \rangle \quad \text{for } f \in L^\infty(Z, \nu),\ \mu \in M(G),\ \phi \in L^1(Z, \nu)$$

and $(\delta_g \square f)(z) = f(gz)$ loc. a.e. ν for each $f \in L^\infty(Z, \nu)$, $g \in G$ (see [11, pp. 298–299]). Let $P_1 = P_1(G) = \{\phi \in L^1(G);\ \phi \geq 0 \int \phi\, d\lambda = 1\}$ where G is equipped with a fixed left Haar measure λ.

Let

$$M = \{\psi \in L^\infty(Z)^*;\ \|\psi\| = \psi(1) = 1\}$$

be the set of means, let

$$IM = \left\{\psi \in M;\ \psi \in M;\ \psi(\delta_g \square f) = \psi(f) \text{ for all } f \in L^\infty(Z),\ g \in G\right\}$$

be the set of invariant means and let

$$TIM = \{\psi \in M,\ \psi(\alpha \square f) = \psi f \text{ for all } \alpha \in P_1,\ f \in L^\infty(Z, \nu)\}$$

be the set of topologically invariant means. As proved in [11, p. 303], $TIM \subset IM$. Let $\{\alpha_n\}_1^\infty$ be a norm dense subset of $P_1 \subset L^1(G)$ (G is second countable) and let t_n: $L^1(Z) \to L^1(Z)$ be given by $t_n(\phi) = \alpha_n * \phi$. Then $t_n^*(f) = \alpha_n \square f$ if $f \in L^\infty(Z)$. Since $\|\mu \square f\| \leq \|\mu\|\,\|f\|$ for each $\mu \in M(G)$ and $f \in L^\infty(Z, \nu)$ [11, p. 297], whenever $\psi \in M$ and $t_n^{**}\psi = \psi$ for all $n \geq 1$ it follows that $\psi \in TIM$. Let

$$K_1 = \left\{\psi \in L^1(Z, \nu);\ \phi \geq 0 \int \phi\, d\nu = 1\right\}$$

and note that since $L^1(Z, \nu)$ is weakly sequentially complete, K_1 when imbedded in $L^\infty(Z, \nu)^*$ satisfies $w^*\text{seq cl } K_1 = K_1$. Furthermore $M = w^*\text{cl } K_1$ see [11, p. 302]. The action $G \times Z \to Z$ is said to be an amenable action if $IM \neq \varnothing$ for some quasiinvariant measure ν on Z, or equivalently if there exists a mean ψ on

$$UCB_l(Z) = \left\{f \in C(Z);\ g \to \delta_g \square f \text{ from } G \text{ to } \left(C(Z), \|\ \|_\infty\right) \text{ is continuous}\right\}$$

such that $\psi(\alpha \square f) = \psi f$ for each $\alpha \in P_1$ and $f \in UCB_l(Z)$ (see [11, p. 299 and p. 302] for equivalent conditions).

THEOREM 2.4. *Let G be second countable, Z locally compact such that $G \times Z \to Z$ is an amenable action with respect to the quasiinvariant Radon measure ν. Let*

$$K \subset K_1 = \left\{\phi \in L^1(Z, \nu);\ \phi \geq 0 \text{ and } \int \phi\, d\nu = 1\right\}$$

be convex and let $\psi_0 \in \{w^*\text{cl } K\} \cap TIM$. Let $f_n \in L^\infty(Z, \nu)$, $\beta_n = \psi_0 f_n$, and $A_0 = \{w^*\text{cl } K\} \cap \{\psi \in TIM;\ \psi f_n = \beta_n\}$.

If Z does not admit a ***finite*** *Randon G-invariant measure or if $TIM \cap L^1(Z, \nu) = \varnothing$ and even if $A_0 \cap$* norm cl *$K = \varnothing$, then A_0 is big, (hence* card *$A_0 \geq 2^c$ and A_0 does not have WRNP).*

Proof. It is enough to prove the theorem under the assumption

$$A_0 \cap \text{norm cl } K = \varnothing.$$

Since $L^1(Z, \nu)$ is weakly sequentially complete we have norm cl $K = w^*$seq cl K in $L^\infty(Z, \nu)^*$. Thus $A_0 \subset w^*$cl $K \sim w^*$seq cl K.

Our Theorem 1.4 applied to the maps $s_n = t_n - I$: $L^1(Z\nu) \to L^1(Z, \nu)$ given by $s_n(\phi) = \alpha_n * \phi - \phi$ where $\{\alpha_n\}$ is dense in $P_1(G)$ implies that there is a sequence $\{u_k\} \subset K$ isomorphic to a canonical l^1 basis such that the map t: $L^\infty(Z, \nu) \to l^\infty$ given by $(tf)(n) = \langle f, u_n \rangle$ is continuous linear onto,

$$t^*: l^{\infty *} \to L^\infty(Z, \nu)^*$$

is a norm isomorphism into and $t^*\mathscr{F} \subset A_0$. Thus A_0 is big.

Now we restrict ourselves to a second countable group G and a *closed subgroup* G_0 and let $Z = G/G_0 = \{gG_0\}$. Then Z admits a unique (up to equivalence) quasiinvariant measure ν; in this case, for any $\alpha \in P_1(G)$ and $f \in L^\infty(Z, \nu)$, $\alpha \square f \in C(Z)$, and, whenever $g_i \to g$ in G,

$$\|\delta_{g_i} \square \alpha \square f - \delta_g \square \alpha \square f\|_{L^\infty(Z)} \to 0$$

(see [11, p. 306, proof of Theorem 3.3]. Thus $P_1(G) \square L^\infty(Z, \nu) \subset UCB_l(Z)$ *in this case*. Furthermore the map j: $C(Z) \to L^\infty(Z, \nu)$ is an injection in this case.

In *addition*, assume that $G \times Z \to Z$ is an amenable action. G need not be amenable for this to happen. For example if $G = SL(2, R)$, $G_0 = SL(2, Z)$ then both G and G_0 are not amenable; however $G \times G/G_0 \to G/G_0$ is an amenable action (see [6, p. 18]. In fact G/G_0 even admits a finite invariant measure but is not compact. However if $G = SL(2, C)$, $G_0 = SL(2, R)$ and $Z = G/G_0$ then $G \times Z \to Z$ is not an amenable action (see [6, p. 56].

Let $f_n \in UCB_l(Z)$ be arbitrary, $\psi_0 \in TIM$ and let $\beta_n = \psi_0 f_n$.

Consider the onto map t: $L^\infty(Z, \nu) \to l^\infty$ of Theorem 2.4. It is given by

$$(tf)(n) = \langle f, u_n \rangle$$

where $u_n \in K \subset \{\phi \in L^1(Z, \nu);\ \phi \geq 0,\ \int \phi \, d\nu = 1\}$ with

$$\|s_n u_k\| = \|\alpha_n^* u_k - u_k\|_{L^1(Z)} \to 0 \textit{ as } k \to \infty \quad \text{for each } n$$

and

$$\langle f_n, u_k \rangle \to \beta_n \text{ if } k \to \infty \quad \text{for each } n.$$

Thus $t(s_n^* f)(k) = \langle f, s_n u_k \rangle \to 0$ if $k \to \infty$ for each fixed n and $f \in L^\infty(Z)$. Thus

$$t[s_n^*(L^\infty(Z))] \subset c_0.$$

However $s_n^* f = \alpha_n \Box f - f$ and if $\alpha \in P_1(G)$ there is a subsequence α_{n_k} such that $\|\alpha_{n_k} - \alpha\|_{L^1(Z)} \to 0$. Thus $\|(\alpha_{n_k} - \alpha)\Box f\|_{L^\infty(Z)} \to 0$. Since t is continuous linear it follows that

$$t(\alpha\Box f - f) = \lim_k t(\alpha_{n_k}\Box f - f) \in c_0.$$

Thus $t(\alpha\Box f - f) \in c_0$ for each $\alpha \in P_1(G)$ and $f \in L^\infty(Z)$.

THEOREM 2.5. *Let G be second countable $G_0 \subset G$ a closed subgroup, $Z = G/G_0$ and assume that $G \times Z \to Z$ is an amenable action. Let $f_n \in UCB_l(Z)$, $\psi_0 \in TIM$ and let $K \subset K_1$ be any convex set such that $\psi_0 \in w^*\text{cl}\, K$. Let*

$$F_0 = \text{norm cl lin}\{f - \alpha\Box f;\ \alpha \in P_1(G);\ f \in UCB_l(Z)\},$$
$$K_u^0 = \{f \in UCB_l(Z);\ \langle f, K \rangle = 0\}.$$

If W is any closed linear subspace of

$$R_0 = \text{norm cl lin}\{C1 + F_0 + \{f_n\}_1^\infty + K_U^0\}$$

then $UCB_l(Z)/W$ has l^∞ as a continuous linear image provided G/G_0 ***does not admit a finite*** *invariant measure.*

Proof. Define $\beta_n = \psi_0(f_n)$. Let $u_n \in K$ be the sequence constructed above for $\{f_n, \beta_n\}$ and let t: $L^\infty(Z, \nu) \to l^\infty, (tf)(n) = \langle f, u_n \rangle$ be the onto map defined above. Let π: $l^\infty \to l^\infty/c$ be the canonical map. If $f \in L^\infty(Z, \nu)$,

$$t(\alpha\Box f - f) \in c_0 \subset c.$$

Thus $\pi t(\alpha\Box f) = \pi tf$. But $\alpha\Box f \in UCB_l(Z))$. Thus $\pi t(UCB_l(Z)) = l^\infty/c$. Clearly $t(F_0) \subset c_0 \subset c$, hence $\pi t(F_0) = 0$. Also $(tf_n)(k) = \langle f_n, u_k \rangle \to \beta_n$ as $k \to \infty$ for each n. Thus $tf_n \in c$ and $\pi tf_n = 0$ for all n. Furthermore $(t1)(k) = \langle 1, u_k \rangle = 1$; thus $\pi t(C1) = 0$. Let $(\pi t)_u$ be πt restricted to $UCB_l(Z)$. It then follows that $W \subset R_0 \subset (\pi t)_u^{-1}(0)$. Clearly $UCB_l(Z)/(\pi t)_u^{-1}(0)$ is *isomorphic* to l^∞/c by the map $(\pi t)_u$ (see [13, p. 40]). Let

$$\pi_W: UCB_l(Z)/W \to UCB_l(Z)/(\pi t)_u^{-1}(0)$$

be the canonical map and let $\pi_1 = (\pi t)_u \pi_W$: $UCB_l(Z)/W \to l^\infty/c$ and let Y_0 be a subspace of l^∞/c which is an isometric copy of l^∞ (see the remark after Lemma 1.5).

If $Y_1 = \pi_1^{-1}Y_0$ then π_1: $Y_1 \to Y_0$ is an onto continuous linear map which has a continuous bounded linear extension onto Y_0 [3, p. 106].

Remarks. (a) Note that $UCB_l(Z)/(\pi t)_u^{-1}(0)$, which is isomorphic to l^∞/c, contains an isomorphic copy of l^∞ (see the remark after Corollary 1.5).

(b) The difference between Corollary 1.5 and Theorem 2.5 is that $UCB_l(Z)$ is not usually a dual space as is X^* in Corollary 1.5.

(c) A stronger result than this is proved by Ching Chou in [1, Theorem 5.2] for the case of a group G with property E (not a coset space G/G_0) and $W(G)$ the space of weakly almost periodic functions on G. It is shown there that $UC(G)/W(G)$ contains an *isometric* copy of l^∞ and hence l^∞ is also a continuous image of $UC(G)/W(G)$ (see [3, p. 106]). This result of Chou was improved to all locally compact G by Dzinotyiweyi in [5, Corollary 2.2, p. 226]. Theorem 2.5 is an improvement of our Theorem 14 in [9], for second countable G.

(d) Note that $R_0 \subset (\pi t)_u^{-1}(0)$ (with notations as in the proof above). By using the full force of Theorem 1.6 and Remark 2 before it, we obtain, as in Theorem 2.5:

THEOREM 2.5′. *Assume that $G/G_0 = Z$ does not admit a finite invariant measure and G is second countable. Let $R_1' \subset l^\infty$ be separable and*

$$R_0' = \operatorname{cl}\operatorname{lin}(AC + R_1').$$

Then for any subspace $W \subset (\pi t)_u^{-1}(R_0'/c)$, $UCB_l(Z)/W$ has l^∞ as a continuous homomorphic image.

Recall that $AC = C1 + \operatorname{cl}\operatorname{lin}\{f - s_1 f;\ f \in l^\infty\}$ where $(s_1 f)(k) = f(k+1)$ is the space of almost convergent sequences. As is well known, AC/c is not norm separable; hence Theorem 2.5′ is a vast improvement on Theorem 2.5.

Part III. We apply now Theorem 1.4 to the case where S is a countable family of linear bounded maps s: $L^1(X\mathscr{S}u) \to L^1(X\mathscr{S}\mu)$ where $(X\mathscr{S}\mu)$ is a σ-finite measure space.

Ching Chou, in solving an open problem of Joe Rosenblatt [19, p. 628] has proved the following in an unpublished paper.

THEOREM. *If G is a countable group of measure preserving maps which act* ***ergodically*** *on the* ***nonatomic probability*** *space $(X, \mathscr{S}, p)$ and if there is some $\phi_0 \in M_G$, the set of G-invariant means on $L^\infty(X)$, such that $\phi_0 \neq p$ then there exists an* ***isometry*** *into t^*: $l^\infty(X)^* \to L^\infty(X)^*$ which is w^*-w^* continuous and such that $t^*(\mathscr{F}) \subset M_G$. Consequently if $H = t^*(\beta N \sim N)$ then* card $H = 2^c$, *H is w^* perfect and H is* ***isometric*** *to a canonical l^1 basis.*

Remarks. By ergodicity, the assumption $\phi_0 \neq p$ implies that

$$\phi_0 \in L^{\infty *} \sim L^1;$$

in particular there are $X_n \subset X$ such that $1_{X_n} \uparrow 1_X$ a.e. and

$$\phi_0(1_{X_n}) = \gamma_n \uparrow \gamma < 1 = \phi_0(1_X)$$

(by the RN theorem).

If $K_1 = \{\phi \in L^1; \phi \geq 0, \int \phi \, dp = 1\}$ then the assumption in Chou's result implies that if

$$A = \{w^*\text{cl } K_1\} \cap \{\phi \in M_G; \phi 1_{X_n} = \gamma_n \text{ for } n \geq 1\}$$

then

$$\phi_0 \in A \subset \{w^* \text{cl } K_1\} \sim w^* \text{seq cl } K_1.$$

In his proof Chou uses Theorem 1.4 of Rosenblatt. This in turn heavily uses ergodicity, the fact that $(X, \mathcal{S}, p)$ is a nontatonic finite measure space and the fact that G is a group of measure preserving point maps on X.

Our Theorem 1.4 yields the following:

THEOREM 2.6. *Let $(X, \mathcal{S}, \mu)$ be σ-finite, G a* ***countable set*** *of maps*

$$g_n\colon L^1(X) \to L^1(X).$$

Let

$$M_G = \{\psi \in L^\infty(X)^*; \psi(1) = \|\psi\| = 1, G^{**}\psi = \psi\}.$$

Assume that there is some $\psi_0 \in M_G \cap \{L^\infty(X)^* \sim L^1(X)\}$ and then let $X_n \subset X$ be such that $1_{X_n} \uparrow 1$ a.e. *and $\psi_0(1_{X_n}) = \gamma_n \uparrow \gamma < 1$. Let*

$$A = \{\psi \in M_G; \psi(1_{X_n}) = \gamma_n \textit{ for } n \geq 1\}$$

Then for any nonvoid w^*G_δ-section A_0 of A there is a w^*-w^* continuous norm isomorphism into t^*: $l^\infty \to L^\infty(X)^*$ such that $t^*(\mathcal{F}) \subset A_0$. If $H = t^*(\beta N \sim N) \subset A_0$ then card $H = 2^c$, *H is w^* perfect and H is* ***isomorphic*** *to a canonical l^1 basis. Thus A_0 does not have the WRNP.*

As proof we only note that the condition $\phi(1_{X_n}) \uparrow \gamma \neq 1$ which holds for each $\phi \in A_0$, implies that $A_0 \subset \{w^*\text{cl } K_1\} \sim w^*\text{seq cl } K_1$.

Remarks. In comparison with Chou's result, we *lost* the fact that H is isometric to a canonical l^1 basis; we only have an isomorphism.

We *gained* the following: G need not be a group arising from point maps on X; a countable set of bounded linear maps g_n: $L^1(X) \to L^1(X)$ will do. G

need not act ergodically; $(X\mathscr{S}\mu)$ may have atoms and is only σ-finite (localis able seems to be enough). Furthermore we could unify the applications I and III by adding a sequence of maps t_n: $L^\infty(X) \to L^\infty(X)$ and $f_n \in L^\infty(X)$. Let $\psi_0(f_n) = \alpha_n$, $\psi_0(t_n f_n) = \beta_n$. Let $K \subset K_1$ be any convex set such that $\psi_0 \in w^*\text{cl } K$. Then the set A_0 in Theorem 2.6 can be taken as

$$A_0 = \{w^*\text{cl } K\}$$
$$\cap \left\{\psi \in M_G;\ \psi 1_{X_n} = \gamma_n,\ \psi(f_n) = \alpha_n,\ \psi(t_n f_n) = \beta_n \text{ for all } n \geq 1\right\}$$

Clearly $\psi_0 \in A_0$ and A_0 is a w^*G_δ section of A included in $\{w^*\text{seq cl } K\} \sim w^*\text{seq cl } K$.

Part IV. We apply now Theorem 1.4 to the case of an algebra of operators on $L^p(G)$, for *second countable* groups G and obtain definitive improvements of results of Ching Chou [1] and of ours in [9] and [10]. The isometric methods of Chou do not seem to work in this case.

Let G be a second countable locally compact group, with unit e, $1 < p < \infty$ and $A_p(G) = A_p$ the Banach algebra of functions $f = \Sigma u_n * v_n^\vee$ where $u_n \in L^{p'}(G)$, $v_n \in L^p(G)$ $(1/p + 1/p' = 1)$ such that $\Sigma_n \|u_n\|_{p'} \|v_n\|_p < \infty$ with norm as the infinum of the last expression over all such representations of f. All the notations and definitions in this application are those of [12] and consistent with [10]. Let $PM_p(G) = PM_p = A_p(G)^*$ (the dual of A_p). If $p = 2$ and G is abelian then $PM_2(G) = L^\infty(\hat{G})$. If G is not abelian then $PM_2(G)$ is the W^* algebra generated by the convolution operators $(\rho f)(g) = f * g$, $g \in L^2(G)$, $f \in C_\infty(G)$ operating on $L^2(G)$. In general PM_p can be identified with an algebra of operators on $L^p(G)$ (see [12]).

Let $S = S_A^p = \{u \in A_p;\ u(e) = 1 = \|u\|\}$. $\psi \in PM_p^*$ is a topologically invariant mean (TIM) if $\psi(u \cdot \phi) = \psi(\phi)$ for each $u \in S, \phi \in PM_p$ where $(u \cdot \phi)(v) = \phi(uv)$ for each $u, v \in A_p$, $\phi \in PM_p$, and furthermore $\|\psi\| = 1 = \psi(I)$ where I is the identity of PM_p. The set of TIM's on PM_p is denoted by $TIM_p(\hat{G}) = TIM_p$. It is easy to see (due to the fact that $S_B^p S_A^p \subset S_A^p$ in the notation of [10, Prop. 1]) that this definition is equivalent to that of [10]. If $p = 2$ and G is abelian then $\psi \in TIM_2$ iff ψ is a topologically invariant mean on $L^\infty(\hat{G})$.

THEOREM 2.7. *Let $K \subset S$ be convex and $A = \{w^*\text{cl } K\} \cap TIM_p \neq \varnothing$. Let $A_0 \neq \varnothing$ be a w^*G_δ section of A. If G is second countable and* ***not*** *discrete then A_0 is big. In particular* card *$A_0 \geq 2^c$ and A_0 does not have the WRNP.*

Proof. We freely use results and notations of [10]. We claim that

$$\{w^*\text{cl } K) \cap TIM_p \subset \{w^*\text{cl } K\} \sim w^*\text{seq cl } K.$$

If not, let $v_n \in K$, $n \geq 1$, and $v_0 \in S$ have compact support E and assume that $v_n \to \psi_0$ in w^* and $\psi_0 \in TIM_p$. Then $v_0 v_n \to v_0 \cdot \psi_0 = \psi_0$ in w^* (see [10, Prop. 4]). This implies that $v_0 v_n$ is a weak Cauchy sequence in

$$A_E^p(G) = \{v \in A_p; \operatorname{supp} v \subset E\}.$$

But A_E^p is weakly sequentially complete by [10, Lemma 18]. Thus $\psi_0 \in A_E^p \subset A_p$. But then G is discrete (see end of proof of Theorem 16 in [10]) which cannot be. Thus

$$A \subset w^*\text{cl}\, K \sim w^*\text{seq}\,\text{cl}\, K.$$

If u_n is norm dense in $S_A^p = S$ and if $t'_{u_n}: A_p \to A_p$ is defined by $t'_{u_n}(v) = u_n v$ and $I': A_p \to A_p$ is the identity then $\psi \in \{w^*\text{cl}\, K\} \cap TIM_p$ iff $\psi \in w^*\text{cl}\, K$ and $(t'_{u_n} - I')^{**}\psi = 0$ for each $n \geq 1$. A direct application of Theorem 1.4 finishes the proof.

A slightly better theorem, for $p = 2$ and second countable G, has been proved by Ching Chou [2, Theorem 3.3] using quite complicated W^* algebra methods (see the proofs of Theorems 3.3 and 2.4 of [2]) which are not available in our case anymore, since PM_p, $p \neq 2$, is not a W^* algebra. In his theorem, Chou gets the set $H = t^*(\beta N \sim N)$ *isometric* (while we have it only isomorphic) to a canonical l^1 basis in TIM_2. We doubt that for $p \neq 2$ one can get an isometric result. Theorem 2.7 is an improvement on our Theorem 17 in [10].

We have defined [10] the algebra $UC_p(\hat{G})$ as norm $\text{cl}\{A_p \cdot PM_p\}$. If

$$F_0 = \text{norm}\,\text{cl}\left\{\sum^n (\phi_i - u_i \cdot \phi_i);\ \phi_i \in UC_p, u_i \in A_p, n \geq 1\right\}$$

and if $L_V = \{\phi \in UC_p; \operatorname{supp} \phi \subset G \sim V\}$ where V is a neighborhood of e (see [10, remarks before Theorem 16]) then, as in Theorem 2.5, we can show the following:

THEOREM 2.8. *Let G be second countable nondiscrete and*

$$R_0 = \text{norm}\,\text{cl}\{F_0 + R_1 + L_V\}.$$

Then for any separable subspace $R_1 \subset UC_p$, any neighborhood V of e and any closed subspace $W \subset R_0$, UC_p/W has l^∞ as a continuous homomorphic image.

This theorem is an improvement of part of our Theorem 16 *in* [10].

A result analogous to Theorem 2.5′ can be obtained in the $UC_p(G)$ context. We omit the details.

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