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EXTREME POINTS OF UNIT BALLS OF QUOTIENTS OF L^∞ BY DOUGLAS ALGEBRAS

BY

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1. Introduction

Let H^∞ be the space of bounded analytic functions in the unit disk D. Identifying with boundary functions, we consider H^∞ as an (essentially) uniformly closed subalgebra of L^∞, the space of bounded measurable functions on the unit circle ∂D with respect to the normalized Lebesgue measure m. Every uniformly closed subalgebra between H^∞ and L^∞ is called a Douglas algebra. In this paper, B always denotes a Douglas algebra. It is well known that $H^\infty + C$ is the smallest Douglas algebra containing H^∞ properly, where C is the space of continuous functions on ∂D. The reader is referred to [5] and [12] for the theory of Douglas algebras, and [4] for uniform algebras.

In this paper, we will study the following problem.

PROBLEM. *For which Douglas algebra B, does* $\operatorname{ball}(L^\infty/B)$ *have extreme points*?

We denote by $\operatorname{ball}(Y)$ the closed unit ball of a Banach space Y. A point x in $\operatorname{ball}(Y)$ is called extreme if $x = (x_1 + x_2)/2$ for x_1, x_2 in $\operatorname{ball}(Y)$ implies $x = x_1 = x_2$. An equivalent condition for a point x in $\operatorname{ball}(Y)$ to be extreme is that the condition $\|x \pm y\| \le 1$, $y \in Y$, implies $y = 0$.

Up to now, we know the following theorems about extreme points of $\operatorname{ball}(L^\infty/B)$.

KOOSIS' THEOREM [9]. $\operatorname{ball}(L^\infty/H^\infty)$ *has an extreme point. A point $f + H^\infty$ in* $\operatorname{ball}(L^\infty/H^\infty)$ *is an extreme point if and only if there is a function h in $f + H^\infty$ such that $|h| = 1$* a.e. *dm and $\|h + g\| > 1$ for every $g \in H^\infty$ with $g \ne 0$.*

AXLER, BERG, JEWELL AND SHIELDS' THEOREM [2]. $\operatorname{ball}(L^\infty/H^\infty + C)$ *does not have extreme points.*

For a subset F of ∂D, we denote by L_F^∞ the space of functions in L^∞ which can be redefined on a set of measure zero so as to become continuous at every

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point of F. Then $H^\infty + L_F^\infty$ is a Douglas algebra [3]. $M(B)$ denotes the maximal ideal space of a Douglas algebra B. Throughout this paper, we let $X = M(L^\infty)$. For a weak peak subset E of X for B, we let

$$B_E = \{ f \in L^\infty;\ f|_E \in B|_E \};$$

then B_E is also a Douglas algebra.

LUECKING AND YOUNIS' THEOREM [10]. *If B is one of the following Douglas algebras,*
(1) $B = H^\infty + L_F^\infty$, *where F is a proper closed or open subset of ∂D,*
(2) $B = H_E^\infty$, *where E is a proper peak subset of X for H^∞, then* $\mathrm{ball}(L^\infty/B)$ *does not have extreme points.*

In [17], Younis showed the corresponding result is also true for $B = (H^\infty + C)_E$, where E is a peak subset of X for $H^\infty + C$. For a closed subset E of X, E is called a support set if E coincides with the support of a representing measure μ_x for some point x in $M(H^\infty + C) \setminus X$. We note that a support set is a weak peak subset for H^∞.

IZUCHI AND YOUNIS' THEOREM [8]. *If E is a support set, then* $\mathrm{ball}(L^\infty/H_E^\infty)$ *has extreme points.*

Every Douglas algebra B which appears in the denominator of quotient spaces in the above theorems has the best approximation property [2], [15], [16]. That is, for each function f in $L^\infty \setminus B$, there is a function h in B such that $\|f + B\| = \|f - h\|$. To prove each theorem above, its authors used this property effectively.

In this paper, we will show the following theorems. We do not assume that B has the best approximation property. We denote by Γ the essential set for B, that is, Γ is the smallest closed subset of X such that every function f in L^∞ which vanishes on Γ belongs to B.

MAIN THEOREM. *Suppose that for each Blaschke product b with $\bar{b} \notin B$, there exists an open-closed subset W of X with $W \cap \Gamma \neq \emptyset$ and a sequence $\{h_n\}_{n=1}^\infty$ in B such that*
(a) $\|\bar{b} - h_n\| \to 1\ (n \to \infty)$ *and*
(b) $\inf\{|h_n(x)|;\ x \in W,\ n = 1,2,\ldots\} > 0$.
Then $\mathrm{ball}(L^\infty/B)$ *does not have extreme points.*

As applications of our main theorem, we will show the following theorem which includes all theorems concerning non-existence of extreme points of $\mathrm{ball}(L^\infty/B)$.

THEOREM. *Let B be one of the following Douglas algebras. Then* $\operatorname{ball}(L^\infty/B)$ *does not have extreme points.*

(1) $B \neq H^\infty$ *and* $\hat{m}(\Gamma) > 0$, *where* $\hat{m}$ *is the lifting measure of* m *onto* X.

(2) $B \neq H^\infty$ *and* Γ *contains a closed* G_δ *subset of* X.

(3) $B = H^\infty + L^\infty_F$ *for a subset* F *of* ∂D.

(4) *There is a sequence* $\{x_n\}_{n=1}^\infty$ *in* $X \setminus \Gamma$ *such that* $\overline{\{x_{n_j}\}} \cap \Gamma \neq \emptyset$ *for every subsequence* $\{x_{n_j}\}$ *of* $\{x_n\}$, *where the bar indicates the closure in* X.

In Section 2, we will prove the main theorem; also we will prove (4) in the above theorem. In Section 3, using Wolff's theorem [14] we will prove (1), (2) and (3) in the above theorem. In Section 4, we will give a condition on B for which $\operatorname{ball}(L^\infty/B)$ has extreme points. Only in Section 4 will we deal with Douglas algebras having the best approximation property.

2. Proof of the main theorem

To prove our theorem, we need the following lemma, a generalization of Axler's factorization theorem [1].

LEMMA 1 (SUNDBERG [13]). *Let* $\{f_n\}_{n=1}^\infty$ *be a sequence in* L^∞. *Then there exists a Blaschke product* b *such that* $bf_n \in H^\infty + C$ *for every* $n = 1, 2, \ldots$.

Proof of the main theorem. Let g be a function in L^∞ such that $\|g + B\| = 1$. Then there is a sequence $\{g_n\}_{n=1}^\infty$ in B such that

$$\|g + g_n\| \to 1 \quad (n \to \infty). \tag{1}$$

We let $G_n = g + g_n$. By Lemma 1, there is a Blaschke product b such that

$$bG_n \in H^\infty + C \quad \text{for every } n = 1, 2, \ldots. \tag{2}$$

Then we get $\bar{b} \notin B$, for if $\bar{b} \in B$, then $G_n \in \bar{b}(H^\infty + C) \subset B$. This means $g \in B$, a contradiction.

Let W be an open-closed subset of X with $W \cap \Gamma \neq \emptyset$ and let $\{h_n\}_{n=1}^\infty$ be a sequence in B satisfying the conditions (a) and (b). By (a), it is clear that $\|1 - bh_n\| \to 1$ $(n \to \infty)$. By (b) and considering a subsequence of $\{h_n\}$, we may assume that the ranges $(bh_n)(W)$ are contained in $\{z \in \mathbf{C};\ |z - 2| < 2\}$. Let

$$f_n = h_n/2 \in B \quad (n = 1, 2, \ldots);$$

then

$$\varlimsup_{n \to \infty} \|1 - bf_n\| \leq 1 \tag{3}$$

and $\sup_n \|1 - bf_n\|_W < 1$, where $\|\cdot\|_W$ is the value of the maximum modulus on W.

Let

$$\sigma = 1 - \sup_n \|1 - bf_n\|_W > 0. \tag{4}$$

Since $W \cap \Gamma \neq \emptyset$, we have $\chi_W L^\infty \not\subset B$, where χ_W is the characteristic function of W. Thus there is a function f such that $f \notin B$, $\|f\| = \sigma$ and $f = 0$ on W^c.

To show our theorem, it is sufficient to show

$$\|g \pm f + B\| \leq 1. \tag{5}$$

To show (5), we let $F_n = f_n b G_n$. Then $F_n \in B$ by (2), and we have

$$\begin{aligned} \|g \pm f + B\| &= \|G_n \pm f + B\| \\ &\leq \|G_n \pm f - F_n\| \\ &\leq \max\{\|G_n - F_n\|_{W^c}, \|G_n - F_n\|_W + \|f\|\} \\ &= \max\{\|(1 - bf_n)G_n\|_{W^c}, \|(1 - bf_n)G_n\|_W + \sigma\} \\ &\leq \max\{\|1 - bf_n\| \|G_n\|, (1 - \sigma)\|G_n\| + \sigma\} \quad \text{by (4)}. \end{aligned}$$

By (1) and (3), we have $\|g \pm f + B\| \leq 1$. This completes the proof.

COROLLARY 1. *Suppose that for each Blaschke product b with $\bar{b} \notin B$, there is a function h in B such that*
(a) $\|\bar{b} - h\| = 1$ *and*
(b) $|h| \neq 0$ *on* Γ.
Then $\mathrm{ball}(L^\infty/B)$ *does not have extreme points.*

Proof. We put $h_n = h$ for $n = 1, 2, \ldots$. By (b), we can take an open-closed subset W of X such that

$$\inf\{|h(x)|;\, x \in W\} > 0 \quad \text{and} \quad W \cap \Gamma \neq \emptyset.$$

Then we can get immediately the conclusion using our main theorem.

DEFINITION. A closed subset E of X is called quasi-G_δ if there is a sequence $\{x_n\}_{n=1}^\infty$ in $X \setminus E$ such that $\overline{\{x_{n_j}\}_{j=1}} \cap E \neq \emptyset$ for every subsequence $\{x_{n_j}\}$ of $\{x_n\}$.

By the definition, it is clear that an open-closed subset is not quasi-G_δ, and if a closed G_δ subset is not open then it is quasi-G_δ.

THEOREM 1. *If* Γ *is quasi-*G_δ*, then* $\mathrm{ball}(L^\infty/B)$ *does not have extreme points.*

Proof. Let $\{x_n\}_{n=1}^\infty$ be a sequence in $X \setminus \Gamma$ such that $\overline{\{x_{n_j}\}} \cap \Gamma \neq \emptyset$ for every subsequence $\{x_{n_j}\}$ of $\{x_n\}$. Let b be a Blaschke product with $\bar{b} \notin B$. We show that there exists an open-closed subset W of X and a sequence $\{h_n\}_{n=1}^\infty$ in B satisfying the conditions of the main theorem.

Let α be a cluster value of $\{b(x_n);\ n = 1, 2, \dots\}$, then $|\alpha| = 1$. We may assume that $b(x_n) \to \alpha$ $(n \to \infty)$ by our assumption of $\{x_n\}$. For each n, we can take an open-closed subset U_n of X such that

$$x_n \in U_n,\ U_n \subset X \setminus \Gamma \quad \text{and} \quad \|b - b(x_n)\|_{U_n} < 1/n.$$

Let $W = \overline{\bigcup_n U_n}$; then W is an open-closed subset of X [4, p. 18] and $W \cap \Gamma \neq \emptyset$. Then we have

$$(1) \qquad b = \alpha \quad \text{on} \quad W \setminus \bigcup_n U_n$$

by our construction.

Let ψ_n be a conformal map from D onto $\{z \in \mathbf{C};\ |1 - z| < 1 + 1/n\}$ such that

$$\psi_n(-\alpha) = -1/n, \quad \psi_n(0) = 0 \quad \text{and} \quad \psi_n(\alpha) = 2 + 1/n.$$

Let $f_n = (\psi_n \circ b)/b$; then $f_n \in H^\infty$,

$$(2) \qquad f_n = (2 + 1/n)\bar{\alpha} \quad \text{on} \quad \{x \in X;\ b(x) = \alpha\}$$

and

$$(3) \qquad \|\bar{b} - f_n\| = \|1 - bf_n\| = \|1 - \psi_n \circ b\| = 1 + 1/n.$$

The idea for this construction of f_n can be found in [10].

Let

$$h_n(x) = \begin{cases} f_n(x) & \text{if } x \in W^c \\ (2 + 1/n)\bar{b}(x) & \text{if } x \in W. \end{cases}$$

Since $W \cap \Gamma \subset W \setminus \bigcup_n U_n$, we have $h_n = f_n$ on Γ by (1) and (2). Then we get $h_n \in B$, because $f_n \in H^\infty \subset B$ and Γ is the essential set for B. Since $|h_n| = 2 + 1/n$ on W, $\{h_n\}$ satisfies condition (b) in the main theorem. Moreover by

(3), we get

$$\begin{aligned}\|\bar{b} - h_n\| &= \max(\|\bar{b} - h_n\|_{\overline{W}}, \|\bar{b} - h_n\|_{W^c}) \\ &\le \max(1 + 1/n, \|\bar{b} - f_n\|) \\ &\le 1 + 1/n.\end{aligned}$$

We already know that $1 = \|\bar{b} + B\| \le \|\bar{b} - h_n\|$. Hence this means that $\{h_n\}$ satisfies condition (a) in the main theorem. Now we can apply our main theorem.

Remark 1. We see the following facts in the proof of Theorem 1.

(1) For each Blaschke product b with $\bar{b} \notin B$, if there exists an open-closed subset W of X and $\alpha \in \partial D$ such that

$$\{x \in \Gamma;\ b(x) = \alpha\} \supset W \cap \Gamma \neq \emptyset,$$

then ball(L^∞/B) does not have extreme points.

(2) If Γ is quasi-G_δ, then the assumptions in (1) above are satisfied.

COROLLARY 2. *A support set is not quasi-G_δ.*

Proof. Let E be a support set of a representing measure μ_x for a point x in $M(H^\infty + C)\setminus X$. By Izuchi and Younis' theorem, ball(L^∞/H_E^∞) has an extreme point. Since E is the essential set for H_E^∞, E is not quasi-G_δ by Theorem 1.

3. An application of Wolff's theorem

As usual, we put $QC = (H^\infty + C) \cap \overline{(H^\infty + C)}$ and $QA = H^\infty \cap QC$. A characterization of QC was given by Sarason [11].

LEMMA 2. $QC = \{f \in L^\infty;$ *f is constant on each support set*$\}$.

In [14], Wolff showed the following theorem.

WOLFF'S THEOREM. *For $f \in L^\infty$, there is an outer function $q \in QA$ such that $qf \in QC$.*

By Wolff's theorem, we can get the following.

THEOREM 2. *If $B \supset H^\infty + C$ and $\hat{m}(\Gamma) > 0$, then* ball(L^∞/B) *does not have extreme points.*

Proof. Let b be a Blaschke product with $\bar{b} \notin B$. We show that there is a function h in B satisfying (a) and (b) in Corollary 1. By Wolff's theorem, there is an outer function $q \in QA$ such that $qb \in QC$. We may assume $\|q\| = 1$. Since $qb \in QC$, b is constant on each support set on which q does not vanish by Lemma 2. Let $h = \bar{b}|q|$; then h is constant on each support set. Again by Lemma 2, we have $h \in QC \subset H^\infty + C \subset B$. Since $q \in QA \subset H^\infty$, q does not vanish on Γ by our assumption $\hat{m}(\Gamma) > 0$. Thus h does not vanish on Γ. Now we have

$$1 = \|\bar{b} + B\| \leq \|\bar{b} - h\| = \|1 - b\bar{b}|q|\,\| = \|1 - |q|\,\| \leq 1.$$

Hence h satisfies (a) and (b) in Corollary 1.

For a subset J of L^∞, we denote by $[J]$ the smallest uniformly closed subalgebra containing J. By Lemma 1, we know that for a sequence $\{f_i\}$ in L^∞, there is a Blaschke production b such that

$$bf_{i_1}^{n_1}f_{i_2}^{n_2}\dots f_{i_k}^{n_k} \in H^\infty + C \quad \text{for } k = 1,2,\dots.$$

Thus we have $b[H^\infty, f_i;\ i = 1,2,\dots] \subset H^\infty + C$. Since X is the essential set for $H^\infty + C$, so is X for $[H^\infty, f_i;\ i = 1,2,\dots]$.

COROLLARY 3. *Let* $B = [H^\infty, f_i;\ i = 1,2,\dots]$. *If* $B \supset H^\infty + C$, *then* $\operatorname{ball}(L^\infty/B)$ *does not have extreme points.*

By Theorem 1 and 2, we have the following.

THEOREM 3. *If* $B \supset H^\infty + C$ *and* Γ *contains a closed* G_δ *subset, then* $\operatorname{ball}(L^\infty/B)$ *does not have extreme points.*

Proof. Let E be a closed G_δ subset with $E \subset \Gamma$. If E is open, then $0 < \hat{m}(E) \leq \hat{m}(\Gamma)$, we can apply Theorem 2. Suppose that E is not open. There is a positive function f in L^∞ such that $\|f\| = 1$ and $E = \{x \in X;\ f(x) = 1\}$. Let

$$E_n = \{x \in X;\ f(x) > 1 - 1/n\};$$

then $E \subset E_n$ and E_n is an open subset. If $E_n \subset \Gamma$ for some n, then $0 < \hat{m}(E_n) \leq \hat{m}(\Gamma)$ and we can apply Theorem 2. So we may assume $E_n \not\subset \Gamma$ for $n = 1,2,\dots$. We take a sequence $\{x_n\}$ in X such that $x_n \in E_n \setminus \Gamma$. Then $f(x_n) \to 1$ $(n \to \infty)$. This shows that $E \cap \overline{\{x_{n_j}\}} \neq \emptyset$ for every subsequence $\{x_{n_j}\}$ of $\{x_n\}$. Thus Γ is quasi-G_δ. Then we can apply Theorem 1.

Remark 2. The proof of Theorem 3 shows that if E is a closed subset of X such that $\hat{m}(E) = 0$ and E contains a closed G_δ subset, then E is quasi-G_δ.

By Remark 2 and Corollary 2, we have the following corollaries.

COROLLARY 4. *If E is a closed G_δ subset of X, then E is not included in any support set.*

COROLLARY 5. *Any support set contained in Γ does not include any peak subset for B.*

The following two corollaries are generalizations of (2) in Luecking and Younis' theorem.

COROLLARY 6. *If $B \neq H^\infty$ and Γ is a peak subset for B, then* $\mathrm{ball}(L^\infty/B)$ *does not have extreme points.*

COROLLARY 7. *If E is a peak subset for B such that $E \subset \Gamma$, then* $\mathrm{ball}(L^\infty/B_E)$ *does not have extreme points.*

Proof. By [7, Proposition 4.1], E is the essential set for B_E.

For $\lambda \in \partial D$, we put $X_\lambda = \{x \in X;\ \hat{z}(x) = \lambda\}$. The following corollary is a generalization of (1) in Luecking and Younis' theorem.

COROLLARY 8. *If $B = H^\infty + L_F^\infty$ for a subset F of ∂D, then* $\mathrm{ball}(L^\infty/B)$ *does not have extreme points.*

Proof. Let $\lambda \in F$. Since $B|_{X_\lambda} = H^\infty|_{X_\lambda}$, we have $X_\lambda \subset \Gamma$. It is trivial that X_λ is a closed G_δ subset of X.

4. A condition for which $\mathrm{ball}(L^\infty/B)$ has extreme points

We give a generalization of Izuchi and Younis' theorem. We need the following theorem which was proved in [8, Theorem 1].

THEOREM. *Suppose that B has the best approximation property. Let $f \in L^\infty$ with $\|f + B\| = 1$. Then the following are equivalent.*

(a) *$f + B$ is an extreme point of* $\mathrm{ball}(L^\infty/B)$.

(b) *$f|_\Gamma$ has a unique best approximation h in $B|_\Gamma$, and $|f|_\Gamma + h| = 1$.*

THEOREM 4. *Suppose that B has the following two properties.*

(a) *B has the best approximation property.*

(b) *There exists a Blaschke product b_0 such that $\bar{b}_0 \notin B$ and*

$$\bigcup\{\operatorname{supp}\mu_x;\ |b_0(x)| \neq 1 \text{ and } x \in M(b)\}$$

is dense in Γ. Then $\mathrm{ball}(L^\infty/B)$ *has an extreme point.*

Proof. Let b_0 be a Blaschke product satisfying (b). By Lemma 1, there exists a Blaschke product b such that

$$(1) \qquad b\bar{b}_0^n \in H^\infty + C \quad \text{for} \quad n = 1, 2, \ldots .$$

If $B = H^\infty$, we get our assertion by Koosis' theorem. So we assume that $B \supset H^\infty + C$. By (1), we have $\bar{b} \notin B$ and $\|\bar{b} + B\| = 1$. We show that $\bar{b} + B$ is an extreme point. We put $h_n = b\bar{b}_0^n$, then $|h_n| = 1$ on X. We fix $x \in M(B)$ with $|b_0(x)| \neq 1$. Then we have

$$(2) \qquad |b(x)| = |b_0^n(x) h_n(x)| \le |b_0(x)|^n \to 0 \quad (n \to \infty).$$

We note that $\operatorname{supp} \mu_x \subset \Gamma$ by condition (b).

To show that $\bar{b} + B$ is an extreme point, we use the above theorem. Since $\bar{b} \notin B$, we have $\|\bar{b}|_\Gamma + B|_\Gamma\| \le \|\bar{b} + B\| = 1$. Then by (2), we have

$$1 = \int (1 + bB)\, d\mu_x \le \|1 + bB|_\Gamma\| = \|\bar{b}|_\Gamma - B|_\Gamma\| \le 1.$$

Thus we have $\|\bar{b}|_\Gamma + B|_\Gamma\| = 1$. By [8, Corollary 1], $B|_\Gamma$ has the best approximation property. Let $h \in B$ with $\|\bar{b} - h\|_\Gamma = 1$, that is,

$$(3) \qquad |1 - bh| \le 1 \quad \text{on } \Gamma.$$

Then we have

$$\begin{aligned} 1 &= \int (1 - bh)\, d\mu_x \quad \text{by (2)} \\ &\le \int |1 - bh|\, d\mu_x \le 1 \quad \text{by (3)}. \end{aligned}$$

This implies that $1 - bh = 1$ a.e. $d\mu_x$, so $h = 0$ on $\operatorname{supp} \mu_x$. By (b), we have $h = 0$ on Γ. Thus $0 \in B|_\Gamma$ is the unique best approximation for $\bar{b}|_\Gamma$, and $\bar{b}|_\Gamma$ satisfies the condition (b) in the above theorem.

COROLLARY 9. *If Γ is a finite union of support sets, then* $\operatorname{ball}(L^\infty / H_\Gamma^\infty)$ *has an extreme point.*

Proof. Let $\Gamma = \bigcup_{n=1}^k \operatorname{supp} \mu_{x_n}$, where $x_n \in M(M^\infty + C) \setminus X$. Since $\operatorname{supp} \mu_{x_n}$ is a weak peak subset for H^∞, so is Γ. Then H_Γ^∞ has the best approximation property by [15], and Γ is the essential set for H_Γ^∞. There is a Blaschke product b_n such that $b_n(x_n) = 0$ by [6, p. 179]. Let $b_0 = \prod_{n=1}^k b_n$; then b_0 satisfies condition (b) in Theorem 4.

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