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# DGA HOMOLOGY DECOMPOSITIONS AND A CONDITION FOR FORMALITY

BY

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## Introduction

The rational homotopy theory of cofibrations has generally been approached via the Lie algebra models of Quillen. Recently, these models also have been used to study fibrations, so it should not be surprising that, dually, Sullivan's differential graded algebra models might find application to cofibrations. This is, then, the point of view adopted in this work.

We begin by developing the notion of a cofibre sequence of DGA's as well as some simple analogues of topological properties of cofibrations. In this framework we then give a new proof of the Mapping Cone Theorem of Felix-Tanre.

Next, we describe the homology decomposition of a minimal DGA in terms of cofibre sequences and note several of its immediate consequences (including the uniqueness result of Toomer). The homology decomposition is then used to give an iterative procedure for deciding whether or not a given DGA (or space) is formal.

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## 0. Preliminaries

For the fundamentals of Sullivan's version of rational homotopy theory the reader is referred to [5], [6], [9] and [13]. We shall recall some relevant definitions and results below, but we shall avoid discussing more technical aspects of the theory such as the notion of DGA homotopy.

Recall that a differential graded algebra (DGA) over $Q$ consists of a graded $Q$-vector space $A = \oplus A^i$ with multiplication $A^i \otimes A^j \to A^{i+j}$ satisfying $ab = (-1)^{ij}ba$ and a graded derivation $d$ of degree 1 with $d^2 = 0$. A DGA $M$ will

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be said to be *minimal* if $M^1 = 0$, $M$ is freely generated by a graded vector space $V$ (written $M = L(V)$) and $d$ is decomposable ($d(M) \subset M^+ \cdot M^+$, where $M^+ = \oplus_{i>0} M^i$).

A DGA map $\phi\colon A \to B$ is called a *quasi-isomorphism* (*$n$-quasi-isomorphism*) if $\phi^*$ is an isomorphism (in degrees $\leq n$). For any DGA $A$ with $H^0(A) = Q$, there is a minimal DGA $M(A)$ which is unique up to isomorphism and a quasi-isomorphism $\rho\colon M(A) \to A$. $M(A)$ is the *minimal model* of $A$. A salient property of minimal DGA's is the following:

LIFTING THEOREM. *Let $M$ be minimal and suppose a DGA map $f\colon M \to B$ is given. Then for any quasi-isomorphism $\phi\colon A \to B$ there is a lift $\tilde{f}\colon M \to A$ which is unique up to homotopy with $\phi\tilde{f} \simeq f$. In particular, there is a bijection of sets of homotopy classes $\phi_*\colon [M, A] \to [M, B]$.*

A minimal DGA is said to be *formal* if there exists a quasi-isomorphism $\theta$: $M \to H^*(M)$. Any such $\theta$ is referred to as a *formalization* of $M$. Subsequently [15], this notion was extended to maps. A DGA map $f\colon M \to N$ is said to be *formal* if there exist formalizations $\theta_M, \theta_N$ such that $f^*\theta_M \simeq \theta_N f$. Now, Sullivan proved that a minimal DGA is formal if and only if there exists a lift of a grading automorphism (see Theorem 12.7 of [13] and [12]). A similar criterion was developed for the detection of formal maps in [4].

To a space $X$, there is associated a DGA $A(X)$ of rational polynomial forms. The minimal model $\rho\colon M(X) \to A(X)$ then captures all the rational homotopy information about the space $X$. Now, there is an adjoint to the rational polynomial form functor called *spatial realization* and denoted $|\ |$: DGA $\to$ TOP. Properties of this functor may be found in [1] or [13]. The only property we shall use is that the adjunction $X \to |M(X)|$ is a rational homotopy equivalence.

A space is *formal*, if its minimal model is. The determination of the formality of a space is an important objective because, in that case, all rational homotopy information resides in the rational cohomology algebra. The notion of formality was first enunciated in [3] where it was shown that compact Kähler manifolds are formal.

A map of spaces is *formal* if any map in the unique DGA homotopy class determined by it is formal. Holomorphic maps of compact Kähler manifolds are examples, as well as $H$-maps and co-$H$ maps of 1-connected co-$H$ spaces.

Throughout this work, all spaces and minimal DGA's are presumed to be 1-connected. Furthermore, spaces are presumed to be of the homotopy type of CW complexes having finite rational betti numbers in each degree.

## 1. DGA cofibre sequences

Let $F\colon A \to B$ be a map of minimal DGA's and suppose $\{y_i\}$ is a set of free generators for $B$. Let $u_i$ correspond to $y_i$ and define a DGA $\bar{A} = A \otimes L(u_i, v_i)$ with $d|_A = d_A$ and $du_i = v_i$. Also, define $\bar{F}\colon \bar{A} \to B$ by $\bar{F}|_A = F$, $\bar{F}(u_i) = y_i$

and $\bar{F}(v_i) = dy_i$. Clearly, $i$: $A \to \bar{A}$ is a quasi-isomorphism and $\bar{F}$ is a surjection. We obtain a commutative diagram

$$\begin{array}{ccccccc} & & A & \xrightarrow{F} & B & & \\ & & {\scriptstyle i}\downarrow & & \| & & \\ 0 \to & K(\bar{F}) & \xrightarrow{I} \bar{A} & \xrightarrow{\bar{F}} & B & \to 0 \end{array}$$

where $K(\bar{F}) = Q \oplus \operatorname{Ker} \bar{F}$.

DEFINITION . A sequence of minimal DGA's

$$M \xrightarrow{J} A \xrightarrow{F} B$$

is a cofibre sequence if there exists a quasi-isomorphism $\gamma$: $M \to K(\bar{F})$ such that $iJ \simeq I\gamma$.

The following result is not hard to prove (see [16] or Prop. 15.18 in [6]).

PROPOSITION 1. *If*

$$X \xrightarrow{f} Y \xrightarrow{j} C_f$$

*is a cofibre sequence of* 1*-connected spaces, then*

$$M(C_f) \xrightarrow{J} M(Y) \xrightarrow{F} M(X)$$

*is a cofibre sequence of DGA's.*

Conversely, we obtain the elementary result:

PROPOSITION 2. *If*

$$M \xrightarrow{J} A \xrightarrow{F} B$$

*is a cofibre sequence of DGA's, then*

$$|B| \xrightarrow{f} |A| \xrightarrow{j} |M|$$

*is a cofibre sequence of spaces.*

*Proof.* By Proposition 1, the cofibre sequence of realizations

$$|B| \xrightarrow{f} |A| \xrightarrow{g} C_f$$

induces a diagram

$$\begin{tikzcd} M(C_f) \arrow[r, "G"] \arrow[d, "\beta"'] & A \arrow[r, "F"] \arrow[d, "i"] & B \arrow[d, equal] \\ K(\bar{F}) \arrow[r, "I"] & \bar{A} \arrow[r, "\bar{F}"] & B \end{tikzcd}$$

with $\beta$ a quasi-isomorphism and $iG \simeq I\beta$. Now, by hypothesis, there exists a quasi-isomorphism $\gamma$: $M \to K(\bar{F})$ with $iJ \simeq I\gamma$. Because $M$ is minimal, there is a lift $\lambda$: $M \to M(C_f)$ with $\beta\lambda \simeq \gamma$. The map $\lambda$, being a quasi-isomorphism of minimal DGA's, is then an isomorphism. Hence we need only show that $G\lambda \simeq J$.

From the homotopies given we obtain $iG\lambda \simeq I\beta\lambda \simeq I\gamma \simeq iJ$. Now $i$ is a quasi-isomorphism, so there is a bijection of homotopy sets,

$$i_*: [M, A] \xrightarrow{\cong} [M, \bar{A}].$$

Hence, $G\lambda \simeq J$ and there is a homotopy commutative diagram

$$\begin{tikzcd} {|B|} \arrow[r, "f"] & {|A|} \arrow[r, "g"] \arrow[dr, "j"'] & C_f \arrow[d, "|\lambda|"] \\ & & {|M|} \end{tikzcd}$$

with $|\lambda|$ a homotopy equivalence. QED

*Remark.* We have given the proof of Proposition 2 in detail since it contains most of the technical results of rational homotopy theory which find use below. The reader is referred to Chapter 10 of [5] for an exposition of these results. Also, note that, given a DGA cofibre sequence $M \to A \to B$, there is an associated long exact sequence in cohomology obtained from the cofibre sequence of spaces $|B| \to |A| \to |M|$. In fact, using the mapping cone of cochain complexes, an equivalent long exact cohomology sequence may be derived directly from the definition of a DGA cofibre sequence.

The following result is the DGA analogue of the mapping extension property for topological cofibre sequences.

COROLLARY 3. *Let*

$$M \xrightarrow{J} A \xrightarrow{F} B$$

*be a cofibre sequence. Suppose $U$ is a minimal DGA and $H$: $U \to A$ is a map with $FH \simeq 0$. Then there exists $\beta$: $U \to M$ with $J\beta \simeq H$.*

*Proof.* Upon taking spatial realizations we obtain

$$\begin{array}{ccccc} |B| & \xrightarrow{f} & |A| & \xrightarrow{j} & |M| \\ & & \downarrow{\scriptstyle h} & & \\ & & |U| & & \end{array}$$

where the top row is a cofibre sequence of spaces and $hf \simeq *$ (since $FH \simeq 0$). By the extension property of cofibre sequences, there exists $b$: $|M| \to |U|$ with $bj \simeq h$. Passing to minimal models and letting $\beta = M(b)$ completes the proof. QED

The following result was proved originally by Felix and Tanré using Quillen's Lie algebra model. As a first application of the DGA approach to cofibre sequences, we present a new proof.

THEOREM 4 [4]. *Let*

$$X \xrightarrow{f} Y \xrightarrow{j} C_f$$

*be a cofibre sequence of 1-connected spaces. If $f$ is a formal map, then $C_f$ is a formal space.*

LEMMA (i) *There exists a formal minimal DGA $A$ and maps*

$$\alpha: A \to M(Y), \quad \beta: A \to M(C_f)$$

*such that $J\beta \simeq \alpha$, $\operatorname{Im} \alpha^* = \operatorname{Im} J^*$ and $\alpha^*$ is injective.*

(ii) *There exists a formal minimal DGA $B$ and a map $\gamma$: $B \to M(\Sigma X)$ such that $\gamma^*$ is injective and*

$$\operatorname{Im} \gamma^* \cong \operatorname{Im}\big(\partial^*: H^*(\Sigma X; Q) \to H^*(C_f; Q)\big).$$

*Proof.* (i) Because $f$ is formal, there is a homotopy commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & M(Y) & \xrightarrow{F} & M(X) \\ {\scriptstyle\rho}\downarrow & & {\scriptstyle\theta_Y}\downarrow & & \downarrow{\scriptstyle\theta_X} \\ K(f^*) & \xrightarrow{i} & H^*(Y) & \xrightarrow{f^*} & H^*(X) \end{array}$$

where $K(f^*) = Q \oplus \operatorname{Ker} f^*$ (with trivial differential) and $(A, \rho)$ is the minimal model of $K(f^*)$. Note that $A$ is formal since $H^*(K(f^*)) = K(f^*)$. Because $A$ is minimal and $\theta_Y$ is a quasi-isomorphism, the lifting theorem for minimal DGA's provides a map $\alpha\colon A \to M(Y)$ with $\theta_Y\alpha \simeq i\rho$.

Without loss of generality it may be assumed that $\theta_X^* = \mathrm{id}$, $\theta_Y^* = \mathrm{id}$. We then have $\operatorname{Im}\alpha^* = \operatorname{Im} i = \ker f^* = \operatorname{Im} J^*$, where the last equality follows by exactness in the cohomology sequence associated to

$$X \xrightarrow{f} Y \xrightarrow{j} C_f.$$

Note that $\alpha^*$ is injective because $\rho^*$ is an isomorphism and $i$ is injective.

Now $\theta_X F\alpha \simeq 0$ since $f^*i\rho = 0$. Because $A$ is minimal and $\theta_X$ is a quasi-isomorphism, we obtain $F\alpha \simeq 0$. By Corollary 3 and Proposition 1 there exists a map $\beta\colon A \to M(C_f)$ with $J\beta \simeq \alpha$. (Hence $\beta^*$ is injective and $\operatorname{Im} J^* \cong \operatorname{Im}\beta^*$.)

(ii) Let $W$ be a complement to $\operatorname{Ker}\partial^*$ in $H^*(\Sigma X)$. Hence $W \cong \operatorname{Im}\partial^*$. Give $W$ the structure of a DGA by requiring multiplication and differential to be trivial. This is compatible with the embedding $W \to H^*(\Sigma X)$ since all cup products in $H^*(\Sigma X)$ vanish. In fact, $\Sigma X$ is rationally equivalent to a wedge of spheres, so is formal. Thus, there is a homotopy commutative diagram

$$\begin{tikzcd} B \arrow[r, "\gamma"] \arrow[d] & M(\Sigma X) \arrow[d, "\theta"] \\ W \arrow[r] & H^*(\Sigma X) \end{tikzcd}$$

where $B$ is the formal minimal model of $W$ and $\gamma$ is a lifting of the embedding. Hence, taking $\theta^* = \mathrm{id}$, we have $\operatorname{Im}\gamma^* = W$. (Thus, $\gamma^*$ is injective and $\operatorname{Im}\gamma^* \cong \operatorname{Ker} J^*$.) QED

If $A$ and $B$ are minimal DGA's, then the product in the category of DGA's is given by $AVB = Q \oplus A^+ \oplus B^+$. In particular, letting $A(\ )$ denote the rational polynomial form functor of Sullivan (see [13] or [2]), there is a homotopy commutative diagram (see [7]) of quasi-isomorphisms

$$\begin{tikzcd} & & M(C_f)VM(\Sigma X) \arrow[d] \\ M(C_fV\Sigma X) \arrow[urr, dashed, "\phi"] \arrow[r] & A(C_fV\Sigma X) \arrow[r] & A(C_f)VA(\Sigma X). \end{tikzcd}$$

If the cohomology of each of the DGA's above is identified with

$$H^*(C_f) \oplus H^*(\Sigma X),$$

then we may take $\phi^* = \mathrm{id}$.

*Proof of Theorem* 4. By the lifting theorem for minimal DGA's there is a homotopy commutative diagram

$$\begin{tikzcd}
 & & M(C_f V \Sigma X) \arrow[d, "\phi"] \\
M(AVB) \arrow[r, hook] \arrow[urr, dashed, "\lambda"] & AVB \arrow[r, "\beta V \gamma"] & M(C_f) V M(\Sigma X)
\end{tikzcd}$$

where $M(AVB)$ is the minimal model of $AVB$ and $\beta, \gamma$ are the maps of the lemma. Note that $\operatorname{Im} \lambda^* = \operatorname{Im} \beta^* \oplus \operatorname{Im} \gamma^* \cong \operatorname{Im} J^* \oplus \operatorname{Ker} J^* \cong H^*(C_f)$.

Now according to [8], there is a cooperation map $c\colon C_f \to C_f V \Sigma X$ which has the effect on cohomology,

$$c^*(u, v) = u + \partial^* v$$

for $u \in H^*(C_f)$, $v \in H^*(\Sigma X)$. Using the induced map on minimal models

$$C\colon M(C_f V \Sigma X) \to M(C_f),$$

we may form the composition

$$C\lambda\colon M(AVB) \to M(C_f).$$

The decomposition of $\operatorname{Im} \lambda^*$ and the effect of $C^* = c^*$ show that $C\lambda$ is a quasi-isomorphism. Because $M(AVB)$ and $M(C_f)$ are minimal, $C\lambda$ is an isomorphism. Now $M(AVB)$ is formal since $A$ and $B$ are, so the proof is complete.

*Remark.* (i) Given a fibration

$$F \to E \xrightarrow{p} B,$$

a not quite dual proof may be given of the dual result: the fibre of a coformal map is a coformal space. The proof uses the operator $\Omega B \times F \to F$ (see [10]).

(ii) Further applications of operators and cooperators to rational homotopy theory may be found in [11].

## 2. The homology decomposition of DGA's

If $M$ is a minimal DGA, then it may be decomposed into a sequence of minimal DGA's,

$$Q \subset M(2) \subset \cdots \subset M(n) \subset \cdots \subset M,$$

where $M(n+1) = M(n) \oplus L_{n+1}(V)$ is an elementary extension. According to rational homotopy theory, this decomposition corresponds to the Postnikov decomposition of a space. In this section we will construct the Eckmann-Hilton dual object, the homology decomposition of a DGA.

Let $M$ be a minimal 1-connected DGA. The pair $(N, \sigma)$ is called a *homology n-section* of $M$ if $H^i(N) = 0$ for $i > n$ and $\sigma\colon M \to N$ is an $n$-quasi-isomorphism.

PROPOSITION 5. *For each $n$, there exists a homology $n$-section of $M$.*

*Proof.* The idea is to "kill" the cohomology above degree $n$ by successive elementary extensions. Let $V_1 = H^{n+1}(M)$ and choose a vector space splitting $\tau_1\colon V_1 \to Z^{n+1}(M)$. Define $M^1 = M \otimes L_n(V_1)$ with differential $d(m \otimes 1) = dm$ and $d(1 \otimes v) = \tau_1(v)$. Plainly, $H^{n+1}(M^1) = 0$ and $M \hookrightarrow M^1$ is an $n$-quasi-isomorphism.

Now assume $M^{k-1}$ has been constructed so that $H^{n+i}(M^{k-1}) = 0$ for $1 \le i \le k-1$ and the inclusion $M \hookrightarrow M^{k-1}$ is an $n$-quasi-isomorphism. Let $V_k = H^{n+k}(M^{k-1})$ and choose a splitting

$$\tau_k\colon V_k \to Z^{n+k}(M^{k-1}).$$

Define $M^k = M^{k-1} \otimes L_{n+k-1}(V_k)$ with differential $d(m \otimes 1) = dm$ and $d(1 \otimes v) = \tau_k(v)$. Then $H^{n+i}(M^k) = 0$ for $1 \le i \le k$ and $M \hookrightarrow M^k$ is an $n$-quasi-isomorphism.

Define $M[n]$ to be the direct limit of the inclusions $M^{k-1} \hookrightarrow M^k$. Clearly, $H^i(M[n]) = 0$ for $i > n$ and $\sigma_n\colon M \to M[n]$ is an $n$-quasi-isomorphism. QED

The rest of this section freely uses the techniques of DGA obstruction theory. The reader is referred to Chapter 10 of [5] for the relevant facts.

If $(M[n], \sigma_n)$ and $(M[n-1], \sigma_{n-1})$ are homology sections as constructed above, then there is a map $\varepsilon_n\colon M[n] \to M[n-1]$ such that $\varepsilon_n\sigma_n \simeq \sigma_{n-1}$. This follows since the obstructions to the existence of $\varepsilon_n$ lie in $H^i(M[n-1]) = 0$ for $i > n$. The same argument shows that, if $(M[n]', \sigma_n')$ is a homology $n$-section obtained by choosing splittings $\tau_k'$, then there is a quasi-isomorphism $\psi\colon M[n] \to M[n]'$ with $\psi\sigma_n = \sigma_n'$. Thus, $M[n]$ and $M[n]'$ have isomorphic minimal models and it is in this sense that $M[n]$ is well defined.

Now, since it is preferable to work with isomorphisms instead of quasi-isomorphisms and minimal DGA's instead of nonminimal ones, we consider the model of $M[n], (M_n, \rho_n)$. The usual lifting property provides maps $\sigma_n\colon M \to M_n$ and $\tilde{\varepsilon}_n\colon M_n \to M_{n-1}$ with $\rho_n\tilde{\sigma}_n \simeq \sigma_n$ and $\rho_{n-1}\tilde{\varepsilon}_n \simeq \varepsilon_n\rho_n$. Because $\rho_{n-1}$ is a quasi-isomorphism, it follows easily that there is a homotopy $\tilde{\varepsilon}_n\tilde{\sigma}_n \simeq \tilde{\sigma}_{n-1}$. We are led to the following:

DEFINITION . A *homology decomposition* of a 1-connected minimal DGA $M$ consists of a homotopy commutative diagram of minimal DGA's,

$$\begin{tikzcd} \cdots \arrow[r] & M_{k+1} \arrow[r, "\varepsilon_{k+1}"] & M_k \arrow[r, "\varepsilon_k"] & M_{k-1} \arrow[r] & \cdots \\ & & M \arrow[ul, "\sigma_{k+1}"] \arrow[u, "\sigma_k"] \arrow[ur, "\sigma_{k-1}"'] & & \end{tikzcd}$$

such that, for each $k$, $(M_k, \sigma_k)$ is a homology $k$-section.

*Remark.* The system $(M_n, \tilde{\sigma}_n, \tilde{\varepsilon}_n)$ above is a homology decomposition of $M$. Also, the homology decomposition of a space induces a homology decomposition of its minimal model.

So far, the notion of a DGA homology decomposition has been rather naive. We would now like to show that the more sophisticated structure of $k'$-invariants may be incorporated into the DGA setting as well.

For this purpose use the technique of Proposition 5 to construct the DGA analogue of a Moore space. Begin with the free algebra $L_{n-1}(V)$ and successively kill cohomology above degree $n-1$. The limit of this construction is denoted by $\bar{L}_{n-1}(V)$ and satisfies $H^i(\bar{L}) = 0$ for $i \neq n-1$, $H^{n-1}(\bar{L}) = V$. Our goal is then to prove the following result.

THEOREM 6. *There exists a map* $k$: $M_{n-1} \to \bar{L}_{n-1}(H^n(M))$ *so that*

$$M_n \xrightarrow{\tilde{\varepsilon}_n} M_{n-1} \xrightarrow{k} \bar{L}$$

*is a cofibre sequence of DGA's.*

*Remark.* Note that this structure is then the precise analogue of the $k'$-invariant cofibre sequence described in [8].

In order to prove Theorem 6, it is convenient to work with a homology $(n-1)$ section which is constructed from $M[n]$, rather than $M$, using the technique of Proposition 5. More precisely, begin with $M[n]$ and kill $V = H^n(M[n])$ by choosing a splitting $\lambda$: $V \to Z^n(M[n])$ and forming $M[n] \otimes L_{n-1}(V)$ with $d(1 \otimes v) = \lambda(v)$. The procedure of Proposition 5 now kills any new cohomology which may be created. The limit of the constructions is then an $(n-1)$ section and is denoted by $M[n-1]'$. Also, we denote the inclusion by $\alpha$: $M[n] \to M[n-1]'$ and the induced map on models by $\tilde{\alpha}$: $M_n \to M'_{n-1}$.

An application of DGA obstruction theory yields a quasi-isomorphism $h$: $M[n-1]' \to M[n-1]$ which restricts to $\varepsilon_n$ on $M[n]$. Hence $\tilde{h}$: $M'_{n-1} \to M_{n-1}$ is an isomorphism with $\rho_{n-1}\tilde{h} \simeq h\rho'$. We have

$$\rho_{n-1}\tilde{h}\tilde{\alpha} \simeq h\rho'\tilde{\alpha} \simeq h\alpha\rho_n \simeq \varepsilon_n\rho_n \simeq \rho_{n-1}\tilde{\varepsilon}_n.$$

Because $\rho_{n-1}$ is a quasi-isomorphism, we obtain $\tilde{h}\tilde{\alpha} \simeq \tilde{\varepsilon}_n$. For convenience, denote $\tilde{h}^{-1}$ by $\theta$.

Now, begin to define a map $\beta$: $M[n-1]' \to \overline{L}_{n-1}(H^n(M))$ by requiring that the restrictions to $M[n]$ and $V = H^n(M)$ be zero and the identity respectively. The obstructions to the existence and uniqueness (up to homotopy) of an extension of $\beta$ to all of $M[n-1]'$ lie in $H^i(\overline{L}_{n-1}(V))$ for $i \geq n$. These groups are zero however, so the desired map $\beta$ exists. We then have

$$M[n] \xrightarrow{\alpha} M[n-1]' \xrightarrow{\beta} \overline{L}$$

with $\beta\alpha = 0$.

Now let $\Delta = \beta\rho'$: $M'_{n-1} \to \overline{L}$ and define $k$: $M_{n-1} \to \overline{L}$ by $k = \Delta\theta$. To reach the conclusion of Theorem 6 we must find a map $\gamma$ which completes the following diagram up to homotopy:

$$\begin{tikzcd} M_n \arrow[r, "\tilde{\varepsilon}_n"] \arrow[d, dashed, "\gamma"'] & M_{n-1} \arrow[r, "k"] \arrow[d, "j"] & \overline{L} \arrow[d] \\ K(\bar{k}) \arrow[r, "I"] & \overline{M}_{n-1} \arrow[r, "\bar{k}"] & \overline{L} \end{tikzcd}$$

*Proof of Theorem* 6. From the lemmas below we obtain a quasi-isomorphism $\tilde{\phi}$: $M_n \to K(\overline{\Delta})$ and a diagram (Fig. 1) where $\hat{\theta}$ is the obvious extension of $\theta$, the right hand cube is commutative and $i\tilde{\alpha} \simeq \psi\tilde{\phi}$. The bottom rows are short exact sequences of DGA's, so the 5-Lemma is applicable to the associated cohomology sequences. Hence $\delta$ is a quasi-isomorphism and the lifting theorem for minimal DGA's provides $\gamma$: $M_n \to K(\bar{k})$ with $\delta\gamma \simeq \tilde{\phi}$. Now

$$\hat{\theta}j\tilde{\varepsilon}_n = i\theta\tilde{\varepsilon}_n \simeq i\tilde{\alpha} \simeq \psi\tilde{\phi} \simeq \psi\delta\gamma \simeq \hat{\theta}I\gamma.$$

Because $\hat{\theta}$ is a quasi-isomorphism, $j\tilde{\varepsilon}_n \simeq I\gamma$. Hence,

$$M_n \xrightarrow{\tilde{\varepsilon}_n} M_{n-1} \xrightarrow{k} \overline{L}$$

is a cofibre sequence. QED

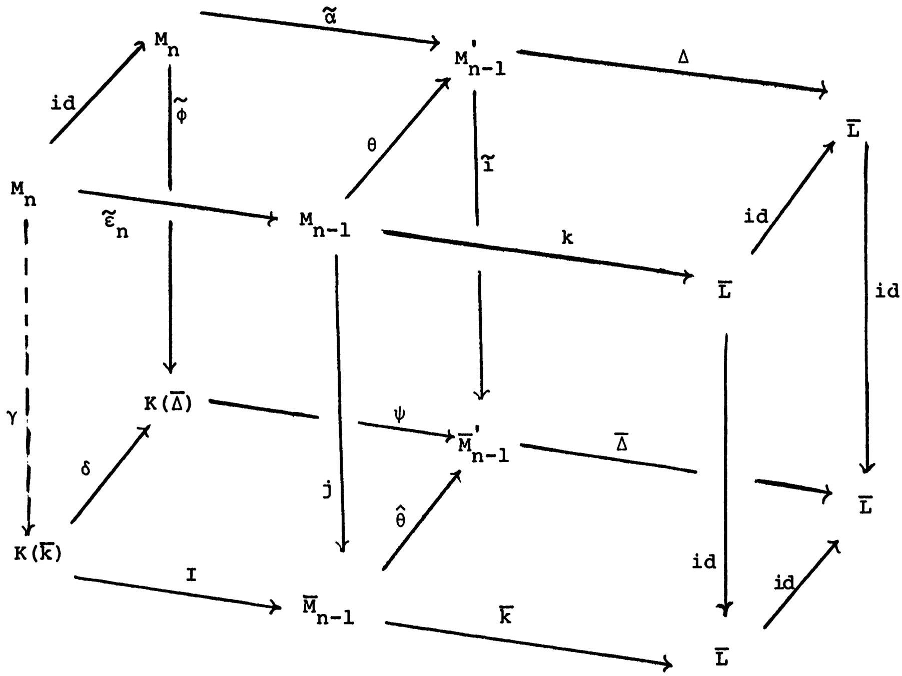


FIG. 1

We now wish to construct the map $\tilde{\phi}$: $M_n \to K(\bar{\Delta})$. Recall that to a map $f$: $A \to B$ of cochain complexes we may associate the *mapping cone complex* $C(f)$ defined by $C(f)^n = A^n \oplus B^{n-1}$ with $d(a, b) = (-da, db + fa)$. If

$$C \xrightarrow{g} A \xrightarrow{f} B$$

is a short exact sequence of complexes, then we may define a map $\hat{g}$: $C \to C(f)$ by $\hat{g}(c) = (gc, 0)$. This map is a cochain map up to sign and thus induces a map on cohomology. It is standard that, in fact, $\hat{g}^*$: $H^*(C) \cong H^*(C(f))$. Now, the map $\beta$: $M[n-1]' \to \bar{L}$ induces a short exact sequence of DGA's,

$$K(\bar{\beta}) \xrightarrow{\lambda} \bar{M}[n-1]' \xrightarrow{\bar{\beta}} \bar{L}$$

Hence, if in general we denote $H^*(C(f))$ by $H^*(f)$, we have the immediate result:

LEMMA. $\hat{\lambda}^*$: $H^*(K(\bar{\beta})) \cong H^*(\bar{\beta})$.

Now,

$$M[n] \xrightarrow{\alpha} M[n-1]' \xrightarrow{\beta} \bar{L}$$

is not short exact, but $\beta\alpha = 0$ and we prove:

LEMMA . $\hat{\alpha}^*$: $H^*(M[n]) \cong H^*(\beta)$.

*Proof.* (i) $H^i(\beta) = 0$ for $i > n$: Let $(a, b) \in Z^i(\beta)$. Then $da = 0$ and $db + \beta a = 0$. Because $H^i(M[n-1]') = 0$ for $i \geq n$, there exists $c$ with $dc = a$ and then $db + \beta a = d(b + \beta c) = 0$. Thus $b + \beta c$ is an $(i-1)$ cocycle in $\bar{L}$ and, since $H^{i-1}(\bar{L}) = 0$ for $i > n$, there exists $e$ with $de = b + \beta c$. Thus, $d(-c, e) = (a, b)$.

(ii) $\hat{\alpha}$ is an $(n-1)$ quasi-isomorphism: If $i < n - 1$, then $\bar{L}^i = 0$. Thus, $C(\beta)^i = M[n-1]'^i$ for $i \leq n-1$. Hence,

$$H^i(\beta) \cong H^i(M[n-1]') \cong H^i(M[n]) \quad \text{for } i \leq n-1.$$

(iii) $\hat{\alpha}$ is a quasi-isomorphism: By the first two parts, it is sufficient to check degree $n$. Suppose $x \in Z^n(M[n])$ such that $\hat{\alpha}(x) = d(c, 0)$ for $(c, 0) \in C(\beta)^{n-1}$. Then $d(-c) = \alpha(x)$ and $\beta c = 0$. The first equality implies that either

$$-c \in V = H^n(M[n]) \quad \text{or} \quad -c = \alpha(c') \in \operatorname{Im} \alpha.$$

Now $\beta|_V = \mathrm{id}$, so $\beta c = 0$ implies the second possibility. Because $\alpha$ is injective, we have $x = dc'$. Hence $\hat{\alpha}^*$ is injective.

Now let $(a, b) \in Z^n(\beta)$. Since $b \in V$, there exists $x \in V = H^n(M[n])$ with $\beta x = b$. Note that $c = a + dx \in Z^n(M[n])$. Then

$$(a, b) = (c - dx, \beta x) = (c, 0) + d(x, 0).$$

Thus $\hat{\alpha}^*([c, 0]) = [a, b]$ and $\hat{\alpha}^*$ is surjective. QED

Now, because $\beta\alpha = 0$, we obtain the commutative diagram

$$\begin{tikzcd} M[n] \arrow[r, "\alpha"] \arrow[d, "\phi"'] & M[n-1]' \arrow[r, "\beta"] \arrow[d, "i"'] & L \arrow[d, equal] \\ K(\bar{\beta}) \arrow[r, "\lambda"] & \bar{M}[n-1]' \arrow[r, "\bar{\beta}"] & \bar{L} \end{tikzcd}$$

Define a map $\Omega$: $C(\beta) \to C(\bar{\beta})$ by $\Omega(a, b) = (ia, b)$. Plainly $\Omega$ is a cochain map and an application of the 5-Lemma shows that $\Omega^*$ is an isomorphism. The definition of $\Omega$ implies that $\Omega^*\hat{\alpha}^* = \hat{\lambda}^*\phi^*$. Then, since $\hat{\alpha}^*$ and $\hat{\lambda}^*$ are also isomorphisms by the lemmas, so is $\phi^*$.

Now, by examining the relevant short exact sequences of DGA's we see that $s\colon K(\overline{\Delta}) \to K(\overline{\beta})$ is a quasi-isomorphism. Hence the lifting theorem provides $\tilde{\phi}\colon M_n \to K(\overline{\Delta})$ with $s\tilde{\phi} \simeq \phi\rho_n$.

Finally, we let $\hat{\rho}'\colon \overline{M}'_{n-1} \to \overline{M}[n-1]'$ denote the extension of

$$\rho'\colon M'_{n-1} \to M[n-1]'$$

and note that

$$\hat{\rho}'\tilde{i}\tilde{\alpha} \simeq i\rho'\tilde{\alpha} \simeq i\alpha\rho_n \simeq \lambda\phi\rho_n \simeq \lambda s\tilde{\phi} \simeq \hat{\rho}'\psi\tilde{\phi},$$

where $\tilde{i}\colon M'_{n-1} \to \overline{M}'_{n-1}$ lifts $i$. Because $\hat{\rho}'$ is a quasi-isomorphism, we have $\tilde{i}\tilde{\alpha} \simeq \psi\tilde{\phi}$.

## 3. Consequences

Again, a simple application of DGA obstruction theory proves:

PROPOSITION 7. *If $M$ is formal, then $M_n$ is formal for each $n$.*

*Remark.* This result is dual to the easily observed fact that if $M$ is coformal (i.e., the differential is quadratic), then $M(n)$ is coformal for each $n$.

In [1], Brown and Copeland give an example of a 1-connected space having two homology decompositions, not all of whose $n$-sections are homotopy equivalent. We now show that this situation cannot arise in the rational category. (Indeed, see [14] for a stronger result!)

PROPOSITION 8. *If $X$ is a rational space, then any two homology decompositions have $n$-sections of the same homotopy type.*

*Proof.* Let $\{X_k\}$ and $\{Y_k\}$ denote homology decompositions of $X$. Passing to minimal models, we find that the DGA obstructions vanish and there is a homotopy commutative diagram

$$\begin{tikzcd} & M(X_k) & \\ M(X) \arrow[ur] \arrow[r] \arrow[dr] & M(x)[k] \arrow[u] \arrow[d] & M_k. \arrow[l] \arrow[ul] \arrow[dl] \\ & M(Y_k) & \end{tikzcd}$$

The right horizontal map and the vertical maps are quasi-isomorphisms, so the compositions $M_k \to M(X_k)$ and $M_k \to M(Y_k)$ are isomorphisms. Then $M(X_k) \simeq M(Y_k)$ and $X_k \simeq Y_k$. QED

Our final (and main) application makes full use of the refined structure provided by Theorem 6 to indicate how a formal space is built up from its formal $n$-sections. Perhaps more importantly, we find certain obstructions to formality which arise in the step-by-step construction of a space from its homology decomposition.

Define a map $\tau: \bar{L} \to \bar{L}$ as follows: let $t \in Q/\{0\}$ and set $\tau(x) = t^n x$ for $x \in V$. Extend freely to $L_{n-1}(V)$. Now, the obstructions to the existence and uniqueness (up to homotopy) of an extension to all of $\bar{L}$ lie in $H^i(\bar{L})$ for $i \geq n$. These groups are, of course, zero. Hence $\tau$ exists and has the cohomological effect $\tau^*[x] = t^n[x]$.

*Remark.* We point out explicitly that $\tau$ is *not* the lift of a grading automorphism (the degrees are wrong), so the following theorem bears no obvious relationship to the characterization of formal maps given by Felix-Tanré in [4].

MAIN THEOREM. *Let*

$$M_n \xrightarrow{\tilde{\varepsilon}_n} M_{n-1} \xrightarrow{k} \bar{L}_{n-1}(H^n(M))$$

*be a cofibre sequence of DGA's and suppose that $M_{n-1}$ is formal. Then $M_n$ is formal if and only if there exists a lift of a grading automorphism $T \in \mathrm{Aut}(M_{n-1})$ such that $\tau k \simeq kT$.*

*Proof.* Note that $\tau$ and $T$ are defined with respect to the same $t \in Q/\{0\}$. Also, recall that $T$ is a lift of a grading automorphism if $T^*[x] = t^i[x]$ for any $[x] \in H^i(M_{n-1})$ and all $i$.

Suppose $T$ exists with $\tau k \simeq kT$. Now $kT\tilde{\varepsilon} \simeq \tau k\tilde{\varepsilon} \simeq 0$ since $k\tilde{\varepsilon} \simeq 0$. Corollary 3 provides a map $R$: $M_n \to M_n$ with $T\tilde{\varepsilon} \simeq \tilde{\varepsilon}R$. Passing to spaces via the functor $|\ \ |$, Proposition 2 gives a homotopy commutative diagram of cofibre sequences,

$$\begin{array}{ccccccc} L & \to & X_{n-1} & \to & X_n & \dashrightarrow & \Sigma L \\ {\scriptstyle\tau}\downarrow & & {\scriptstyle T}\downarrow & & {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle\Sigma\tau} \\ L & \to & X_{n-1} & \to & X_n & \dashrightarrow & \Sigma L \end{array}$$

where we have abused notation for convenience and the last square is the natural extension of the Puppe sequence. Now, clearly $(R^*)^n = (\Sigma\tau^*)^n$ by exactness of the cohomology sequences and $(\Sigma\tau^*)^n[\Sigma x] = \Sigma(\tau^*[x]) = \Sigma t^n[x] = t^n[\Sigma x]$. Now, $(R^*)^i = (T^*)^i$ for $i < n$ and $T^*$ is a grading automorphism. Hence $M_n$ is formal by Sullivan's condition (see Theorem 12.7 of [13]).

Now suppose that $M_n$ is formal and take a homology decomposition. Passing to spaces via $| \quad |$, there is a homotopy commutative diagram

$$\begin{tikzcd} \to X_{k-1} \arrow[r] \arrow[dr] & X_k \arrow[r] \arrow[d] & X_{k+1} \to \arrow[dl] \\ & X_n & \end{tikzcd}$$

Convert these maps to cofibrations (inclusions) and then, using the homotopy extension property, convert the homotopy commutative triangles to strictly commutative ones. The resulting system is a homology decomposition of $X_n = |M_n|$ homotopy equivalent to the original system. Now, applying the full strength of Toomer's naturality results (see [14]) we see that a lift of a grading automorphism $R$ (which exists since $M_n$ is formal) induces a homotopy commutative diagram

$$\begin{tikzcd} K' \arrow[r] \arrow[d, "\tau"'] & X_{n-1} \arrow[r] \arrow[d, "T"] & X_n \arrow[d, "R"] \\ K' \arrow[r] & X_{n-1} \arrow[r] & X_n \end{tikzcd}$$

where $T = R|$ and $K' = K'(H_n X_n, n - 1)$. Comparing the long exact cohomology sequences, we see that $\tau^*[x] = t^n[x]$ since $R^*$ has this property. Since $M(K') \simeq \overline{L}$ and the homology decomposition is equivalent to the original system, then we obtain $\tau k \simeq kT$. QED

*Example.* Consider the homology decomposition

$$S^5 \to S^3 V S^3 \to S^3 \times S^3$$

where the first map is given by the Whitehead product of the inclusions $i_1, i_2$: $S^3 \to S^3 V S^3$. This translates into a DGA cofibre sequence

$$M(S^3 \times S^3) \to M(S^3 V S^3) \xrightarrow{k} M(S^5)$$

where

$$M(S^3 \times S^3) = L_3(x, y), \qquad M(S^5) = L_5(w)$$

and

$$M(S^3 V S^3) = L_3(x, y) \otimes L_5(z) \otimes \cdots$$

with $dz = xy$. Now $k$ is trivial on $x$ and $y$ and $k(z) = w$. Clearly $k$ is not homotopic to zero and therefore is *not* formal since $(k^* = 0)$.

Now define $\tau$ on $M(S^5)$ by $\tau(w) = t^6 w$ and $T$ on $M(S^3 V S^3)$ by $Tx = t^3 x$, $Ty = t^3 y$ and extending. Then $dT = Td$ implies $Tz = t^6 z$ and $kT \simeq \tau k$. Hence our Main Theorem says that $M(S^3 \times S^3)$ is formal, which is certainly the case.

*Remark.* The discussion above furnishes an example of a map which is not formal but induces a formal cofibre. This also occurs, for example, in the case of the Hopf map $S^3 \to S^2$. It is hoped that our Main Theorem sheds some light on this phenomenon.

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