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AN INDEX THEOREM FOR FOLIATIONS

BY

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The purpose of this paper is to prove a theorem relating the index theory to exotic characteristic numbers for an important class of foliations.

One of the problems in foliation theory is to relate the transverse geometry to its topological invariants, the exotic classes. As our representative of the transverse geometry, we study an invariant arising from geometrically interesting transverse operators to leaves.

A Lie group acting by isometries with constant orbit dimension generates a Riemannian foliation. In this paper we study the case of R^n acting locally freely by isometries, this being an interesting class of foliations and also the study of a much larger class of actions can be reduced to that of R^n. Let P be an invariant transversely elliptic differential operator to the action. We construct, using index (P), an R/Z analytic invariant called virtindex ${}_{R^n}(P)$ and relate this to the Simons characteristic numbers of the resulting Riemannian foliation. A special case has appeared in [8].

We first state the situation for $n = 1$. Let R act by isometries with no fixed points on the compact $4k - 1$ oriented manifold. Let f be a symmetric, homogeneous polynomial of degree k in $2k$ indeterminates. Let D^+ be the transverse signature operator, u_f the virtual representation of [2, p. 596], and $D^+ \otimes u_f$ the (virtual) operator on $\Lambda^+(\nu) \otimes u_f(\nu)$, ν being the normal bundle to the foliation F generated by the R action. Let $S_f(F)$ be the Simons class associated to f and the Riemannian foliation F.

THEOREM 1.

$$\operatorname{Virtindex}_R\left(D^+ \otimes u_f\right) = -2^{2k-1} S_f(F)[M] \quad \operatorname{mod} Z\left[\tfrac{1}{2}\right]$$

Now consider R^n acting on M locally freely by isometries, generating a codimension $4k - 1 - n$ oriented Riemannian foliation F. Each θ in R^n determines an R action and a foliation F_θ. Let D_θ^+ be the transverse signature operator to the R action and D^+ to the R^n action.

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THEOREM 2. (a) *For generic θ,* $\operatorname{Virtindex}_R(D_\theta^+ \otimes u_f)$ *is constant and*

$$\operatorname{Virtindex}_R\left(D_\theta^+ \otimes u_f\right) = -2^{2k-1}S_f(F)[M] \quad \operatorname{mod} Z\left[\tfrac{1}{2}\right].$$

(b) *For $n > 2$,*

$$\begin{aligned}\operatorname{Virtindex}_R\left(D_\theta^+ \otimes u_f\right) &= \operatorname{Virtindex}_{R^n}\left(D^+ \otimes u_f\right) \\ &= S_f(F)[M] = 0 \quad \operatorname{mod} Z\left[\tfrac{1}{2}\right]\end{aligned}$$

As a simple application we get:

THEOREM 3. *Let R act locally freely by isometries on the compact oriented $4k-1$ manifold M. Let F be the resulting codimension $4k-2$ Riemannian foliation and suppose $S_f(F)[M] \neq 0$ mod $Z[\frac{1}{2}]$. Then this action cannot be extended to a locally free isometric action of R^n for $n > 2$.*

The characteristic classes can be expressed in terms of the residues of f along the singular set of the action extended to an oriented manifold which bounds M. Suppose W is a compact oriented manifold which bounds M. Consider any isometric extention of the R^n action to W (with the metric a product near M). Let Γ_j be the connected components of the fixed set on W.

THEOREM 4. *Consider generic θ.*
(a) $\operatorname{Virtindex}_R(D_\theta^+ \otimes u_f) = -2^{2k-1}\Sigma_j \operatorname{Res}(f, \theta, \Gamma_j)$ mod Z.
(b) *For $n > 2$,* $\Sigma \operatorname{Res}(f, \theta, \Gamma_j) = 0$ mod Z.

COROLLARY 5. *Let R act isometrically on the closed oriented $4k$ manifold M, with fixed set $\cup \Gamma_j$. Let D_j^+ be the transverse signature operator to the codimension $4k-2$ Riemannian foliation on the boundary of a tubular neighborhood of Γ_j. Then* $\Sigma_j \operatorname{Virtindex}_R(D_j^+ \otimes u_f) = 0$ mod Z.

In §1 we review some basic facts about foliations and about transverse index theory. In §2 we introduce and study the virtindex. §3 contains an outline of the proofs, and §§4, 5 contain details of proofs.

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1. Riemannian foliation and the transverse signature operator

We recall some basic facts about Riemannian foliations and the transverse signature operator. We assume all our foliations are oriented. Riemannian foliations were first described in [13]; however we shall follow the approach in

[12]. A codimension q Riemannian foliation of M is given by a family $\{U_\alpha, f_\alpha, h_{\alpha\beta}, g_\alpha\}$ where $\{U_\alpha\}$ is an open covering of M, $f_\alpha : U_\alpha \to R^q$ a smooth submersion, g_α a Riemannian metric on R^q, and for each $x \in U_\alpha \cap U_\beta$, $h^x_{\alpha\beta}$ is an isometry of a neighborhood of $f_\alpha(x)$ with the metric g_α onto a neighborhood of $f_\beta(x)$ with the metric g_β satisfying $f_\beta = h_{\beta\alpha} f_\alpha$. If a Lie groups acts on a Riemannian manifold M with constant orbit dimension, by isometries, the orbits form a Riemannian foliation.

With this description of a Riemannian foliation we can readily produce a signature complex. First, the normal bundle ν of the foliation is obtained from the union of TR^q using identifications given by the differentials of the $h_{\beta\alpha}$ and the metrics g_α yield a metric on ν. We now assume q is even. The metric g on ν yields a star operator on $\Lambda\nu^*$ (the complexified exterior algebra of the dual normal bundle) which in turn yields a splitting, sections $\Lambda\nu^* = \Omega_+ \oplus \Omega_-$. This is the same splitting as we get by using the star operator in R^q relative to g_α and making identifications using the $dh_{\beta\alpha}$. Thus, for each α let D^+_α be the signature operator on R^q relative to g_α. Then $f_\alpha^{-1}(D^+_\alpha)$ can be interpreted as an operator on $\Omega_+ | U_\alpha$ and $f_\alpha^{-1}(D^+_\alpha) = f_\beta^{-1}(D^+_\beta)$ on $U_\alpha \cap U_\beta$ because the $h_{\beta\alpha}$ are local isometries. The resulting operator

$$D^+ : \Omega_+ \to \Omega_-$$

is the transverse signature operator. We will want D^+ to act on sections of $\Lambda\nu^*$ with coefficients in a vector bundle V (sections of $\Lambda\nu^* \otimes V$) and on the resulting $\Omega_\pm(V)$. We can extend D^+ to $\Omega_\pm(V)$ by the construction of [11, p. 87]. We denote the resulting operator by $D^+ \otimes V$. We could also construct D^+ *directly* by constructing a transverse d and a transverse $*d*$ using a connection on ν.

We can compute the symbol of $(D^+)^2$ just as one does in the case of the ordinary signature operator and $\sigma((D^+)^2)_v$ is multiplication by $\|\pi v\|$, where π is the orthogonal projection on ν^* induced by a compatible metric on M. Thus $\sigma(D^+)_v$ and $\sigma(D^+ \otimes V)_v$ are injective for non-zero v which vanish on leaves.

Exotic characteristic classes for a foliation are cohomology classes that come from the cohomology of the appropriate classifying space for foliations. A general Riemannian foliation has classes corresponding to the Simons construction [14] applied to the unique Riemannian torsion-free connection on the normal bundle and appropriate polynomials. If the normal bundle is stably trivial we can study such classes by the methods of [9]. Let f be a homogeneous symmetric polynomial of degree k in $2k$ indeterminates, $f = \Sigma\alpha_I \sigma_{i_1} \ldots \sigma_{i_r}$. Let $\phi(A)$ be the polynomial function $\Sigma\alpha_I c_{2i_1}(A) \ldots c_{2i_r}(A)$ where c_j are described by $\Sigma t^j c_j(A) = \mathrm{Det}(I - (t/2\pi i)A)$. We will be concerned with the following data. Given a Riemannian foliation of M of codimension $4k - 2$ or less, the Simons construction applied to ϕ yields a class

$$S_f \in H^{4k-1}(M, R/Z).$$

The same polynomial f yields a virtual representation u_f of $SO(4k)$ (and by restriction, of $SO(4k-n-1)$) which is described in [2, p. 596]. We can take the associated virtual vector bundle $u_f(\nu^*)$ to u_f and the principal normal coframe bundle of the foliation.

2. Transverse index theory

We recall briefly the transverse index theory of Atiyah and Singer [1], [15]. Let G be a Lie group acting smoothly on a compact manifold M, E and F smooth G bundles over M and D a G invariant operator from E to F. D is said to be transversally elliptic to the G action if $\sigma(D)_v$ is an isomorphism for all non-zero v which vanish on orbits. Introducing metrics, one extends D as an unbounded operator on L_2. Let $f \in C_0(G)$ and let ρ be the representation of G on $L_2(E)$. Let T_f be the operator on $\ker(D)$ given by $T_f(v) = P\int_G \bar{\rho}(g)f(g)v\,dg$ where P is projection on $\ker(D)$. Perform a similar construction on $\ker(D^*)$ to obtain U_f. Then $T_f - U_f$ is a trace class operator, the $\operatorname{index}_G(D)$ is defined to be the function $f \to \operatorname{Trace}(T_f - U_f)$ and this index is a distribution (see [15] for a detailed treatment). If G is a compact group or an abelian group acting by isometries, the techniques of [1] are applicable. Let G be a Lie group acting by isometries on a compact oriented Riemannian manifold with constant orbit dimension. The orbits are then the leaves of an oriented Riemannian foliation and the transverse signature operator (with any coefficients) is transversally elliptic to this action.

We will be concerned with actions of R^n by isometries on a compact oriented $4k-1$ manifold M. In this case R^n acts via a torus T which acts on M. Let R^n be the universal covering space of T and let $R^n \to R^N$ be the inclusion corresponding to the original action. Let D be transversely elliptic to the R^n action and hence also to the R^N action and the T action. $\operatorname{Index}_{R^N}(D)$ is C^∞ in directions transverse to R^n in R^N and hence restricts (in the sense of distributions [16, p. 71]) to $\operatorname{index}_{R^n}(D)$. If $\operatorname{index}_{R^N}(D)$ is the limit of C^∞ functions, then $\operatorname{index}_{R^n}(D)$ is just the limit of the corresponding restricted functions. The same remarks apply if D is transversely elliptic to the orbits of a subtorus T' of T.

We recall from [1] that for G compact the index_G of D is determined by its symbol class in $K_G(T_GM)$ and

$$\operatorname{index}_G\colon K_G(T_GM) \to D'(G). \tag{2.1}$$

For $G = S^1$ we have the $R(G)$ map [1, lecture 5]

$$J\colon R(G)_\lambda \to D'(G)$$

where $R(G)_\lambda$ is all quotients of $R(G)$ with powers of λ in the denominator.

Now let $G = T^N$ and let $t_j^k(\theta_1, \dots, \theta_N) = e^{ik\theta_j}$. We also have a map J from rational functions with denominator a power of $(1 - t_1^{m_1})\dots(1 - t_N^{m_N})$ for $m_j > 0$ to distributions on T^N given by $J(h) = h^+ - h^-$ where $h^\pm$ is the distribution

$$h^\pm\{t_1^{-k} \dots t_N^{-k_N}\} = \frac{1}{(2\pi i)^N} \int_{|t_j| = 1 \pm \varepsilon} h t_1^{-k_1} \dots t_N^{-k_N} \frac{dt_1}{t_1} \dots \frac{dt_N}{t_N}.$$

The virtual index. We introduce an analytic R/Z invariant which will be defined for some operators transversely elliptic to an R^n or T^n action on a compact oriented manifold. Of course topologically interesting operators have a virtual index.

We first start with S^1 actions. *Let* $G = S^1$. Let G act on compact oriented M and let D be transversely elliptic to this action. Let $\psi = \text{index}_G(D)$.

Assume that there are functions $h_0(z)$ and $h_1(z)$ which satisfy:

(1) $h_0(z)$ is analytic in $0 < |z| < 1$ with at worst a pole at $z = 0$, has integer coefficients in its Laurent expansion around $z = 0$, and has a removable singularity at $z = 1$.

(2) $h_1(z)$ is analytic in $|z| > 1$ with at worst a pole at ∞, has integer coefficients in its Laurent expansion around ∞, and has a removable singularity at $z = 1$.

(3) $\psi = h_0^+ + h_1^+$ (see [1, p. 38]).

DEFINITION.

$$\text{Virtindex}_G(D) = \lim_{\substack{|z|<1 \\ z \to 1}} h_0(z) - \lim_{\substack{|z|>1 \\ z \to 1}} h_1(z) \mod Z.$$

Remarks. This is well defined as is easily seen by looking at the Laurent expansions of h_0 and h_1. Since index_G depends only on the symbol in $K_G(T_G M)$ so also does the virtual index (when it is defined). Thus, in this definition, D can be any element of $K_G(T_G M)$.

Now, directly from the definitions of J and virtindex we get:

LEMMA (2.2). *If* $\text{index}_G(D) = J(h)$ *where* h *has a removable singularity at* $z = 1$, *then* $\text{Virtindex}_G(D) = 2h(1)$.

Next let R act by isometries on a compact oriented Riemannian manifold M; then R acts via a torus T^N acting on M. Let the inclusion of R in T^N corresponding to this action be given by

(2.3) $$s \to (e^{i\lambda_1 s}, \dots, e^{i\lambda_N s}).$$

If we are given another action parameterized by $\lambda' = (\lambda'_1, \dots, \lambda'_N)$ we say that the actions are near if λ' is near λ in R^N/Z^N. When $\lambda_1, \dots, \lambda_N$ are rational numbers (in lowest terms) let m be the least common multiple of the denominators. We call $\theta \to (e^{im\lambda_1\theta}, \dots, e^{im\lambda_N\theta})$ the *associated* G action to the given R action.

Let D be transversely elliptic to the given R action. Let $r_{(k)}$ be a sequence of rational N-tuples approaching λ. Each defines an R action and we consider the associated G action. D is transversely elliptic to nearby actions. Assume that for each such rational sequence approaching λ, $\lim_k \mathrm{Virtindex}_G(D)$ exists and this limit is independent of the sequence. Then,

DEFINITION. $\mathrm{Virtindex}_R(D) = \lim_k \mathrm{Virtindex}_G(D)$.

Note. In this definition, D can be a formal sum of transversely elliptic operators.

Let u be a virtual representation of $T = T^N$ with $ch(u) = p(x_1, \dots, x_N)$ + higher degree terms in $H^{**}(BT; Q)$, where p is a homogeneous polynomial of degree Nl. Let $\alpha = ((1 - t_1)\dots(1 - t_N))^l$ and $h = u/\alpha$.

LEMMA (2.3). *Let D be transversely elliptic to the R action and assume* $\mathrm{index}_T(D) = J(h)$. *Then* $\mathrm{Virtindex}_R(D)$ *exists and equals h restricted to R and evaluated at $t = 1$.*

Proof. R acts by $\theta \to (e^{i\lambda_1\theta}, \dots, e^{i\lambda_N\theta})$ for some λ's none equal to zero (assuming that the image of R is dense in T). Let $(r_1, \dots, r_N)$, with $r_j \neq 0$ and r_j rational, parametrize a nearby action relative to which D is transversely elliptic. Consider the associated S^1 action. Let $k_j = mr_j$. By restriction, $\mathrm{index}_{S^1}(D) = J(h)$, where h is a rational function in z. To see that h has a removable singularity, at $z = 1$ replace t_j in u by $e^{it_j\theta}$ to get

$$u = i^{Nl}\theta^{Nl}p(k_1, \dots, k_N) + A\theta^{Nl+1}$$
$$\alpha = i^{Nl}\theta^{Nl}k_1^l \dots k_N^l + B\theta^{Nl+1}.$$

Thus h has a removable singularity at $z = 1$ and

$$h(1) = \frac{p(k_1, \dots, k_N)}{(k_1 \dots k_N)^l} = \frac{p(r_1, \dots, r_N)}{(r_1 \dots r_N)^l}$$

by homogeneity. Then the result follows by continuity and the definition.

Finally, let R^n act by isometries on the compact oriented M and let D be transversely elliptic to this action. First consider $T = T^n$ acting. Assume

$\psi = \text{index}_T D$ satisfies (2.1), where now h_0 and h_1 are functions of $z_1, \ldots, z_n$ with at worst poles in $\bigcup_j \{0 < |z_j| < 1\}$ and $\bigcup_j \{1 < |z_j|\}$ respectively. Then:

DEFINITION.

$$\text{Virtindex}_T(D) = \lim_{\substack{z_j \to 1 \\ |z_j| < 1}} h_0(z_1, \ldots, z_n) - \lim_{\substack{z_j \to 1 \\ |z_j| > 1}} h_1(z_1, \ldots, z_n) \mod Z$$

(*if this limit exists*).

For the general R^n action, R^n acts via a torus T^N in which it is dense. Let R^N be the universal cover of T^N. We have $R^n \to R^N$ by

$$(t_1, \ldots, t_n) \to (\ldots, a_{j1}t_1 + \cdots + a_{jn}t_n, \ldots).$$

Approximate the points $(a_{j1}, \ldots, a_{jn})$ by points P_j in R^N with rational coordinates in such a way that D remains transversely elliptic to the action of R^n determined by the P_j. Multiply by a lcm to obtain points Q_j in R^n with integer coordinates. Call the action of R^n determined by the Q_j the action *associated* to that determined by the P_j. We can find a basis $\{Y_1, \ldots, Y_N\}$ of $Z^N \subset R^N$ such that $Q_j = k_j Y_j$. Relative to this basis we have

$$R^N = R^n \times R^{N-n}, \quad Z^N = Z^n \times Z^{N-n}, \quad T^N = T^n \times T^{N-n}$$

and the associated action of R^n is given by

$$R^n \xrightarrow{\pi} T^n \xrightarrow{\text{a covering}} T^n \subset T^N$$

where π is the universal covering. Thus, for the associated action $\text{index}_{R^n}(D)$ is determined by $\text{index}_{T^n}(D)$. Thus:

DEFINITION.

$$\text{Virtindex}_{R^n}(D) = \lim \text{Virtindex}_{T^n}(D) \mod Z$$

if this limit exists when taken over nearby associated actions.

We observe that if $R \subset R^n$ and if D is transversely elliptic to the R action then if $\text{Virtindex}_{R^n}(D)$ exists, so does $\text{Virtindex}_R(D)$ and they are equal, since $\text{index}_{R^n}(D)$ restricts to $\text{index}_R(D)$. The converse is, of course, false. Virtindex_R may exist but not Virtindex_{R^n}.

Simple Example. Let R act on S^3 by

$$\theta \cdot (z_1, z_2) = (e^{i\lambda_1\theta}z_1, e^{i\lambda_2\theta}z_2), \ \lambda_1, \lambda_2 \text{ positive integers.}$$

Let $f = X_1 + X_2$. A calculation using [1] shows

$$\operatorname{index}(D^+ \otimes u_f) = (t^{-\lambda_1} - t^{\lambda_1}) \sum_{k \in Z} t^{k\lambda_2} + (t^{-\lambda_2} - t^{\lambda_2}) \sum_{k \in Z} t^{k\lambda_1}.$$

We can take

$$h_0 = \frac{(t^{-\lambda_1} - t^{\lambda_1})}{1 - t^{\lambda_2}} + \frac{(t^{-\lambda_2} - t^{\lambda_2})}{1 - t^{\lambda_1}}$$

$$h_1 = \frac{(t^{-\lambda_1} - t^{\lambda_1})t^{-\lambda_2}}{(1 - t^{-\lambda_2})} + \frac{(t^{-\lambda_2} - t^{\lambda_2})t^{-\lambda_1}}{(1 - t^{-\lambda_1})}$$

and a simple calculation shows

$$\lim_{t \to 1} h_0 - \lim_{t \to 1} h_1 = -4\frac{\lambda_1^2 + \lambda_2^2}{\lambda_1 \lambda_2}.$$

In general if R acts on S^{4k-1} by

$$\theta(z_1, \ldots, z_{2k}) = (e^{i\lambda_1\theta} z_1, \ldots, e^{i\lambda_{2k}} z_{2k})$$

then [8] shows that

$$\operatorname{Virtindex}(D^+ \otimes u_f) = \pm 2^{2k-1} \frac{f(\lambda_1^2, \ldots, \lambda_{2k}^2)}{\lambda_1 \ldots \lambda_{2k}} \mod Z.$$

3. Outline of proofs

Let R^n act by isometries locally freely on the compact oriented $4k - 1$ manifold M. The image of R^n in the isometry group of M is dense in a torus T^N. Let R^N be the universal cover of T^N. We can assume $R^n \subset R^N$. A point θ in R^n will be called *generic* if it projects to a generator of T^N. θ determines an action of R on M and this action will be called generic. From now on $G = T^N$.

THEOREM (3.1) (reduction). *Let $E \to Z$ be a complex G vector bundle of dimension $2k$ over the trivial G manifold Z. Assume that Z is connected and oriented and that for $z \in Z$, E_z does not contain the trivial representation of G, and that R acts generically on $S(E)$. If Theorem 1 is true for this action then it is true in general.*

This theorem will follow from a sequence of theorems about the index and about Simons classes. First by [10] some multiple of M by a power of 2 bounds an oriented G manifold W. If Theorem 1 is true for $2^s M$, it is true for M. Thus we can assume M bounds W. Let Γ be the fixed set of the G action on W and $\{\Gamma_i\}$ the connected components. Let $\{N_i\}$ be disjoint invariant closed tubular neighborhoods of $\{\Gamma_i\}$. Then the R action on W induces a locally free action on ∂N_i. Let $D_{\theta i}^+$ be the transverse signature operator to this action. Note that Γ_i is orientable and ∂N_i inherits an orientation from W.

The action of R on W determined by θ is given by a map to R^N,

$$s \to \left(e^{i\lambda_1 s}, \dots, e^{i\lambda_N s}\right), \tag{3.2}$$

and by a change of co-ordinates on R^N we can assume that, for $1 \leq \mu_j \leq \lambda_j$,

$$s \to \left(e^{i\mu_1 s}, \dots, e^{i\mu_N s}\right) \tag{3.3}$$

has no fixed points on M or ∂N_i.

LEMMA (3.4). *Let D^+ be the transverse signature operator relative to the standard action of R (all $\mu_i = 1$). The symbols of $D^+ \otimes u_f$ and $D_\theta^+ \otimes u_f$ in $K_G(T_G M)$ and of $D_i^+ \otimes u_f$ and $D_{\theta i}^+ \otimes u_f$ in $K_G(T_G \partial N_i)$ are equal.*

Note. u_f is always taken for the appropriate normal bundle.

Since the R index is the restriction of the G index we can replace D_θ^+ by D^+.

THEOREM (3.5). $\operatorname{index}_G(D^+ \otimes u_f) = \Sigma_i \operatorname{index}_G(D_i^+ \otimes u_f)$.

For the Simons classes, let F be the foliation on M and F_i on ∂N_i induced by the R action.

THEOREM (3.6). $S_f(F)[M] = \Sigma S_f(F_i)[\partial N_i] \bmod Z$.

Now, the normal bundle to Γ_i in W satisfies the hypotheses of (3.1), thus (3.4), (3.5), (3.6) and the definition of virtindex will yield (3.1). Finally we have:

THEOREM (3.7). For $E \to Z$ as in (3.1),

$$\operatorname{Virtindex}_R\left(D^+ \otimes u_f\right) = -2^{2k-1} S_f(F)[M] \bmod Z\left[\tfrac{1}{2}\right].$$

This will prove Theorem 1. Regarding Simon classes we further show:

THEOREM (3.8). *Let $\tilde{F}$ be the Riemannian foliation of M generated by the R^n action. Then $S_f(F) = S_f(\tilde{F}) \bmod Z$.*

As a consequence of our work on the Simons class and of the theorems of [4] and [3] we will have:

THEOREM (3.9). $S_f(F_i)[\partial N_i] = \operatorname{Residue}(f, \theta, \Gamma_i) \bmod Z$.

Thus

$$2^{2k-1}S_f(\tilde{F})[M] = 2^{2k-1}S_f(F)[M] = -\text{Virtindex}_R(D_\theta^+ \otimes u_f) \mod Z[\tfrac{1}{2}]$$

and this will establish (2a) and (4a).

THEOREM (3.10). *Let R^n act locally freely by isometries on M compact and oriented, and assume $n > 2$. Let P be transversely elliptic to the orbits of a generic R action. Then* $\text{index}_G(P) = 0$.

THEOREM (3.11). *Let D_θ^+, D_n^+, D^+ be the transverse signature operators to the R, R^n, G actions respectively. Their symbols are the same in $K_G(T_G M)$.*

Since the R^n and R index can be obtained from the R^N index by restriction, and the G index determines the R^N index we have (2b) and (4b).

Finally for a closed oriented $4k$ manifold M, $f[M]$, the Pontryagin number (an integer) equals the sum of the residues around the Γ_i and so Corollary 5 follows from

$$2^{2k-1}S_f(F_i)[\partial N_i] = 2^{2k-1}\text{Res}(f, \theta, \Gamma_i) = -\text{Virtindex}(D_i^+ \otimes u_f) \mod Z.$$

In §4 we present proofs of (3.1), (3.4), (3.5), in §5 of (3.6), (3.8), (3.9), and in §6 of (3.7), (3.10), (3.11).

4. Proofs of theorems about index

We have $R \subset R^n \subset R^N$ and R^N acts on W. First we can change coordinates on R^N so that all $\lambda_j > 0$. Let $(r_1, \ldots, r_N)$ be rational numbers which parameterize a nearby action such that for $\mu_j = tr_j + (1-t)\lambda_j$, the action given by (3.3) has no fixed points on M or ∂N_i for $0 \le t \le 1$. Let m be the lcm of the denominators of the r_j and $k_j = mr_j$. Then $h: R^N \to R^N$ given by

$$h(x_1, \ldots, x_N) = (k_1 x_1, \ldots, k_N x_N)$$

induces a map

$$R^N \xrightarrow{h} R^N \to \text{isometries}(W).$$

Let $R \to R^N$ be given by $s \to (\lambda_1 s/k_1, \ldots, \lambda_N s/k_N)$. With these new coordinates the desired condition is satisfied.

Proof of (3.4). Let $\lambda_j(t) = (1-t)\lambda_j + t$ and $\lambda(t) = (\lambda_1(t), \ldots, \lambda_N(t))$. The $\lambda(t)$'s determine a one parameter family of actions with no fixed points and hence a smooth family of Riemannian foliations. Let $\nu_t \subset TM$ be the normal bundle to the t^{th} action. Let π_t be orthogonal projection on ν_t. $\lambda(t)$ positive implies $\pi_t : \nu_1 \to \nu_t$ is an isomorphism. Let D_t^+ be the transverse signature operator relative to the t^{th} action and let $\pi_t(D_t^+) = \pi_t^{-1} D_t^+ \pi_t$. Direct computation shows $\sigma(D_1^+)_v = \sigma(\pi_t(D_t^+))_v$ for $v \in T_G^*M$ and from this it follows that their symbol classes in $K_G(T_G^*M)$ or $K_G(T_G SE_i)$ are the same. Thus $\sigma(D_1^+)$ and $\sigma(\pi_0(D_0^+))$ and $\sigma(D_0^+)$ represent the same element. If u is a virtual representation of $SO(4k)$, $u(\nu_0^*)$ and $v(\nu_1^*)$ are isomorphic vector bundles. Tensoring the above symbol classes with such an isomorphism yields the result.

Proof of (3.5). Let Y be a $4k-1$ oriented Riemannian manifold on which G acts and on which the diagonal S^1 acts without fixed points. Let $X = Y \times R$ with the trivial G action on R and the product metric, and $i : Y \to X$ given by $i(y) = (y, 0)$. Let D^+ be the transverse signature operator to the S^1 action on Y and let η be the ordinary signature symbol on X. Referring to [1, p. 44–45] we have the trivializations $\pm$ and $\eta^\pm$ will be taken on X relative to the S^1 action. η^+ and η^- are elements in $K_G(T_{S^1}X)$ and hence in $K_G(T_G X)$. Precisely, $f : T_{S^1}(X) \to TX$,

$$f(x, v) = \left(x, v + \rho(x) g(|v|) A_x^\pm\right)$$

where ρ is zero on Y and positive otherwise. If σ is the symbol of η on TX, $f^*(\sigma)$ represents $\eta^\pm$ in $K_G(T_{S^1}X)$. Now,

$$T_G(Y \times R) = T_G(Y) \times \mathbf{C}$$

hence we have a Thom isomorphism

$$i_* : K_G(T_G Y) \to K_G(T_G X)$$

THEOREM (4.1). $2 i_*(\sigma(D^+)) = \eta^+ - \eta^-$.

COROLLARY (4.2). $2 i_*(\sigma(D^+ \otimes u_f)) = \eta^+ \otimes u_f(TX) - \eta^- \otimes u_f(TX)$.

Proof. Theorem (4.1) is stated in [1, p. 54] and so we will merely give a sketch of the proof. The symbol of D^+ pulled up to $T_G(Y) \times \mathbf{C}$ is represented by

$$\alpha : \pi^* \Lambda^+(\nu) \to \pi^* \Lambda^-(\nu),$$
$$\alpha(v, x + iy)(e) = (v, x + iy, v \wedge e + v \lrcorner e)$$

where

$$v \lrcorner (e_1 \wedge \cdots \wedge e_n) = \sum (-1)^j (v, e_j) e_1 \wedge \cdots \overset{j}{\bigwedge} \cdots \wedge e_n.$$

ν of course is the normal bundle to the S^1 action. The Thom class is

$$\beta: T_G(Y) \times \mathbf{C}^2 \to T_G(Y) \times \mathbf{C}^2 \quad \text{where } \beta(v, z_1, z_2) = (v, z_1, z_1 z_2).$$

Then $i_*(\sigma(D^+))$ is represented by

$$(4.3)\quad \pi^*\Lambda^+(\nu) \otimes \underline{\mathbf{C}} + \pi^*\Lambda^-(\nu) \otimes \underline{\mathbf{C}} \to \pi^*\Lambda^+(\nu) \otimes \underline{\mathbf{C}} + \pi^*\Lambda^-(\nu) \otimes \underline{\mathbf{C}}$$

where $\underline{\mathbf{C}} = T_G(Y) \times \mathbf{C}^2$ and the map is

$$\begin{pmatrix} \alpha(v) \otimes 1 & -1 \otimes z \\ 1 \otimes \bar{z} & \alpha(v) \otimes 1 \end{pmatrix}$$

at (v, z).

To relate this to η on X we write

$$T(Y \times R) = \nu \oplus \left\{ \frac{\partial}{\partial u}, A \right\}$$

where A is the unit tangent to the S^1 orbits and u is the R coordinate. Let $V = \{\partial/\partial u, A\}$ with $\{du, a\}$ the dual basis. The metric on $T(Y \times R)$ is the product of a metric on V in which $\{\partial/\partial u, A\}$ is an orthonormal basis and the metric on ν. Thus

$$\Lambda^+(TX) = \Lambda^+(\nu) \otimes \Lambda^+(V) + \Lambda^-(\nu) \otimes \Lambda^-(V)$$

and similarly $\Lambda^-(TX) = \Lambda^+ \otimes \Lambda^- + \Lambda^- \otimes \Lambda^+$. We can take $1 + i\,du \wedge a$ and $du + ia$ as a basis for $\Lambda^+(V)$ and $1 - i\,du \wedge a$ and $du - ia$ for $\Lambda^-(V)$.

Now, $\eta^+ - \eta^-$ on $T_G(Y) \times \mathbf{C}$ can be represented, similar to [1, p. 63], as the pullback of the signature symbol on X by the map

$$T_G(X) = T_G(Y) \times \mathbf{C} \to T(Y) \times \mathbf{C} = T(X)$$

given by $(v, x + iy) \to (v, xa + y\,du)$. Thus $\eta^+ - \eta^-$ is represented by

$$\pi^*\Lambda^+(X) \to \pi^*\Lambda^-(X).$$

with the map given by

$$(v, x + iy, e) \to (v, x + iy, (xa + y\,du) \wedge e + (xa + y\,du) \lrcorner e).$$

Then direct computation shows that this complex splits as a sum of (4.3) and another complex with the same bundles as (4.3) but with map

$$\begin{pmatrix} \alpha(v) \otimes 1 & 1 \otimes z \\ 1 \otimes -\bar{z} & \alpha(v) \otimes 1 \end{pmatrix}.$$

However, a simple argument shows this is equivalent to (4.3). To add coefficients we just notice that if Q and P are the principal bundles of ν and TX on X, ρ a representation of $SO(4k)$, and ρ_0 its restriction to $SO(4k-2)$, then the vector bundles $\rho(P)$ and $\rho_0(Q)$ are equal.

Now we relate $\eta^{\pm}\otimes u_f$ on M and ∂N_i. Let us suppress the coefficients for clarity. We let $V = W - \cup\,\mathrm{int}(N_i)$. Let U_0 and U_1 be disjoint invariant neighborhoods of M and ∂N ($N = \cup N_i$). We can "trivialize" η along the infinitesmal generator A^+ or A^- of the S^1 action on U_0 and U_1 to get $\eta_0^{\pm}$ in $K_G(T_H U_0)$, $\eta_1^{\pm}$ in $K_G(T_H U_1)$ where $H = S^1$. To do this, say, for U_0 let $h: T_H U_0 \to TU_0$ be defined as in [1, p. 45] by

$$h(x, v) = (x, v + \rho(x)g(|v|)A_x^+) \tag{4.4}$$

where ρ is zero only on M. If σ represents the symbol of η, then $h^*(\sigma)$ represents η_0^+. Similarly for η_0^-, η_1^+, η_1^-. We can also define h on all of v by taking ρ to be zero on $M \cup \partial N$. This gives $h^*(\sigma)$ representing $\eta^{\pm}$ in $K_G(T_H W|V)$. Let $U = U_0 \cup U_1$. We can apply excision [1, (3.7)]

$$E: K_G(T_H U) \to K_G(T_H W|V)$$

and since $h^*(\sigma)$ is an isomorphism outside $TW|M \cup \partial N$, the construction of E allows to conclude $E(\eta_1^+ - \eta_0^+) = \eta^+$ and similarly for η^-. Since σ is elliptic, it defines an element in $K_G(TW|V)$ and a simple K theory argument shows that σ, η^+, η^- all represent the same element. Thus

$$E(\eta_0^+ - \eta_0^-) = E(\eta_1^+ - \eta_1^-).$$

Since index commutes with excision we have

$$\mathrm{Index}_G(\eta_0^+ - \eta_0^-) = \mathrm{Index}_G(\eta_1^+ - \eta_1^-)$$

Now from (4.1), (4.2), and explicating coefficients

$$2(\eta_0^+ - \eta_0^-) \otimes u_f(TX) = i_*(D^+ \otimes u_f),$$
$$2(\eta_1^+ - \eta_1^-) \otimes u_f(TX) = \sum i_*(D_j^+ \otimes u_f).$$

Thus $0 = i_*(D^+\otimes u_f) - \Sigma i_*(D_j^+ \otimes u_f)$. Finally apply index_G which commutes with the Thom isomorphism to get Theorem (3.5).

Now let $E \to Z$ be a complex $2k$ G bundle over the trivial oriented G manifold Z as in the statement of (3.1). We can assume the diagonal S^1 acts without fixed points on $E - Z$. $E = E_1 \otimes \rho_1 + \cdots + E_r \otimes \rho_r$ where E_j is a complex vector bundle, and ρ_j is an irreducible representation of G. The weights of the ρ_j determine a map $\phi: R^N \to R^r$ and by a change of coordinates on R^r with jacobian one we can assume that the diagonal R of R^N is carried into that of R^r. A simple argument using restriction and change of variables in a multiple integral shows that we can obtain the T^N or R^N index of an operator transversely elliptic to the diagonal from the T^r index by ϕ^{-1}, and so we might as well assume $N = r$ and $\rho_j = t_j$ where $t_j(\theta_1, \dots, \theta_N) = e^{i\theta_j}$.

Consider the signature operator of E (as a manifold) with coefficients $u_f(TE)$. E is not compact, so there is no index theorem, but we can consider $L(E)$, the right hand side of the G signature theorem applied to this operator, which is a rational function in characters of T^N with, as we will see, denominator a power of $(1 - t_1) \cdots (1 - t_N)$.

THEOREM (4.5). *Let D^+ be the transverse signature operator to the diagonal S^1 action. Then* $-2\,\mathrm{index}_G(D^+ \otimes u_f) = J(L(E))$.

COROLLARY (4.5a). $-2\,\mathrm{Virtindex}_R(D^+ \otimes u_f) = L(E)_{t=1}$.

Corollary (4.5a) follows from (2.3). We will obtain a further reduction. Let V be the canonical line bundle of E. Form $B(E) \cup (-B(V))$ along $S(V) = S(E)$. The right hand side of the G signature theorem with coefficients in u_f for this manifold is $L(E) - L(V)$. But since $B(E) \cup (-B(V))$ is a closed manifold, this is a character. Hence $J(L(E)) = J(L(V))$.

Thus to prove (4.5) it is enough to show:

THEOREM (4.6). *Let V be the canonical line bundle of E. Then*

$$-2\,\mathrm{index}_G\big(D^+ \otimes u_f\big) = J\big(L(V)\big).$$

Proof. First assume no coefficients u_f. The general case is just a technical modification. Let $x_1, \dots$ be such that the Pontryagin classes of Z are symmetric functions of the x^2 and $y_{jk}, \dots$ such that the Chern classes of E_j are symmetric functions of the y's. The fixed set of the G action on $CP(E)$ is $\cup\, CP(E_i)$. Directly from the G signature theorem we get

$$L(V) = \sum_i - C_i\left(\frac{1 + e^{y_{i1}}t_i}{1 - e^{y_{i1}}t_i}\right)[CPE_i]. \tag{4.7}$$

where γ_i is the class

$$\prod \frac{e^{-x} - e^{x}}{e^{-x} + e^{x} - 2}$$

of $TCPE_i$ and

$$C_i = \gamma_i \prod_{\substack{k \neq i \\ j}} - \left(\frac{t_i + e^{y_{kj}}t_k}{t_i - e^{y_{kj}}t_k}\right).$$

To compute $\text{index}_G(D^+)$, we consider $\pi: S(V) \to CP(E)$. D^+ is the pull-back of the ordinary signature operator of $CP(E)$, the normal bundle to the diagonal action of S^1 is the pullback of $TCP(E)$ and we can apply (3.3) of [1] with $H = S^1$, $a = D^+$ on $CP(E)$. Apply the G signature theorem to each index_G and we get

$$\text{Index}_G(D^+) = \sum_{\alpha \in Z} \sum_i C_i e^{-\alpha y_{i1}} t_i^{-\alpha} [CPE_i]. \tag{4.8}$$

Recall from [1, p. 39],

$$\left(\frac{1}{1-t}\right)^+ = \sum_{\alpha \geq 0} t^\alpha, \qquad \left(\frac{1}{1-t}\right)^- = \sum_{\alpha < 0} t^\alpha.$$

If it makes sense to replace t by $e^{y_{i1}} t_i$ in those formulas and if C_i were unchanged by $\pm$ we could apply J to (4.7) and use

$$e^y t \sum_{\alpha \in Z} (te^y)^\alpha = \sum_{\alpha \in Z} (te^y)^\alpha$$

to obtain $-2J(L(V)) = \text{index}(D^+)$. We now justify this.

First, a simple induction argument shows

$$(1 - e^y t)^{-1} = \frac{p_n(t, y)}{(1-t)^{n+1}}, \tag{4.8}$$

$$(t_i - e^y t_k)^{-1} = \frac{r_n(t_i, t_k, y)}{(1-t)^{n+1}}$$

where p_n is a polynomial in t and y, r_n is a polynomial homogeneous in t_i and t_k. These formulas hold in $H^*(CP(n)) \otimes R(G)_S$ where S is the multiplicative set $\{(1 - t_i^{\pm 1})^k\}$ and $R(G)_S$ is all quotients by elements of S. Thus those elements are in the domain of $1 \otimes \pm$. Since $+$ is an $R(G)$ module map,

$$(1 - e^y t)\left[(1 - e^y t)^{-1}\right]^+ = 1$$

and also

$$(1 - e^y t) \sum_{\alpha \geq 0} e^{\alpha y} t^\alpha = 1.$$

Similar considerations hold for $-$, hence

$$\left[(1 - e^y t)^{-1}\right]^+ = \sum_{\alpha \geq 0} e^{\alpha y} t^\alpha, \quad \left[(1 - e^y t)^{-1}\right]^- = \sum_{\alpha < 0} e^{\alpha y} t^\alpha. \tag{4.9}$$

Also, $L(E) - L(V)$ is the signature of $B(E) \cup (-B(V))$ and then applying the right hand side of the G signature theorem to E, we have

$$L(V) = \frac{P}{\lambda^{n+1}} \tag{4.10}$$

where $n \gg N$, $\lambda = \prod_{j=1}^{N}(1 - t_j)$ and P is a polynomial in the t's. Let

$$Q_i = \prod_{k \neq i} (t_i - t_k)^{n+1}, \quad Q = \prod Q_i,$$

$$f_i = (1 - t_i)^{n+1}(1 - e^{y_{i1}} t_i)^{-1} \quad q_i = Q_i C_i(1 + e^{y_{i1}} t_i).$$

From (4.8) f_i, q_i and Q_i/Q are all polynomials. From (4.7), (4.10),

$$L(V) = -\sum \frac{Q}{Q_i} q_i f_i [CPE_i] \frac{1}{(1 - t_i)^{n+1}} = \frac{QP}{\lambda}.$$

Now we can apply $\pm$ to obtain

$$L(V)^{\pm} = \sum \frac{Q}{Q_i} q_i f_i [CPE_i] \frac{1}{\left[(1 - t_i)^{n+1}\right]^{\pm}} = \frac{QP}{\lambda^{\pm}}. \tag{4.11}$$

Now replace q_i, f_i by their expressions in terms of the t's and y's to get the result.

The general case, where there are coefficients u_f, goes the same way. In (4.7) we get, for each i, a term $u_{f,i}$ and in (4.8) a term $u'_{f,i}$. Then the proof proceeds exactly as for no coefficients, but in the last step we use

$$(1 - e^{y} t) \sum_{\alpha \in Z} e^{\alpha y} t^{\alpha} = 0$$

to obtain the result.

5. Simons classes

Let ∇ be a connection and ϕ an ad-invariant polynomial function (of degree $2k$ in our discussion) on matrices. The Simons class $S_{\phi}(\nabla)$ is an R/Z cochain which represents a cohomology class when the differential form $\phi(\nabla) \equiv 0$. $\phi(\nabla)$ is ϕ applied to the curvature matrix of ∇. The principal reference is [14].

We also recall some other notions about connections. Given an infinitesmal isometry X we call an invariant connection on the tangent bundle an X-connection if outside a neighborhood of the singular set of X, ∇ is given by

$\nabla_A B = \omega(A)[X, B] + D_{\pi A} B$, where D is the Riemannian connection, π is projection on the complement to X, and $\omega(A) = (A, X)/(X, X)$. Such connections are easy to construct. We have the λ construction from [5, p. 64]. Given two connections ∇^0, ∇^1 and an invariant polynomial ϕ, $\lambda(\nabla^1, \nabla^0)(\phi)$ is a differential form whose exterior derivative is $\phi(\nabla^1) - \phi(\nabla^0)$.

As in §§3 and 4, $M = \partial W$ with the product metric, F is the foliation on M induced by the given R action, $\tilde{F}$ by the R^n action. ∇^L is the unique Riemannian torsion free connection on the normal bundle ν to F [12], X is the infinitesimal generator of the R action, ∇^{fl} is the connection on (X) which is flat relative to $X/|X|$, and $\tilde{\nabla}$ an X-connection on TM. Then $\tilde{\nabla}$ and $\nabla^{fl} \oplus \nabla^L$ are connections on TM.

LEMMA (5.1). *$\lambda(\tilde{\nabla}, \nabla^{fl} \oplus \nabla^L)(\phi) \equiv 0$ if ϕ has degree $2k$.*

Proof. Choose local co-ordinates

$$\{x, y_1, \ldots, y_q\} = U$$

where $q = 4k - 2$ with $X = \partial/\partial x$. Let us consider the local framing

$$\left\{X/|X|, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_q}\right\}$$

and $f : U \to R^q$, $q = 4k - 2$ given by $f(x, y) = y$. Let θ and θ_0 be the local connection matrices of $\tilde{\nabla}$ and $\nabla^{fl} \oplus \nabla^L$. We show that θ and θ_0 lie in $f^*\Omega(TR^{4k-2})$; then a direct computation shows

$$\lambda(\tilde{\nabla}, \nabla^{fl} \oplus \nabla^L)(\phi)$$

is a sum of terms $\phi(\sigma^{2i+1} \wedge (d\sigma + [\sigma, \theta_0])^j \wedge \Omega_0^l)$ where $\sigma = \theta - \theta_0$ is in $f^*\Omega^1(R^q)$, Ω_0 and $d\sigma + [\sigma, \theta_0]$ are in $f^*\Omega^2(R^q)$ and $i + j + l = 2k - 1$. Thus each summand is of degree at least $4k - 1$, hence zero.

To show θ is in the given ideal use the definitions of $\tilde{\nabla}$ and the invariance of this connection. Then a direct computation shows $i(X)$ and L_X annihilate θ. For θ_0, introduce p, the orthogonal projection on X. Then

$$(\nabla^{fl} \oplus \nabla^L)h = \left(p\nabla^{fl}ph, \pi\nabla^L\pi h\right).$$

A direct calculation together with the facts that $\pi\tilde{\nabla} = \nabla^L$ on ν and that ∇^L is locally pulled back, via f, from R^q will yield the result. The computations are similar to those in [9].

LEMMA (5.2). *$S_\phi(\nabla^L) = S_\phi(\nabla^{fl} \oplus \nabla^L)$.*

Proof. This is (1.2) of [9].

LEMMA (5.3). $S_\phi(\tilde{\nabla}) = S_\phi(\nabla^{fl} \oplus \nabla^L)$.

Proof. According to [14, p. 31], $S_\phi(\tilde{\nabla}) - S_\phi(\nabla^{fl} \oplus \nabla^L)$ as a function on $4k-1$ cycles is given by integrating

$$2k\int_0^1 \phi\left(\sigma_t' \wedge \Omega_t^{2k-1}\right) dt$$

where $\sigma_t' = \theta - \theta_0$, $\Omega_t = t^2\sigma^2 + t(d\sigma + [\sigma, \theta_0]) + \Omega_0$. It is then easily seen that this $4k-1$ form is just $\lambda(\tilde{\nabla}, \nabla^{fl} \oplus \nabla^L)(\phi)$ which is zero.

Now let ∇ be an X connection on W. ∇ is also a connection on TW restricted to M.

LEMMA (5.4). $S_\phi(\nabla) = S_\phi(\tilde{\nabla})$ *in* $H^{4k-1}(M; R/Z)$.

Proof. Let D and $\tilde{D}$ be the Riemannian connections on W and M. $\nabla = \omega \otimes L_X + D_\pi$. On a collar of ∂W the metric is a product and $D = \tilde{D} + D^{fl}$ where D^{fl} is flat relative to $\partial/\partial u$. Then $\nabla = \tilde{\nabla} \oplus D^{fl}$ and so the lemma follows from (1.2) of [9].

Thus we have:

PROPOSITION (5.5). $S_\phi(F) = S_\phi(\nabla)$.

Proof. $S_\phi(F)$ is by definition $S_\phi(\nabla^L)$. Now apply (5.1)–(5.4).

THEOREM (5.6). $S_\phi(F)[M] = \int_W \phi(\nabla) \bmod Z$.

Proof. By (3.19) of [14],

$$\begin{aligned} S_\phi(\nabla)[M] - S_\phi(D)[M] &= \int_M \lambda(\nabla, D)(\phi) \\ &= \int_W d\lambda(\nabla, D)(\phi) \\ &= \int_W \phi(\nabla) - \int_W \phi(D). \end{aligned}$$

By Theorem (5.15) of [14] we have $S_\phi(\tilde{D})[M] = \int_W \phi(D)$. Since $D = \tilde{D} \oplus D^{fl}$, $S_\phi(\tilde{D}) = S_\phi(D)$. Thus $S_\phi(F)[M] = S_\phi(\nabla)[M] = \int_W \phi(\nabla)$.

Proof of (3.6). ∇ is an X-connection on W and we can construct it so $\nabla = \omega \otimes L_X + D_\pi$ outside $\cup N_i$. Then outside $\cup N_i$ the proof of (5.1) shows

$\phi(\nabla) \equiv 0$ so by (5.5),

$$S_\phi(F)[M] = \sum \int_{N_i} \phi(\nabla) = \sum S_\phi(F_i)[N_i].$$

Proof of (3.8). Again X is the generator of the R action on M. Take $X = X_1, X_2, \ldots, X_n$ to be commuting vector fields which, at each point, generate $\tilde{F}$. Let ν and $\tilde{\nu}$ be the normal bundles to F and $\tilde{F}$ and ∇ and $\tilde{\nabla}$ the Riemannian torsion free connections on ν and $\tilde{\nu}$. We have

$$\nu = \tilde{\nu} \oplus \{X_2, \ldots, X_n\}$$

and we let ∇^{fl} be flat relative to $X_2, \ldots, X_n$. Then by (5.2)–(5.5),

$$S_\phi(F) = S_\phi(\nabla), \quad S_\phi(\tilde{F}) = S_\phi(\tilde{\nabla}), \quad S_\phi(\tilde{\nabla} \oplus \nabla^{fl}) = S_\phi(\tilde{\nabla})$$

and

$$S_\phi(\nabla) - S_\phi(\tilde{\nabla}) = \int_M \lambda(\nabla, \tilde{\nabla} \oplus \nabla^{fl})(\phi)$$

Thus it will be sufficient to show $\lambda \equiv 0$.

We choose local co-ordinates $U = \{x_1, \ldots, x_n, y_1, \ldots, y_q\}$ such that $X_j = \partial/\partial x_j$ and $y_j =$ constant describe the local leaves of $\tilde{F}$. Then $x_2 = \cdots = x_n = y_1 = \cdots = y_q =$ constant describe those of F. Let $f: U \to R^q$ and $g: U \to R^{4k-2}$ be given by

$$f(x, y) = y, \quad g(x, y) = (x_2, \ldots, x_n, y_1, \ldots, y_q).$$

Let $\tilde{\omega}$ be the local connection matrix of $\tilde{\nabla}$ relative to

$$\left\{\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_q}\right\}$$

and ω of ∇ relative to

$$\left\{\frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_q}\right\}.$$

Then

$$\tilde{\omega} \in f^*\Lambda(dy_1, \ldots, dy_q), \quad \omega \in g^*\Lambda(dx_2, \ldots, dx_n, dy_1, \ldots, dy_q).$$

Let $I = g^*\Lambda(dx_2,\ldots, dy_q)$. $\tilde{\omega}$ and ω are in I. We have

$$\lambda(\nabla,\tilde{\nabla} \oplus \nabla^{fl})(\phi) = (2k)\int_0^1 \phi\left(\sigma \wedge \Omega_t^{2k-1}\right) dt$$

where $\sigma = \tilde{\omega} - \omega$, $\Omega_t = t^2\omega^2 + t(d\sigma + [\sigma, \omega]) + d\omega + \omega^2$. Since σ, ω are in I, Ω_t is in I^2 and so $\phi(\sigma \wedge \Omega_t^{2k-1}) \in I^{4k-1} \equiv 0$.

The residue of an infinitesmal isometry along a component of the singular set can be defined by the right hand side of (5.6) in [3], or for isolated singularities by the left hand side of Theorem 2 of [4]. Then from (5.5) and (5.6) of [3] and our (5.5), (5.6) we have:

THEOREM (5.7). $S_\phi(F)[M] = \Sigma\,\mathrm{Res}(\phi, \theta, \Gamma_i)$.

Note. In the statement we have used the notation $\mathrm{Res}(\phi, \theta, \Gamma_i)$ instead of $\mathrm{Res}(f, \theta, \Gamma_i)$. They mean the same thing. The polynomial f determines ϕ as in §1.

6. Remaining proofs

Proof of (3.7). From (5.6) and the proof of (3.6),

$$S_\phi(F)[M] = \sum \int_{N_i} \phi(\nabla).$$

Now $\int_{N_i}\phi(\nabla)$ is given by a local expression near Γ_i by (5.6) and (2.2)–(2.3) of [3]. From [2, p. 598 (top line)] and (8.9) it follows that this local expression equals $(1/2^{2k})L_{t=1}$. That is the local expressions of [3] and [2] are the same by examination of the terms of degree equal to the dimension of Γ_i. Then from (4.6),

$$-2\,\mathrm{Virtindex}_R\left(D^+\otimes u_f\right) = L_{t=1} = 2^{2k}S_\phi(F)[M].$$

Proof of (3.10). Since $n > 2$, we can assume n is odd. As in the proof of (3.8) let $X = X_1, X_2,\ldots, X_n$ be commuting vector fields arising from the R^n action on M, X the generator of the R action. Let V be the trivial bundle on M generated by $X_2,\ldots, X_n$. Since n is odd, V is complex. Let $T_R(M)$, $T_{R^n}(M)$ be the normal bundle to the R, R^n action respectively (notation consistent with [1]). Then $T_R(M) = T_{R^n}(M) \oplus V$. Let j be the composite

$$T_G(M) \xrightarrow{g} T_{R^n}(M) \xrightarrow{i} T_{R^n}(M) \oplus V = T_R(M).$$

The symbol $\sigma(P)$ lies in $K_G(T_R M)$ and, by the restriction principle,

$$\text{index}_R(P) = \text{index}_G(j^*\sigma(P)).$$

By the Thom isomorphism for V, $\sigma(P) = i^*(u)$ for some u. Then $j^*\sigma(P) = g^*i^*i_*(u) = g^*(u \otimes \Lambda_{-1}(V))$. Here $u \otimes \Lambda_{-1}(V)$ makes sense since V is a trivial bundle and $\otimes$ is the action of $R(G)$. But V is also a trivial G bundle, hence $\Lambda_{-1}(V) = 0$.

Proof of (3.11). For any of $R, R^n, G = H$, the symbol of the transverse signature operator to H orbits is given by

$$\sigma_v(e) = \pi(v) \wedge e - \pi(v) \lrcorner e$$

where $\pi : TM \to T_H M$ is orthogonal projection at each point of M. Thus the symbols of D_θ^+, D_n^+, D^+ all agree on $T_G E$.

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