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ODD PRIMARY bo-RESOLUTIONS AND K-THEORY LOCALIZATION

BY

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1. Introduction

In this paper we adapt to odd primes p Mahowald's theory of bo-resolutions [17] and apply it to calculate the homotopy groups of some nonconnected spectra which are obtained as direct limits of some spectra constructed from $B\Sigma_p$. The calculation of their homotopy groups is the main step in proving that these spectra are, in fact, K-theory localizations of certain Moore spectra, analogous to the situation for real projective spaces established in [8].

Throughout this paper, p is a fixed odd prime and $q = 2p - 2$. The symbols $\mathbf{Z}_p$ and $\mathbf{Z}/p$ are used interchangeably, A is the mod p Steenrod algebra, and $H^*(X) = H^*(X; \mathbf{Z}_p)$. We let bu (resp. bo) denote the spectrum for connective complex (resp. real) K-theory, localized at p. Adams [2] obtained a splitting

$$\mathrm{bu} = \bigvee_{i=0}^{p-2} \Sigma^{2i} l,$$

from which the splitting

$$\mathrm{bo} = \bigvee_{i=0}^{(p-3)/2} \Sigma^{4i} l$$

is easily derived. This l is often called $BP\langle 1\rangle$ [9]. In Section 3 we utilize Kane's splitting of $l \wedge l$ [13] to show that all the theorems of bo-resolutions can be adapted to l. In fact, the situation is simpler here, and the reader who has had difficulty with [17] and [6] may find this paper more understandable.

Let $B\Sigma_p$ denote the classifying space for the symmetric group of p letters localized at p [3]. Then

$$H^i(B\Sigma_p) \approx \begin{cases} \mathbf{Z}_p, & i \equiv -1, 0 \bmod q, \quad i \geq 0, \\ 0, & \text{otherwise} \end{cases}$$

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by [21]. Let $\beta' : \Sigma_p \to U(p-1)$ be the representation obtained by restricting the permutation action on $\mathbf{C}^p$ to the hyperplane $z_1 + \cdots + z_p = 0$. Let β_k denote the bundle

$$B\Sigma_p^{(kq)} \xrightarrow{B\beta'} BU(p-1)$$

on the (kq)-skeleton of $B\Sigma_p$. For any integers $b < t$, t possibly infinite, let $B_b^t = T((b-1)\beta_{t-b+1})/S^{(b-1)q}$, where $T(\ \ )$ denotes the Thom spectrum.

These spectra are (except when $b < 0$ and $t = \infty$) stunted $B\Sigma_p$'s by the following result.

PROPOSITION 1.1. (i) *If $b \geqq 0$, then B_b^t is isomorphic to the suspension spectrum of $B\Sigma_p^{(tq)}/B\Sigma_p^{((b-i)q)}$.*

(ii) *If $N \equiv 0(p^{t-b+1})$ then $\Sigma^{qN}B_b^t \simeq B_{b+N}^{t+N}$.*

This is proved in Section 2, along with the following result, which is used in forming our desired spectra $\overline{B}_b^t$.

PROPOSITION 1.2. *If $t \in \mathbf{Z} \cup \{\infty\}$ and $b < t$, then there are maps g_b^t trivial in $H^*(\ \ ;\mathbf{Z}_p)$ and nontrivial on the bottom cell, so that the diagrams*

$$\begin{array}{ccc} B_b^t & \xrightarrow{g_b^t} & B_{b-1}^{t-1} \\ \downarrow & & \downarrow \\ B_b^\infty & \xrightarrow[g_b^\infty]{} & B_{b-1}^\infty \end{array}$$

commute. The maps g_b^t induce isomorphisms in $K_*(\ \ )$ and $K^*(\ \ )$. The composite

$$B_{b-1}^{t-1} \xrightarrow{i \circ c} B_b^t \xrightarrow{g_b^t} B_{b-1}^{t-1}$$

is $.p$.

DEFINITION 1.3. $\overline{B}_b^t$ is the mapping telescope of

$$B_b^t \xrightarrow{g_b^t} B_{b-1}^{t-1} \xrightarrow{g_{b-1}^{t-1}} B_{b-2}^{t-2} \to \cdots.$$

We often write B_b for B_b^∞, and $\overline{B}_b$ for $\overline{B}_b^\infty$.

Let J denote the fibre of a map

$$l \xrightarrow{\theta} \Sigma^q l$$

which is nontrivial in $H^q(\ ;\mathbf{Z}_p)$; a lifting of an Adams operation $\Psi^r - 1$ is one way of constructing such a θ. Let $\nu(\ )$ denote the exponent of p

THEOREM 1.4.

$$J_i(\overline{B}_b) = \begin{cases} \mathbf{Z}/p^{\nu(i+1)+1}, & i \equiv -1 \bmod q,\ i \neq -1, \\ \mathbf{Z}/p^{\infty}, & i = -1, -2, \\ 0, & \textit{otherwise}, \end{cases}$$

$$J_i(\overline{B}_b^t) = \begin{cases} \mathbf{Z}/p^{m+1}, & i = a \cdot p^m q - \varepsilon,\ \varepsilon = 1 \text{ or } 2,\ \nu(a) = 0,\ m \leqq t - b, \\ \mathbf{Z}/p^{t-b+1}, & i = a \cdot p^m q - \varepsilon,\ \varepsilon = 1 \text{ or } 2,\ \nu(a) = 0,\ m \geqq t - b, \\ 0, & i \not\equiv -1, -2 \bmod q. \end{cases}$$

THEOREM 1.5. *If $p > 3$, the Hurewicz homomorphism*

$$\pi_i(\overline{B}_b^t) \xrightarrow{h} J_i(\overline{B}_b^t)$$

is an isomorphism.

The surjectivity of h in 1.5 is proved for all odd primes in Section 2; the injectivity for $p > 3$, which uses l-resolutions, is proved in Section 3. It seems very likely that h is 1.5 is injective also when $p = 3$, particularly since Miller's proof [18] of 1.5 when $t = b$ works when $p = 3$.

The following results are now easily derived, similarly to the case $p = 2$ in [8]. Let X_K denote the K_*-localization of a spectrum X [4]. For other notation, see Section 4, where these results are proved.

THEOREM 1.6. *$\overline{B}_b^t$ is K_*-local if $p > 3$.*

COROLLARY 1.7. *If $p > 3$, $\overline{B}_b^t = (B_b^t)_K$ and is independent of the choice of the maps $g_{b,t}$; $\overline{B}_b^t = \overline{B}_{b+n}^{t+n}$ for all integers n.*

THEOREM 1.8. *If $p > 3$, $\overline{B}_b^t = (S^{-1}/p^{t-b+1})_K$ for t finite or infinite.*

COROLLARY 1.9. *If $p > 3$, there are cofibrations*

$$S^{-1}Q \to \overline{B}_1 \to S_{K_{(p)}} \quad \textit{and} \quad S^{-2}Q \vee S^{-1}Q \to \overline{B}_1 \to \mathscr{J}_{(p)}.$$

The significance is that the spectrum $\overline{B}_1^{\infty}$, since it is constructed from geometric spaces $B\Sigma_p$ bearing no apparent relation to K, is in some sense a simpler model for K-theory localizations than Bousfield's, which is constructed from $\mathcal{J}$ and hence ultimately from K.

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2. The spectra $\overline{B}_b^t$ and their homotopy groups

In this section we prove 1.1, 1.2, 1.4, and the surjectivity part of 1.5.

Proof of 1.1. The restriction of β to $B\mathbf{Z}_p$ is the bundle α induced by the reduced regular representation. The bundle α is equivalent to the sum of the $p-1$ nontrivial complex line bundles over $B\mathbf{Z}_p$, which have equivalent sphere bundles, and hence the Thom spectrum $T(\alpha)$ is equivalent to the Thom spectrum of $(p-1)\lambda$ where λ is the canonical complex line bundle over $B\mathbf{Z}_p$ [20]. Thus for any integer k, there is a map

$$T\big(k(p-1)\lambda_{mq}\big) \to T(k\beta_m), \tag{2.1}$$

where λ_n denotes the restriction of λ to the n-skeleton of $B\mathbf{Z}_p$, which is the lens space L^n. We index our lens spaces and stunted lens spaces with real dimensions, rather than the complex dimensions used in [12]. Thus $L_b^n = B\mathbf{Z}_p^{(n)}/B\mathbf{Z}_p^{(b-1)}$.

We use the result of [12] that identifies Thom spectra of multiples of λ as stunted lens spaces, so that, after collapsing a skeleton, (2.1) gives a map

$$L_{(k+1)q-1}^{(k+m)q} \to B_{k+1}^{k+m}. \tag{2.2}$$

The transfer [10] gives a map of suspension spectra

$$B\Sigma_p \xrightarrow{t} B\mathbf{Z}_p,$$

injective in $H_*(\ ;\mathbf{Z}_p)$ since $|\Sigma_p:\mathbf{Z}_p|$ is prime to p. This t induces

$$B\Sigma_p^{((k+m)q)}/B\Sigma_p^{(kq)} \to L^{((k+m)q)}/L^{(kq)} \to L_{(k+1)q-1}^{(k+m)q},$$

which when followed by the map of (2.2) induces an isomorphism in $H_*(\ ;\mathbf{Z}_p)$ and hence an equivalence of p-localized spectra.

(ii) follows immediately from the fact that

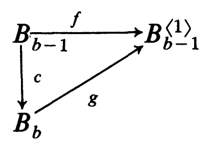
$$\widetilde{KU}^0\big(B\Sigma_p^{(t-b+1)q}\big) \approx \mathbf{Z}/p^{t-b+1},$$

which is easily proved by the Atiyah-Hirzebruch spectral sequence or the

Adams spectral sequence for $ku_0(D(B\Sigma_p^{((t-b+1)q)}))$, where $D(\ )$ denotes an S-dual. □

A map has $(H\mathbf{Z}_p)$-filtration $\geqq s$ if it can be written as a composite of s maps, each trivial in $H^*(\ ;\mathbf{Z}_p)$. If X is any spectrum, a map $S^m \to X$ has filtration s iff it is detected by an element of $\mathrm{Ext}_A^{s,s+m}(H^*X)$. We delete $\mathbf{Z}_p$ from second component of Ext(,). Let $X^{\langle s\rangle}$ denote the spectrum obtained from X by killing $\mathrm{Ext}^i(H^*X)$ for $i < s$.

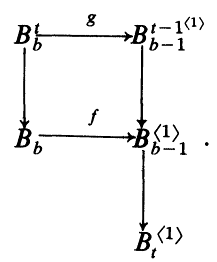
Proof of 1.2. (Entirely analogous to [8; 2.1].) There is a diagram



where f is a lifting of $\cdot p$, and g exists because

$$\dim(\mathrm{fibre}(c)) < \mathrm{conn}\left(B_{b-1}^{\langle 1\rangle}\right).$$

Similarly one obtains



□

We now work toward the proof of 1.4. $H^*(l) \approx A/\!/E$, where E is the exterior subalgebra generated by $Q_0 = \beta$ and $Q_1 = P^1\beta - \beta P^1$. Then $l_* = \pi_*(l)$ is calculated from a spectral sequence beginning with $\mathrm{Ext}_E(\mathbf{Z}_p)$, and is a polynomial algebra over $\mathbf{Z}_{(p)}$ on a generator α of degree q and Adams filtration 1. We have

$$H^i\left(B_b^t\right) = \begin{cases} \mathbf{Z}_p, & i = 0, -1\ (q),\ bq - 1 \le i \le tq, \\ 0, & \text{otherwise}, \end{cases}$$

with Q_0 and Q_1 nonzero wherever possible. $l_*(B_b^t)$ is calculated from $\mathrm{Ext}_E(H^*B_b^t)$ and its chart is shown in Fig. 1. In a chart such as this, column i is $\pi_i(\ )$ (in this case $l_i(\ ) = \pi_i(\ \wedge l)$), and vertical lines connecting dots correspond to multiplication by p. The following result reformulates the chart.

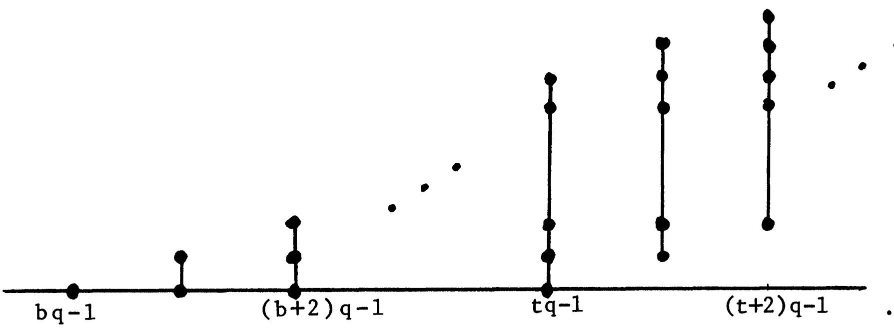


FIG. 1

PROPOSITION 2.3. *$l_*(B_b^t)$ is an l_*-module on generators $y_i \in l_{iq-1}(\ )$ for $b \le i \le t$ with relations py_b and $\alpha y_i = py_{i+1}$ for $b \le i < t$.*

The following result could be proved using an Adams-operation interpretation of θ, but we prefer the more homotopy-theoretic version given below.

PROPOSITION 2.4. *The composite*

$$l_{iq-1}(B_b^t) \xrightarrow{\theta_*} l_{iq-1}(\Sigma^q B_b^t) \xrightarrow{\alpha_*} l_{iq-1}(B_b^t)$$

is (up to a unit in $\mathbf{Z}_{(p)}$) multiplication by $p^{\nu(i)+1}$.

Proof. Define $\theta : l \to \Sigma^q l$ to be the second component of

$$l \xrightarrow{\iota\wedge 1} l \wedge l \xrightarrow{K} l \vee \Sigma^q l \vee \cdots,$$

where K is Kane's splitting [13], which will be discussed in more detail in Section 3. The only pertinent fact about θ is that $H^q(\theta) \neq 0$.

Use the method of [18; 2.11] to show $\alpha_*\theta_* : l_{iq-1}(S^0) \to l_{iq-1}(S^0)$ is multiplication by $p^{\nu(i)+1}$. Let $\lambda : B_1 \to S^0$ denote a map of the type considered in [10] or [3]. Then $l_*(S^0 \cup_\lambda CB_1)$ has the chart in Fig. 2. and the exact sequence $0 \to l_*(S^0) \to l_*(S^0 \cup_\lambda CB_1) \to l_*(\Sigma B_1) \to 0$ enables deducing θ_* in $l_*(\Sigma B_1)$ from θ_* in $l_*(S^0)$. Then θ_* in $l_*(B_b^t)$ follows by naturality and the equivalences of 1.1(ii). □

The homomorphisms

$$l_*(B_b^t) \xrightarrow{g_{b*}^t} l_*(B_{b-1}^{t-1})$$

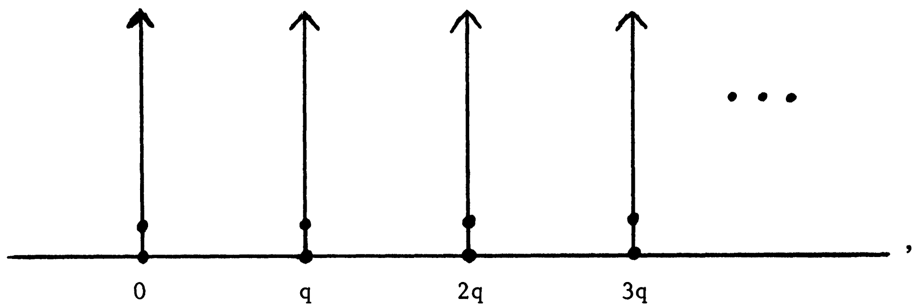


FIG. 2

are injections onto all classes of positive filtration. Thus

$$l_i(\overline{B}^t_b) = \begin{cases} \mathbf{Z}/p^{b-t+1}, & i \equiv -1\ (q), \\ 0, & \text{otherwise,} \end{cases}$$

and 1.4 follows from the exact sequence

$$\xrightarrow{\theta_*} l_{*-q+1}(\overline{B}^t_b) \to J_*(\overline{B}^t_b) \to l_*(\overline{B}^t_b) \xrightarrow{\theta_*} l_{*-q}(\overline{B}^t_b) \to$$

together with 2.4. □

Similarly to [8], $J_*(\overline{B}^t_b)$ may be conveniently represented by a chart which incorporates negative, as well as positive, filtrations. This is achieved by defining

$$E^{s,\tau}(l_*(\overline{B}^t_b)) = \varinjlim E^{s+i,\tau+i}(l_*(B^{t-i}_{b-i})).$$

then

$$E_1^{s,\tau}(J_*(\overline{B}^t_b)) = E^{s,\tau}(l_*(\overline{B}^t_b)) \oplus E^{s-1,\tau}(l_*(\Sigma^q \overline{B}^t_b)),$$

and finally inserting differentials to reflect the homorphism θ_*. For example, if $p = 3$, $J_*(\overline{B}^3_1)$ would have the chart in Fig. 3.

The proof of surjectivity in 1.5 is analogous to that in [8]. The following lemma will be useful. $D(\ )$ always refers to a (stable) 0-dual; i.e., $DX \wedge X \to S^0$.

LEMMA 2.5. *$D(B^t_b) = \Sigma B^{-b}_{-t}$ for any integers b and t.*

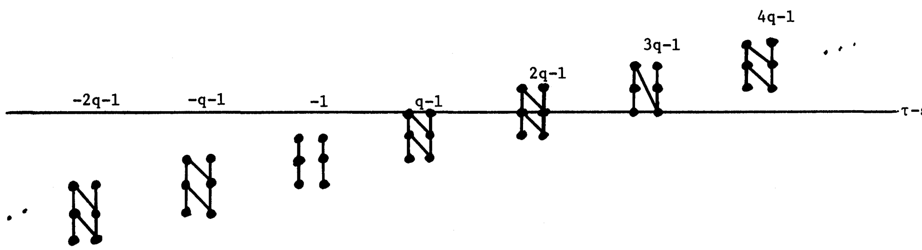


FIG. 3

Proof. This follows from the analogous result for stunted lens spaces [11] together with the splitting maps, (2.2) and t, used in the proof of 1.1. □

Now we prove the surjectivity in 1.5.

Case 1. $t = \infty$, $i \equiv -1$ (q). Since $p^{k-a}\beta_{k-a}$ is trivial, there is a filtration-0 map $T(a\beta_{k-a}) \to S^{aq}$ if $\nu(a) \geq k - a$. Lemma 2.5 translates this to a filtration-0 map $S^{-aq-1} \to B_{-a-\nu(a)}$. Since

$$\Sigma^{qpa} B^{-(p-1)a}_{-(p-1)a-\nu(a)} \simeq B^{a}_{a-\nu(a)}$$

by 1.1(ii), there are filtration-0 maps $S^{aq-1} \to B_{a-\nu(a)}$ for all integers a. Hence the first vertical arrow in the commutative diagram below is surjective.

$$\begin{array}{ccccc} \pi_{aq-1}\big(B_{a-\nu(a)}\big) & \xrightarrow{g_{a-\nu(a)}} & \pi_{aq-1}\big(B_{a-\nu(a)-1}\big) & \xrightarrow{g_{a-\nu(a)-1}} & \cdots \\ \downarrow h & & \downarrow h & & \\ J_{aq-1}\big(B_{a-\nu(a)}\big) & \xrightarrow{g_{a-\nu(a)}} & J_{aq-1}\big(B_{a-\nu(a)-1}\big) & \xrightarrow{g_{a-\nu(a)-1}} & \cdots \end{array}$$

All arrows along the bottom are isomorphisms, and hence all vertical arrows are surjective. Therefore $\pi_{aq-1}(\overline{B}_b) \to J_{aq-1}(\overline{B}_b)$ is surjective for any b.

Case 2. $t = \infty$, $i = -2$. Let $a \in \pi_{p^eq-2}(B^{p^e-1}_{p^e-2e})$ denote the attaching map for the top cell of $(B^{p^e}_{p^e-2e})^{(p^eq-1)}$. The splitting map constructed in the second sentence of the proof of Case 1 implies that a maps to 0 in $\pi_{p^eq-2}(B^{p^e-1}_{p^e-e})$. The image of a in $J_{p^eq-2}(B^{p^e-1}_{p^e-2e})$ has Adams filtration $e + 1$, by the exact sequence in $J_*(\ )$ of

$$S^{p^eq-2} \xrightarrow{a} B^{p^e-1}_{p^e-2e} \longrightarrow \left(B^{p^e}_{p^e-2e}\right)^{(p^eq-1)}.$$

Thus a pulls back to an element of $\pi_{p^eq-2}(B^{p^e-e-1}_{p^e-2e})$ whose image in $J_*(\ )$

has filtration $e+1$. By 1.1(ii), the same is true of $\pi_{-2}(B_{-2e}^{-e-1})$, and hence of $\pi_{-2}(B_{-2e})$. Using the maps g of 1.2, we deduce that all elements of $J_{-2}(\bar{B}_k)$ of filtration $\geq 1-k-e$ are in $\mathrm{im}(h)$, and e can be chosen arbitrarily large.

Case 3. t finite. This follows from the case $t=\infty$ exactly as in [8; 3.8], using a diagram which applies $\pi_*(\ ) \to J_*(\ )$ to the cofibration $B_b^t \to B_b \to B_{t+1}$, and using the vanishing line of [15] for $\pi_*(B_{t+1})$. □

3. Odd-primary bo-resolutions

In this section we adapt Mahowald's theory of bo-resolutions to the spectrum l, and apply it to prove the injectivity in 1.5. Lellman [14] has developed a very nice theory for resolutions with respect to the spectrum $k(1)=l/p$, but it does not seem to be quite appropriate for our application to the spectra $\bar{B}_b$, which have arbitrarily large p-torsion in $\pi_*(\ )$.

Let $\mathscr{H}=\mathscr{H}_p$ denote the category of locally-finite wedges of Eilenberg-MacLane spectra $\Sigma^n H/p$. A map of spectra $f: X\to Y$ is an *equivalence mod* $\mathscr{H}$ if there are spectra $H_1, H_2 \in \mathscr{H}$ and equivalences h_1 and h_2 so that the $(X_1 \to Y_1)$-component of

$$X_1 \vee H_1 \underset{\simeq}{\overset{h_1}{\to}} X \overset{f}{\to} Y \underset{\simeq}{\overset{h_2}{\to}} Y_1 \vee H_2$$

is an equivalence. If f is such a map, there is a map $g: Y\to X$ which is also an equivalence mod $\mathscr{H}$.

Kane [13] constructed complexes $C(n)$ (we have changed the name from his $K(n)$ to avoid confusion with Morava K-theory) of dimension $2n-\Sigma n_s$ if $n=\Sigma n_s p^s$ with $0\leq n_s<p$. Let $E\subset A$ be as in Section 2, and let $L(m)$ denote the E-module with generators G_i of degree qi for $0\leq i\leq m$ and relations Q_0G_0, $Q_1G_i=Q_0G_{i+1}$ $(0\leq i<m)$, Q_1G_m. Then $H^*C(n)$ splits over E as $F\oplus L(\nu(n!))$, where F is a free E-module. We note that

$$\nu(n!) = \Sigma n_s\left(p^{s-1}+\cdots+p+1\right) = \Sigma n_s\frac{p^s-1}{p-1}.$$

The following analogue of [6; 3.9] will be useful later and gives a simple visualization of $l_*(C(n))$.

PROPOSITION 3.1. *There is an equivalence* mod $\mathscr{H}$:

$$C(n)\wedge l \to l^{\langle \nu(n!)\rangle}.$$

Proof. The proof is analogous to that of [6; 3.9] given on [6; pp. 51–52]. We begin with the case $n=p^s$. Let $h=\nu(p^s!)=(p^s-1)/(p-1)$. The map

a below is defined by the cofibration sequence

$$\Sigma^{-1}\bar{C} \xrightarrow{a} S^0 \to C = C(p^s) \to \bar{C}.$$

$S = S^0$ is the sphere spectrum.

$$\begin{array}{ccccccc}
 & \overset{\bar{a}}{\nearrow} & S^{\langle 1\rangle} & \xrightarrow{\bar{f}} & S^{\langle h+1\rangle} & & \\
 & & \downarrow & & \downarrow & & \\
\Sigma^{-1}\bar{C} & \xrightarrow{a} & S & \xrightarrow{f} & S^{\langle h\rangle} & \xrightarrow{i} & l^{\langle h\rangle} \\
 & & & \underset{p^h}{\searrow} & \downarrow & & \downarrow \\
 & & & & S & \longrightarrow & l
\end{array}$$

Since $\dim(\Sigma^{-1}C) = 2p^s - 2$ is less than the connectivity of $S^{\langle h+1\rangle}$ by [15], $\overline{fa}$ is trivial and hence so is fa. Thus fa factors through a map

$$C \xrightarrow{\tilde{f}} S^{\langle h\rangle},$$

and $i\tilde{f}: C \to l^{\langle h\rangle}$ has the property that

$$L(h) \approx H^*(l^{\langle h\rangle}) \xrightarrow{(i\tilde{f})^*} H^*(C) \approx L(h) \oplus F \xrightarrow{\pi_1} L(h)$$

is the identity. The composite f_s,

$$C \wedge l \xrightarrow{i\tilde{f}\wedge l} l^{\langle h\rangle} \wedge l \xrightarrow{\mu} l^{\langle h\rangle},$$

is our desired map when $n = p^s$.

In the general case, write $n = \Sigma n_s p^s$ with $0 \leqq n_s < p$, and consider the diagram

$$\begin{array}{ccc}
\bigwedge_s C(p^s)^{\wedge n_s} & \xrightarrow{\wedge f_s^{n_s}} \bigwedge_s \left(l^{\langle \nu(p^s!)\rangle}\right)^{\wedge n_s} & \xrightarrow{\mu} l^{\nu(n!)} \\
\downarrow m & & \\
C(n) & &
\end{array}$$

where m is the pairing constructed in [13; 5.5] and μ uses

$$l^{\langle h\rangle} \wedge l^{\langle h'\rangle} \to l^{\langle h+h'\rangle}.$$

By [13; 15:3:4], when $\wedge l$ is applied to m, an equivalence mod $\mathscr{H}$ is obtained;

i.e., we have

$$\begin{array}{ccccc} X_1 \vee H_1 \xrightarrow{\simeq} \wedge C(p^s)^{\wedge n_s} \wedge l & \longrightarrow & l^{\langle \nu(n!)\rangle} \wedge l & \xrightarrow{\mu} & l^{\langle \nu(n!)\rangle} \\ \downarrow & & & & \\ Y_1 \wedge H_2 \xrightarrow{\simeq} C(n) \wedge l & & & & \end{array}$$

with

$$X_1 \xrightarrow{h} Y_1$$

an equivalence. Then

$$C(n) \wedge l \longrightarrow Y_1 \xrightarrow{h^{-1}} X_1 \longrightarrow l^{\langle \nu(n!)\rangle}$$

is the desired equivalence mod $\mathscr{H}$. □

If $\bar{n} = (n_1, \ldots, n_k)$, let $|\bar{n}| = \Sigma n_i$,

$$C(\bar{n}) = C(n_1) \wedge \cdots \wedge C(n_k) \quad \text{and} \quad C_{\bar{n}} = \Sigma^{q|\bar{n}|} C(\bar{n}).$$

COROLLARY 3.2. *If $\bar{n} = (n_1, \ldots, n_k)$, there is an equivalence* mod $\mathscr{H}$:

$$C(\bar{n}) \wedge l \to l^{\langle \Sigma\nu(n_i!)\rangle}.$$

Let R_s denote the set of s-tuples of positive integers. Let I and $\bar{l}$ be defined by the cofiber sequence

$$I = \Sigma^{-1}\bar{l} \longrightarrow S^0 \xrightarrow{\iota} l \xrightarrow{p} \bar{l}.$$

Let $d_s = \bar{l}^{\wedge s} \wedge p \wedge \iota : \bar{l}^{\wedge s} \wedge l \to \bar{l}^{\wedge (s+1)} \wedge l$. As in [13; p. 89], let L be the Thom spectrum of the obvious spherical fibration over ΩS^{2p-1}, and $\phi : L \to l$ the rational equivalence defined on [13; p. 90].

THEOREM 3.3. *For $s \geq 1$, there is an equivalence*

$$h_s : \bigvee_{\bar{n} \in R_s} C_{\bar{n}} \wedge l \to \bar{l}^{\wedge s} \wedge l$$

and a map g_s so that the following diagram commutes mod elements of filtration

$\geqq 2$:

$$\begin{array}{ccc}
\bigvee_{\bar{n}\in R_s} S^{q|\bar{n}|}\wedge L & \xrightarrow{g_s} & \bigvee_{\bar{m}\in R_{s+1}} S^{q|\bar{m}|}\wedge L \\
\Big\downarrow {\scriptstyle j\wedge\phi} & & \Big\downarrow {\scriptstyle j\wedge\phi} \\
\bigvee_{\bar{n}\in R_s} C_{\bar{n}}\wedge l & \xrightarrow{h_{s+1}^{-1}d_s h_s} & \bigvee_{\bar{m}\in R_{s+1}} C_{\bar{m}}\wedge l
\end{array}$$

and, in $H_*(\ ;\mathbf{Z}_{(p)})$,

$$g_{s*}(x^{qi_1}\otimes\cdots\otimes x^{qi_{s+1}})$$
$$=\sum_{j,a}(-1)^j\binom{i_j}{a}x^{qi_1}\otimes\cdots\otimes x^{qi_{j-1}}\otimes x^{qa}\otimes x^{q(i_j-a)}\otimes\cdots\otimes x^{qi_{s+1}}.$$

Proof. We omit the lengthy proof of the following result, [13; 11.1 and 23.6], which is analogous to [6; 3.18]:

There are (stable) equivalences f, g, and h so that the following diagram commutes mod elements of filtration $\geqq 2$:

$$\begin{array}{ccc}
\bigvee_{n\geqq 0} S^{qn}\wedge L & \xrightarrow{g\wedge L} & \Omega S_+^{2p-1}\wedge L \\
 & & \Big\downarrow {\scriptstyle f} \\
\Big\downarrow {\scriptstyle j\wedge\phi} & & L\wedge L \\
 & & \Big\downarrow {\scriptstyle \phi\wedge\phi} \\
\bigvee_{n\geqq 0} C_n\wedge l & \xrightarrow{h} & l\wedge l
\end{array}$$

The failure to commute [13; pp. 56–59] is due to elements of $[\Sigma^{2p^{s+1}-3}M, l\wedge l]$ for $s\geqq 1$, where M is the mod p Moore spectrum, and any such elements in positive filtration have filtration $\geqq s+1$, the first being in the $(C_{p^s-1}\wedge l)$-summand of $l\wedge l$.

This diagram is iterated as in [6; p. 55] to yield the diagram of the theorem. The homology statement is a consequence of [16; 2.3]. □

Let D^* denote the cochain complex

(3.4)

$$\begin{array}{ccccccccc}
\pi_*(l) & \xrightarrow{d_{0*}} & \pi_*(\bar{l}\wedge l) & \xrightarrow{d_{1*}} & \pi_*(\bar{l}^{\wedge 2}\wedge l) & \xrightarrow{d_{2*}} & \pi_*(\bar{l}^{\wedge 3}\wedge l) & \xrightarrow{d_{3*}} & \cdots \\
\| & & \| & & \| & & \| & & \\
D^0 & & D^1 & & D^2 & & D^3 & &
\end{array}$$

The main theorem of l-resolutions is:

THEOREM 3.5. *If $s \geq 2$, $H^s(D^*)$ is a $\mathbf{Z}_p$-vector space, all elements of which have $(H\mathbf{Z}_p)$-filtration* 0 *or* 1. $H^1(D^*)$ *differs from*

$$\operatorname{coker}\left(\pi_*(l) \xrightarrow{\theta_*} \pi_*(\Sigma^q l)\right)$$

by a $\mathbf{Z}_p$-vector space, all elements of which have filtration 0 *or* 1.

Proof. We calculate $H^*(D^*))$ using what has been called a geometric May spectral sequence (GMSS) in [19] and [14]. The E_1-term is $\oplus_s \pi_*(\hat{\bar{l}}^s \wedge l)$, which is given (above filtration 0) by 3.2 and 3.3. The δ_1-differential is the $(H\mathbf{Z}_p)$-filtration preserving part of d_{s*}.

Remark 3.6. We first single out for special attention the

$$\pi_*(l) \to \pi_*(C_1 \wedge l)$$

component of d_{0*}. This was calculated in the proof of 2.4; it increases filtration by $\nu(k)$ in $\pi_{kq}(\ )$. This can be interpreted as $\delta_{\nu(k)+1}$-differential in the GMSS, but we shall omit it from our subsequent consideration of this spectral sequence; its behavior is anomalous.

The following lemma is well known (e.g., [5]).

LEMMA 3.7. *Let U be a $\mathbf{Z}_p$-vector space with basis $\{u_n : n > 0\}$, and let $U^m = U \otimes \cdots \otimes U$ with m factors. Define*

$$\Delta(u_n) = \Sigma_{i=1}^{n-1} \binom{n}{i} u_{n-i} \otimes u_i$$

and

$$d_s(u_{n_1} \otimes \cdots \otimes u_{n_s}) = \Sigma_{i=1}^{s} (-1)^i u_{n_1} \otimes \cdots \otimes \Delta(u_{n_i}) \otimes \cdots \otimes u_{n_s}.$$

Then

$$U \xrightarrow{d^1} U^2 \xrightarrow{d_2} U^3 \xrightarrow{d_3} \cdots$$

is a cochain complex U^* with

$$H^*(U^*) \approx E\left[u_{p^j} : j \geq 0\right] \otimes \mathbf{Z}_p\left[v_{p^{j+1}} : j \geq 0\right]$$

where

$$v_{p^{j+1}} = \Sigma_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} u_{p^j(p-i)} \otimes u_{p^j i}.$$

The cochain complex in 3.7 is naturally isomorphic to the complex obtained by applying $H_*(\ ;\mathbf{Z}_{(p)}) \otimes \mathbf{Z}_p$ to the maps g_s of 3.3 restricted to the bottom cells. By 3.3, the same is true of the complex obtained by applying $\pi_*(\ )/\mathrm{Torr} \otimes \mathbf{Z}_p$ to the maps $h_{s+1}^{-1}d_s h_s$ of 3.3, restricted to the bottom cells. The $C_{\bar{n}} \wedge l \to C_{\bar{m}} \wedge l$ component, f, will be nonzero on the bottom cell if $\bar{m}$ is obtained from $\bar{n}$ by splitting some n_i as $(a, n_i - a)$. If

$$\binom{n_i}{a} \equiv \alpha \not\equiv 0 \mod p,$$

then tightness of the A-module $H^*(C_{\bar{m}} \wedge l)$ implies that (ignoring split $H\mathbf{Z}_p$'s)

$$\mathrm{Ext}_A^{*,*}(H^*(C_{\bar{n}} \wedge l)) \xrightarrow{f_*} \mathrm{Ext}_A^{*,*}(H^*(C_{\bar{m}} \wedge l))$$

is $.\alpha$ in all nonzero groups.

We elaborate slightly upon the preceding sentence. Both $H^*(C_{\bar{m}} \wedge l)$ and $H^*(C_{\bar{n}} \wedge l)$ as A-modules have generators and relations corresponding to those of the same E-module $L(h)$ defined before 3.1. The relations $Q_0G_{i+1} = Q_1G_i$ imply that $f^*G_i = \alpha G_i$ for all i. The elements in the minimal resolution are given by relations $Q_1x_{i-1} + \varepsilon Q_0x_i$, and so the map of minimal resolutions inductively assures that all Ext-maps are $\cdot\alpha$.

A similar argument, similar to [6; p. 56], shows that if

$$\binom{n_i}{a} \equiv 0(p),$$

then

$$f_* = 0 : \mathrm{Ext}_A^{*,*}(H^*(C_{\bar{n}} \wedge l)) \to \mathrm{Ext}_A^{*,*}(H^*(C_{\bar{m}} \wedge l)),$$

except that elements of $\mathrm{Ext}_A^0(H^*(C_{\bar{m}} \wedge l))$ corresponding to split $H\mathbf{Z}_p$'s might be in $\mathrm{im}(f_*)$. Lemma 3.7 and the paragraphs which follow it imply the following result.

LEMMA 3.8. *Let $\overline{E} = (e_0, e_1, m_1, e_2, m_2 \dots)$ be a finite sequence of nonnegative integers with $e_i \leq 1$, corresponding to the generator*

$$v(\overline{E}) = u_1^{e_0} \prod_{j \geq 1} u_{p^j}^{e_j} v_{p^j}^{m_j}$$

of H^*U^* of 3.7. *Define $\bar{n}(\overline{E})$ to be the sequence of positive integers below, where the numbers beneath a number or pair of numbers indicates the number of occurrences of that number or pair of numbers*:

$$(\underbrace{p^0}_{e_0}, \underbrace{p^1}_{e_1}, \underbrace{p^0, p^1 - p^0}_{m_1}, \underbrace{p^1, p^2 - p^1}_{m_2}, \dots).$$

Let $\mathscr{E}$ be the set of sequences $\bar{E}$ with $e_i > 0$ or $m_i > 0$ for some $i > 0$. Then the E_2 – term of the GMSS (modulo Remark 3.6) in positive ($H\mathbf{Z}_p$)-filtration agrees with $\oplus_{\bar{E}\cdot\mathscr{E}}\pi_*(C_{\bar{n}(\bar{E})} \wedge l)$.

Proof. The chain complex U^* of 3.7 splits as a direct sum of subcomplexes $U_{(S_1, S_2)}$ spanned by $u_{\bar{n}} = u_{n_1} \otimes \cdots \otimes u_{n_k}$ with

$$\bar{n} \in N(S_1, S_2) = \{\bar{n} : \Sigma n_i = S_1, \Sigma \nu(n_i!) = S_2\}.$$

For all $\bar{n} \in N(S_1, S_2)$, the charts $l_*(C_{\bar{n}})$ are isomorphic above filtration 0 by 3.2. In the GMSS, (E_1, δ_1) splits as a direct sum of subcomplexes $E_1(S_1, S_2)$, and the subcomplex in each $\mathrm{Ext}^{s,t}(\ )$-bigrading is either 0 or $U_{(S_1, S_2)}$. □

Here we have, rather arbitrarily, selected $C_{p^{r-1}} \wedge C_{p^r - p^{r-1}} \wedge l$ to represent the homotopy classes corresponding to a v_{p^r}-factor. Next we analyze the δ_2-differential in the GMSS, utilizing the following lemma, which uses some notation of 3.8.

LEMMA 3.9. *Let $\mathscr{E}' = \{\bar{E} = (e_1, m_1, e_2, m_2, \dots) : (0, \bar{E}) \in \mathscr{E}\}$. Let V^* be a graded $\mathbf{Z}_p$-vector space with basis $\{v(\bar{E}) : \bar{E} \in \mathscr{E}'\}$, where $v(\bar{E})$ has grading $\Sigma_{j \geqq 1} e_j + 2m_j$. Suppose V^* has a differential d such that $d \circ d = 0$ and*

$$d(v(\bar{E})) = \sum_{j : e_j = 1} \alpha_j v(\dots, m_{j-1}, 0, m_j + 1, e_{j+1}, \dots)$$

with $\alpha_j \not\equiv 0 \bmod p$. Then $H^*V^* = 0$.

SUBLEMMA 3.10. *We say a differential vector space is of n-type if it has basis $w_{(\varepsilon_1, \dots, \varepsilon_n)}$, $\varepsilon_i = 0$ or 1, and*

$$d\left(w_{(\varepsilon_1, \dots, \varepsilon_n)}\right) = \sum_{\varepsilon_j = 1} \alpha_j w(\dots, \varepsilon_j - 1, \dots)$$

with $\alpha_j \neq 0$. If (W, d) is of n-type, then $H^*W = 0$.

Proof. If $W' = \langle w_{\bar{\varepsilon}} : \varepsilon_1 = 0\rangle$, then W' and W/W' are of $(n-1)$-type, and so the result follows by induction and the exact cohomology sequence. □

Proof of 3.9. $V^* = \oplus_K W(K)$, where K ranges over finite sequences of nonnegative integers, and $W(K) = \langle v(\bar{E}) : e_i + m_i = k_i\rangle$. Each $W(K)$ is of n-type, where n is the number of nonzero k_i's, and the ε_i's are the e's associated to nonzero k's. □

The following result is analogous to [6; 3.7(ii)] as corrected in [7].

LEMMA 3.11. *Suppose $v(\overline{E}_1)$ occurs with nonzero coefficient in $d(v(\overline{E}))$ in 3.9, and $\bar{n} = \bar{n}(\overline{E}) = (n_1, \ldots, n_k)$ is as in 3.8. Let*

$$s = \Sigma_{i=1}^{k} \nu(n_i!).$$

Let $r = r_1 \vee r_2$ denote the following composite, which utilizes the equivalences mod $\mathscr{H}$ *of* 3.2 *and the maps of* 3.3:

$$\Sigma^{|\bar{n}|} l^{\langle s \rangle} \to C_{\bar{n}} \wedge l \xrightarrow{h_{k+1}^{-1} d_k h_k} C_{\bar{n}(\overline{E}_1)} \wedge l \underset{\simeq}{\xrightarrow{g}} \Sigma^{|\bar{n}|} l^{\langle s-1 \rangle} \vee H.$$

Then r_1 lifts to an equivalence $\Sigma^{|\bar{n}|} l^{\langle s \rangle} \to (\Sigma^{|\bar{n}|} l^{\langle s-1 \rangle})^{\langle 1 \rangle}$.

Proof. Since

$$\nu \binom{p^j}{p^{j-1}} = 1,$$

3.3 implies that the restriction of r to the bottom cell lifts to

$$(\Sigma^{|\bar{n}|} l^{\langle s-1 \rangle})^{\langle 1 \rangle},$$

where it is nonzero in $H^*(\ ; \mathbf{Z}_p)$. By tightness of A-module structure, $H^*(r_1; \mathbf{Z}_p)$ must be 0, so that r_1 lifts to $(\Sigma^{|\bar{n}|} l^{\langle s-1 \rangle})^{\langle 1 \rangle}$, and, as before, analysis of relations in the minimal resolution shows that all Ext-classes map by multiplication by the same nonzero element of $\mathbf{Z}_p$. □

We can now deduce 3.5. First note that, similarly to 3.11 and [6; 3.7(iii)], 3.3 and tightness of A-structure imply that if $\bar{v}(\overline{E}_1)$ does not occur with nonzero coefficient in $d(v(\overline{E}))$ in 3.9, then the

$$C_{\bar{n}(\overline{E})} \wedge l \to C_{\bar{n}(\overline{E}_1)} \wedge l$$

component of δ_2 is zero, i.e., $\mathrm{Ext}^{s,t} \to \mathrm{Ext}^{s+1,t+1}$ is 0. This and 3.11 imply that the chain complex (E_2, δ_2) in the GMSS, except in filtration 0 and perhaps 1, and with the exception discussed in Remark 3.6, splits as $\oplus A(K)$, where K ranges over finite sequences of nonnegative integers as in the proof of 3.9. Here

$$A(K) = \bigoplus_{\overline{E} :\, v(\overline{E}) \in W(K)} \mathrm{Ext}_A^{*,*}\big(H^*(C_{\bar{n}(\overline{E})} \wedge l)\big),$$

where $W(K)$ is as in the proof of 3.9. Similarly to the proof of 3.8, for any K

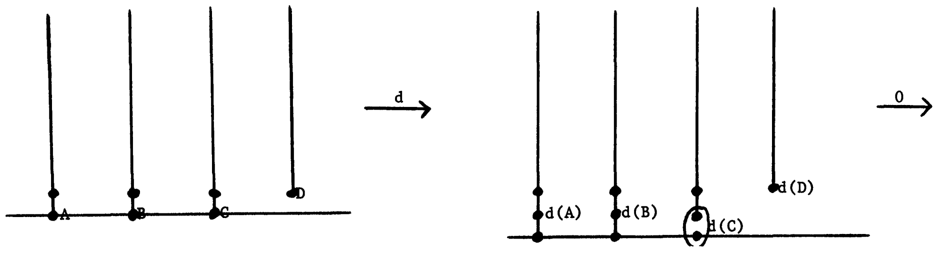


FIG. 4

and (s, t), the intersection with $A(K)$ of the sequence

$$\bigoplus_{\bar{n} \in R_1} \mathrm{Ext}^{s,t}(H^*(C_{\bar{n}} \wedge l)) \to \bigoplus_{\bar{n} \in R_2} \mathrm{Ext}^{s+1,t+1}(H^*(C_{\bar{n}} \wedge l)) \to \bigoplus_{\bar{n} \in R_3} \mathrm{Ext}^{s+2,t+2}(H^*(C_{\bar{n}} \wedge l)) \to \cdots$$

is either 0 or $W(K)$, and hence is acyclic. Since the E_∞-term of the GMSS is an associated graded for H^*D^*, this implies 3.5. The filtration 1 terms, which were overlooked in [17] and [6], can occur due to the situation in Fig. 4 (see [7]), where $d(C)$ is the sum of the indicated classes:

A result similar to [6; 3.6] adapting 3.5 to any spectrum X for which the Adams spectral sequence $\mathrm{Ext}_A(H^*(X \wedge l)) \Rightarrow \pi_*(X \wedge l)$ collapses could probably be proved, but an adaptation of the proof of that result seems tedious at best. We are content to prove that it holds for the spectra B_b^t of 1.1.

THEOREM 3.12. *If $X = B_b^t$ with t possibly infinite, then the chain complex D_X^*,*

$$\begin{array}{ccccccc} \pi_*(X \wedge l) & \xrightarrow{d_{0*}} & \pi_*(X \wedge \bar{l} \wedge l) & \xrightarrow{d_{1*}} & \pi_*(X \wedge \bar{l}^{\wedge 2} \wedge l) & \xrightarrow{d_{2*}}, \\ \| & & \| & & \| & \\ D_X^0 & & D_X^1 & & D_X^2 & \end{array}$$

has the following properties:

(i) *If $s \geq 2$, $H^s D_X^*$ is a $\mathbf{Z}_p$-vector space consisting of classes of $(H\mathbf{Z}_p)$-filtration 0 or 1.*

(ii) $H^1(D_X^*)$ agrees with

$$\mathrm{coker}\left(\pi_*(X \wedge l) \xrightarrow{\theta_*} \pi_*(X \wedge l \wedge l)\right)$$

except for classes of filtration 0 or 1.

Proof. We first show that if the $l_*(C_{\bar{n}}) \to l_*(C_{\bar{m}})$ component of $h_{k+1}^{-1} d_k h_k$ is $.\alpha$ except for filtration-0 $\mathbf{Z}_p$'s, then the same is true of $l_*(X \wedge C_{\bar{n}}) \to l_*(X \wedge C_{\bar{m}})$. This is proved for $X = B_1^t$ by the argument of 2.4. It is then deduced for B_b^t with $b \geq 1$ by use of the collapsing map

$$B_1^t \xrightarrow{c} B_b^t,$$

noting that

$$l_*\left(B_1^t \wedge C_{\bar{n}}\right) \xrightarrow{c_*} l_*\left(B_b^t \wedge C_{\bar{n}}\right)$$

is surjective. Next it is deduced for all B_b^t with t finite using the equivalences of 1.1(ii). Finally it is deduced for B_b^∞ by using B_b^t to study $\pi_i(\ \ )$ for $i \leq qt$. Then Lemma 3.8 remains valid for the E_2-term of the GMSS converging to $H^* D_X^*$, and Lemma 3.11 remains valid when $X \wedge$ is applied, so that we can deduce that the $E_3 = E_\infty$-term (modulo Remark 3.6) of the GMSS is as claimed. □

We are now prepared to prove the injectivity of $\pi_*(\bar{B}_b^t) \to J_*(\bar{B}_b^t)$ in 1.5. After possibly reindexing, it suffices to show that if $\alpha : S^n \to B_b^t$ becomes trivial in $B_b^t \wedge J$, then for some k the composite

$$S^n \xrightarrow{\alpha} B_b^t \xrightarrow{g_b^t} B_{b-1}^{t-1} \xrightarrow{g_{b-1}^{t-1}} \cdots \xrightarrow{g_{b-k+1}^{t-k+1}} B_{b-k}^{t-k}$$

is trivial. It suffices to do this when t is finite, for if $\alpha \in \pi_n(B_b^\infty)$ choose $t > [n/q]$ and use the commutative diagram (from 1.1)

$$\begin{array}{ccccccc}
 & \overset{\alpha'}{\nearrow} & B_b^t & \longrightarrow & \cdots & \longrightarrow & B_{b-k}^{t-k} \\
 & & \downarrow & & & & \downarrow \\
S^n & \xrightarrow{\alpha} & B_b^\infty & \longrightarrow & \cdots & \longrightarrow & B_{b-k}^\infty
\end{array}$$

to deduce the result for α.

By duality it is equivalent to show that if $f : \Sigma^{n+1} B_{-t}^{-b} \to S^0$ becomes trivial in J, then for some k the composite

$$S^0 \xleftarrow{f} \Sigma^{n+1} B_{-t}^{-b} \underset{\textstyle \overset{\|}{g_1}}{\xleftarrow{g_{-t+1}^{-b+1}}} \cdots \underset{\textstyle \overset{\|}{g_k}}{\xleftarrow{g_{-t+k}^{-b+k}}} \Sigma^{n+1} B_{-t+k}^{-b+k} \qquad (3.13)$$

is trivial. We use the l-resolution

$$\begin{array}{ccccccccc}
S^0 & \xleftarrow{p_0} & I & \xleftarrow{p_1} & I^{\wedge 2} & \xleftarrow{p_2} & I^{\wedge 3} & \longleftarrow & \cdots \\
\downarrow{\scriptstyle q_0} & \swarrow{\scriptstyle q_1} & & & \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle q_3} & & \\
l & I \wedge l & & & I^{\wedge 2} \wedge l & & I^{\wedge 3} \wedge l & &
\end{array}$$

where $I = \Sigma^{-1} \bar{l}$.

LEMMA 3.14. *If $s \geq 2$ and f_s below is any map,*

$$\begin{array}{ccccccc}
 & & \Sigma^{-1} I^{\wedge(s-1)} \wedge l & & I^{\wedge(s+1)} & & \\
 & & & \searrow^{j} & \downarrow p_s & & \\
B^{\tau}_{\beta} & \xrightarrow{g^{\tau-1}_{\beta-1} \circ g^{\tau}_{\beta}} & B^{\tau-2}_{\beta-2} & \xrightarrow{f_s} & I^{\wedge s} & \xrightarrow{q_s} & I^{\wedge s} \wedge l \\
 & & & & \downarrow P_{s-1} & & \\
 & & & & I^{\wedge(s-1)} & &
\end{array}$$

then there exists $\tilde{f}: B^{\tau}_{\beta} \to I^{\wedge(s+1)}$ such that

$$P_{s-1} \circ P_s \circ \tilde{f} = P_{s-1} \circ f_s \circ g^{\tau-1}_{\beta-1} \circ g^{\tau}_{\beta}.$$

The conclusion is also true if $s = 1$ and

$$B^{\tau-2}_{\beta-2} \xrightarrow{f_1} I \longrightarrow \Sigma^{q-1} l$$

is trivial, where the last arrow is the first component of the splitting of $I \wedge l$.

Proof. Dual to $f_s \circ g^{\tau-1}_{\beta-1} \circ g^{\tau}_{\beta}$ is an element $\gamma \in \pi_{-1}(B^{-\beta}_{-\tau} \wedge I^{\wedge s})$ of $H\mathbf{Z}_p$-filtration ≥ 2. Since

$$g_s\gamma \in \ker\left(\pi_{-1}\left(B^{-\beta}_{-\tau} \wedge I^{\wedge s} \wedge l\right) \xrightarrow{d_{s*}} \pi_{-1}\left(B^{-\beta}_{-\tau} \wedge I^{\wedge(s+1)} \wedge l\right)\right),$$

3.12 implies that there exists $\delta \in \pi_{-1}(B^{-\beta}_{-\tau} \wedge \Sigma^{-1} I^{\wedge(s-1)} \wedge l)$ such that $q_s(j\delta + \gamma) = 0$. Thus there is $F \in \pi_{-1}(B^{-\beta}_{-\tau} \wedge I^{\wedge(s-1)})$ such that $p_s F = j\delta + \gamma$ and hence $p_{s-1} p_s F = p_{s-1}\gamma$. Dualize again to obtain the desired result. □

Now suppose f is as in 3.13. Its triviality in J implies that it lifts to a map f_1 satisfying the hypothesis in the last sentence of 3.14. Then 3.14 implies that $f_1 g_1 \dots g_{2k}$ lifts to a map

$$\Sigma^{n+1} B^{-b+2k}_{-t+2k} \xrightarrow{f_{k+1}} I^{\wedge(k+1)}.$$

By [14; 3.6],

$$I^{\wedge(k+1)} \xrightarrow{p^{k+1}} S^0$$

is null-homotopic on the $(\frac{1}{2}(k+1)pq - 1)$-skeleton if $k+1$ is even. The dimension of the domain of $f_1 g_1 \dots g_{2k}$ is $n + 1 - bq + 2kq$. If $p/2 > 2$, then, for k sufficiently large, $p^{k+1} f_{k+1}$ is trivial for dimensional reasons, establishing triviality of 3.13 and hence injectivity in 1.5. □

Lellman ([14]) also required the condition $p > 3$ in his proof of 1.5 when $b = t$.

4. K-theory localization

In this section we prove 1.6, 1.7, 1.8, and 1.9. As the arguments are totally analogous to the case $p = 2$ handled in [8], details will be kept to a minimum.

Analogues of the proofs of [8; 4.1] could be given. Since both of those use results of Bousfield, we prefer the following self-contained proof, mimicking the proof of [4; 4.2].

Proof of 1.6. Let

$$\mathscr{J} = \text{fibre}\left(K_{(p)} \xrightarrow{\Psi^r - 1} K_{(p)} \right)$$

be as in [4; p. 269]. Then $X \wedge \mathscr{J}$ is K_*-local for any spectrum X, since it is a cofibre of K-module spectra. The map

$$S^0 \xrightarrow{\iota} J \longrightarrow \mathscr{J}$$

induces a commutative diagram

$$\begin{array}{ccccc} B_b^t & \xrightarrow{g_b^t} & B_{b-1}^{t-1} & \xrightarrow{g_{b-1}^{t-1}} & \cdots \\ \downarrow & & \downarrow & & \\ B_b^t \wedge \mathscr{J} & \xrightarrow{g_b^t \wedge \mathscr{J}} & B_{b-1}^{t-1} \wedge \mathscr{J} & \longrightarrow & \cdots \end{array} .$$

Since

$$\Psi^r - 1 : K_{aq-1}\left(B_{b-k}^{t-k}\right) \to K_{aq-1}\left(B_{b-k}^{t-k}\right)$$

is multiplication by $r^{a(p-1)} - 1$ on $\mathbf{Z}/p^{t-b+1}$ (using [11]), and

$$\nu_p\left(r^{a(p-1)} - 1\right) = \nu_p(a) + 1$$

by [1], the groups $\pi_*(B_{b-k}^{t-k} \wedge \mathscr{J})$ are isomorphic to $\pi_*(\overline{B}_b^t)$ given by 1.4 and 1.5, and the maps $g_b^t \wedge \mathscr{J}$ induce isomorphisms in $\pi_*(\ )$. Since $\pi_i(B_b^t \wedge J) \to \pi_i(B_b^t \wedge \mathscr{J})$ is an isomorphism for $i \geq tq$, it follows from 1.5 that $\overline{B}_b^t \to B_b^t \wedge \mathscr{J}$ is an equivalence, and hence $\overline{B}_b^t$ is K_*-local.

Corollary 1.7 follows immediately from 1.6 and the fact that the maps g_b^t of 1.2 are K_*-equivalences. The independence of g_b^t is a consequence of the uniqueness of Bousfield's localization.

In 1.8, S^{-1}/p^n is the Moore spectrum whose only nonzero integral homology group is $H_{-1}(S^{-1}/p^n) \approx \mathbf{Z}/p^n$.

Proof of 1.8. As in the proof of surjectivity in 1.5, there is a degree-1 map

$$S^{-1} \xrightarrow{f} B^0_{b-t}.$$

The map $\cdot p^{t-b+1}$ on B^0_{b-t} is trivial because

$$\operatorname{Ext}_A^{s,\tau}\left(H^* B^0_{b-t}\right) = 0$$

for $s \geq t - b + 1$, $\tau - s \leq 0$ by [15]. Thus f factors through a map

$$S^{-1}/p^{t-b+1} \xrightarrow{\tilde{f}} B^0_{b-t}.$$

That $K_*(\tilde{f})$ is an isomorphism can be shown using either the Atiyah-Hirzebruch spectral sequence or the Adams spectral sequence for $ku_*(\ \ )$ as in [8]. Thus $(S^{-1}/p^{t-b+1})_K \simeq (B^0_{b-t})_K \simeq \overline{B}^t_b$.

A K_*-localization map S^{-1}/p^∞ is obtained as in [8] by inductively constructing a diagram

$$\begin{array}{ccc} \downarrow & & \\ S^{-1}/p^n & \searrow & \\ \downarrow & & \overline{B}_0, \\ S^{-1}/p^{n+1} & \nearrow & \\ \downarrow & & \end{array}$$

using (from 1.4 and 1.5) $\pi_{-1}(\overline{B}_0) \approx \mathbf{Z}/p^\infty$. □

The first cofibration in 1.9 follows from 1.8 and the fact that $\mathbf{Z}/p^\infty \approx Q/\mathbf{Z}_{(p)}$, while the second follows from Bousfield's description of $S_{K_{(p)}}$.

Added in proof. An alternate approach to 3.1, 3.2, and part of 3.3 can be found in W. Lellman, *Operations and cooperations in odd-primary connective K-theory*, J. London Math. Soc., vol. 29 (1984), pp. 562–576. An alternate approach to 1.6 can be found in D. M. Davis, M. Mahowald, and H. R. Miller, *Mapping telescopes and K_*-localizations*, to appear in Proc. John Moore Conference, Princeton Univ. Press.

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