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ORDERED FIELDS SATISFYING ROLLE'S THEOREM

BY

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1. Introduction

Throughout this paper F will denote an ordered field. We say F *satisfies Rolle's theorem for polynomials* (*with respect to the given ordering*) if for each polynomial $f \in F[x]$ with distinct roots a and b in F, the formal derivative f' of f has a root in F between a and b. Note that the usual proof from elementary calculus shows that such a field will also satisfy the mean value theorem for polynomials.

The purpose of this paper is to give a complete characterization of the ordered fields which satisfy Rolle's theorem and to investigate some of their properties. The main result of the paper is Theorem 2.1, which characterizes the fields satisfying Rolle's theorem for polynomials as fields which are Henselian with respect to some valuation with real closed residue class field and with a value group which is n-divisible for each odd integer n. The reader is referred to Ribenboim [16] or Endler [10] for the concepts we need from valuation theory. This paper completes the partial results obtained by Craven and Csordas [9]. The fields were already known to be pythagorean (every sum of squares is again a square in the field). They are now shown to be hereditarily pythagorean (and hence superpythagorean [1, Theorem 2, p. 89]). These properties make such fields attractive as local objects in the study of general fields. Indeed we indicate in Section 5 how their Witt rings of quadratic forms [13] can be used in a natural way in the "exact" representation theory of the reduced Witt ring [8, §3] in place of the "Witt rings modulo a fan arising from a place into $\mathbf{R}$".

In Section 3 we show that a field satisfies Rolle's theorem for all rational functions if and only if it is real closed. We raise, but don't solve, the question of when Rolle's theorem will hold for all functions which are integral over the polynomial ring of our field. The brief fourth section contains an approximate version of Rolle's theorem for arbitrary fields (and certain families of orderings); our less-than-inspiring result does serve to indicate the limits of our techniques in the absence of special hypotheses on the field.

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In the last section of the paper we characterize those fields with n orderings which have exactly $2n - 1$ minimal proper extensions in some fixed algebraic closure. This result implies that fields with $n < \infty$ orderings will satisfy Rolle's theorem for polynomials if and only if they have exactly $2n - 1$ minimal proper extensions and admit exactly one place into $\mathbf{R}$. The number s of minimal proper extensions of a field F turns out (cf., Proposition 6.5) to be the unique integer s such that F is "s-maximal" in the sense of McCarthy [14]. Fields which are 1-, 2-, or 3-maximal have been studied by several authors (e.g., see [11]); the key idea of considering subfields of an algebraically closed field which are maximal with respect to the exclusion of certain elements is attributed by Lang to E. Artin.

2. Rolle's Theorem for polynomials

In this section we characterize the family of fields which satisfy Rolle's theorem for polynomials. Let π denote the unique place from F into the real numbers $\mathbf{R}$ with $\pi(P) \geq 0$ [2]. Let $v: F^{\cdot} \to \Gamma$ be the associated valuation onto the additively written value group Γ and let $V = \pi^{-1}(\mathbf{R})$ be the associated valuation ring. We shall call an additively written group Δ *odd divisible* if $n\Delta = \Delta$ for all odd integers n.

THEOREM 2.1. *The following are equivalent*:

(a) *F satisfies Rolle's theorem for polynomials*;

(b) *F is Henselian with respect to some valuation with real closed residue class field and odd divisible value group*;

(c) *if $f \in V[x]$ and if $\beta \in \mathbf{R}$ is a root of $\pi(f)$ of odd multiplicity, then there is a root α of f in F with $\pi(\alpha) = \beta$.*

In (c) above, we have let π act on $V[x]$ in the obvious way, i.e., on coefficients. One should note that a Henselian field with real closed residue class field admits a unique place into $\mathbf{R}$ (it is Henselian with respect to its canonical real extended absolute value [4, Corollary, p. 395]). Indeed any place into $\mathbf{R}$ on a Henselian field must factor through the place with respect to which the field is Henselian.

In the proof below, we often write $\bar{f}$ for $\pi(f)$.

Proof of Theorem 2.1. (a) $\Rightarrow$ (c) We can write

$$\bar{f}(x) = \sum_{i=m}^{n} s_i(x - \beta)^i \in \mathbf{R}[x]$$

where m is odd and $s_m \neq 0$. We may assume $\beta \neq 0$ (otherwise replace $f(x)$ by $f(x + 1)$) and that for some even integer t we have $s_t \neq 0$ (otherwise replace

$f(x)$ by $xf(x)$). Assume t is the least even integer with $s_t \neq 0$. Then for all $\varepsilon > 0$,

$$A_\varepsilon := \int_{\beta-\varepsilon}^{\beta+\varepsilon} \bar{f}(x)\,dx = \int_{-\varepsilon}^{\varepsilon} \sum_{i=m}^{n} s_i u^i\,du = 2s_t\varepsilon^{t+1}(t+1)^{-1} + \varepsilon^{t+2}h(\varepsilon)$$

where $h(x) \in \mathbf{R}[x]$; similarly, for some $k(x) \in \mathbf{R}[x]$,

$$B_\varepsilon := \int_{\beta-\varepsilon}^{\beta+\varepsilon} (x-\beta)\bar{f}(x)\,dx = 2s_m(m+2)^{-1}\varepsilon^{m+2} + \varepsilon^{m+3}k(\varepsilon).$$

Hence, as ε approaches zero, $B_\varepsilon/A_\varepsilon$ approaches either ∞ (if $t > m+1$) or s_m/s_t (if $t = m+1$). Hence by choosing ε sufficiently small we can guarantee that

$$\varepsilon < |B_\varepsilon/A_\varepsilon| = \left|\beta - \int_{\beta-\varepsilon}^{\beta+\varepsilon} x\bar{f}(x)\,dx \Big/ \int_{\beta-\varepsilon}^{\beta+\varepsilon} \bar{f}(x)\,dx\right|$$

and also that

$$A_\varepsilon \neq 0, \quad B_\varepsilon \neq 0 \quad \text{and} \quad \bar{f}(\delta) \neq 0 \quad \text{for all } \delta \in (\beta-\varepsilon, \beta+\varepsilon),\ \delta \neq \beta.$$

Hence we can find $p, q \in \mathbf{Q}$, with $\beta - \varepsilon < p < \beta < q < \beta + \varepsilon$ such that

$$\bar{A} := \int_p^q \bar{f}(x)\,dx \neq 0, \quad \bar{B} := \int_p^q x\bar{f}(x)\,dx \neq 0 \quad \text{and} \quad |\beta - (\bar{B}/\bar{A})| > \varepsilon.$$

It follows that

$$A := \int_p^q f(x)\,dx \text{ and } B := \int_p^q xf(x)\,dx$$

are units in V and

$$\pi(B/A) = \bar{B}/\bar{A} \notin (\beta-\varepsilon, \beta+\varepsilon).$$

Since

$$\int_p^q f(x)(x - BA^{-1})\,dx = B - (BA^{-1})A = 0,$$

we may apply Rolle's theorem to $\int_p^x f(t)(t - BA^{-1})\,dt \in F[x]$ to find $\alpha \in F$ with $p < \alpha < q$ and $f(\alpha)(\alpha - BA^{-1}) = 0$. But $\alpha \neq BA^{-1}$ since otherwise

$$\bar{B}/\bar{A} = \pi(BA^{-1}) = \pi(\alpha) \in (p,q) \subset (\beta-\varepsilon, \beta+\varepsilon),$$

a contradiction. Hence $f(\alpha) = 0$. Further, $\bar{f}(\pi(\alpha)) = 0$ and

$$\pi(\alpha) \in [p, q] \subset (\beta + \varepsilon, \beta + \varepsilon),$$

so $\pi(\alpha) = \beta$ by the choice of ε.

(c) $\Rightarrow$ (b). Let $\beta \in \mathbf{R}$ be algebraic over the residue class field $\bar{F} := \pi(V)$. Then there is a monic irreducible polynomial $f \in V[x]$ with $\bar{f}$ the irreducible polynomial of β over $\bar{F}$. By hypothesis, f has a root $\alpha \in F$ with $\pi(\alpha) = \beta$. Thus $\beta \in \bar{F}$. This shows that $\bar{F}$ is real closed. Next suppose $a \in \pi^{-1}(0)$ and n is an odd positive integer. By hypothesis $x^n - a$ has a root, say α, in F and clearly $v(\alpha) = v(a)/n$. Thus Γ is odd divisible. Finally, F is Henselian at v by Corollary 16.6(iv) of [10].

(b) $\Rightarrow$ (a). Let $a < b$ be roots in F of $f \in F[x]$. We must show that f' has a root between a and b. We may suppose $a = 0$ and $b = 1$ (otherwise replace $f(x)$ by $f((b - a)x + a)$), so

$$f(x) = x(x - 1)\sum_i s_i x^i \quad \text{for some } s_i \in F.$$

We may also suppose $f \in V[x]$ and $\bar{f} \neq 0$ (otherwise replace f by $s^{-1}f$ where $s = \max|s_i|$). Since $\bar{f}(0) = \bar{f}(1) = 0$, $\bar{f}'$ has a root $\beta \in \mathbf{R}$ of odd multiplicity between 0 and 1. Since $\bar{F}$ is real closed, $\beta \in \bar{F}$, whence $\beta = \pi(\alpha_0)$ for some $\alpha_0 \in F$. Hence for some $g \in V[x]$ we can write $\bar{f}'(x) = \pi((x - \alpha_0)^m g(x))$ where m is odd, $m + \deg g \leq \deg f'$ and $x - \beta = \pi(x - \alpha_0)$ does not divide $\pi(g)$. Thus by Hensel's lemma [16, 3) on p. 186] there is a divisor $h \in F[x]$ of f' of degree m with $\pi(h) = (x - \beta)^m$. The polynomial h must therefore have an irreducible factor of odd degree; let α be a root of any such factor (in some algebraic closure of F). By a theorem of Ostrowski [16, p. 232], the odd number $[F(\alpha): F]$ is the product of the ramification index "e_v" and the ramification degree "f_v" of the extension $F(\alpha)/F$, so both f_v and e_v are odd. But they are also both powers of two since $\bar{F}$ is real closed and Γ is odd divisible. Hence both equal 1 and therefore $\alpha \in F$. In fact $\alpha \in V$ (since V is integrally closed and the leading coefficient of h is a unit), so $\pi(\alpha) = \beta \in (0, 1)$. Thus α is a root of f' between 0 and 1, as required.

The first example of a field satisfying the hypotheses of Theorem 2.1 but not real closed was given by the third author [15]. In his example the residue class field is the field of real algebraic numbers and the value group is isomorphic to the additive group of integers localized at 2.

COROLLARY 2.2. *If F satisfies Rolle's theorem for polynomials with respect to some ordering, then it does so with respect to all of its orderings.*

The corollary is immediate from condition (b) of Theorem 2.1, which is independent of the ordering on F. The proof of (b) $\Rightarrow$ (a) above yields another corollary. (We'll squeeze a bit more out of this argument in §4 below.)

COROLLARY 2.3. *Suppose F satisfies Rolle's theorem for polynomials and a and b are distinct roots in F of $f \in F[x]$. Then f' has a root c in F which is between a and b with respect to every ordering of F; in fact, $(a - c)(c - b) \in F^{\cdot 2}$.*

Proof. We have seen that the derivative of $f((b - a)x + a)$ has a root $\alpha \in F$ with $0 < \pi(\alpha) < 1$. But then $c := (b - a)\alpha + a$ is a root of $f'(x)$ and $0 < \alpha < 1$ for every ordering of F (since they all induce the same place π); hence c is between a and b for all orderings of F. Since F is pythagorean, $F^{\cdot 2}$ is precisely the set of elements of F which are positive in every ordering. Thus $(a - c)(c - b) \in F^{\cdot 2}$.

COROLLARY 2.4. *If F satisfies Rolle's theorem for polynomials, then so does every (ordered) algebraic extension of F. Hence F is hereditarily pythagorean and superpythagorean.*

The first sentence follows from the fact that the conditions of Theorem 2.1(b) are all preserved by ordered algebraic extensions. The same reasoning shows that all Henselian fields with real closed residue class field are not only pythagorean, but are even hereditarily pythagorean (and hence superpythagorean [1, p. 89]). The following observation follows immediately from Theorem 2.1(b) or (c).

COROLLARY 2.5. *An archimedean real closed field which satisfies Rolle's theorem is real closed.*

COROLLARY 2.6. *Let F be a field which satisfies Rolle's theorem for polynomials. Then every polynomial of odd degree over F has a root in F. In particular, every element of F has an m-th root in F for each odd positive integer m.*

Proof. Let $f \in F[x]$ be a polynomial of odd degree. Dividing by the leading coefficient, we may assume f is monic. Let m be the degree of f and let c be a coefficient of f of minimum value. Replacing $f(x)$ by $c^{-m}f(cx)$, we may consider the polynomial $\bar{f} \in \mathbf{R}[x]$. Since $\bar{f}$ has odd degree, it has a root $\beta \in \mathbf{R}$ of odd multiplicity. Thus f has a root in F by Theorem 2.1(c).

3. Rolle's Theorem for other functions

Throughout this section f will denote a function whose domain and range are in our ordered field F and which is continuous in the order topology. The "derivative" of f can be defined as in elementary calculus: $f'(a)$ denotes the unique number (if there is one) such that for all positive ε in F there exists a positive δ in F such that if b is in the domain of f and $0 < |b - a| < \delta$, then

$$|f'(a) - (f(b) - f(a))(b - a)^{-1}| < \varepsilon.$$

Most of the familiar rules for calculating derivatives are valid in this generalized setting, since their proofs do not depend on the completeness axiom or the Archimedean property of the real numbers.

We say F *satisfies Rolle's theorem for* f if whenever f is zero on the endpoints of a bounded closed interval of F and differentiable at every interior point, then f' has a zero in the interior.

THEOREM 3.1. *The field F satisfies Rolle's theorem for all rational functions (i.e., quotients of polynomial functions) if and only if F is real closed.*

Proof. The assertion that F satisfies Rolle's theorem for rational functions is equivalent to a countable family of elementary statements, each of which is valid for the real numbers, and hence is valid for any real closed field by the Tarski principle [17]. Now suppose F satisfies Rolle's theorem for rational functions. It then satisfies Rolle's theorem for polynomials, and hence (b) of Theorem 2.1. Moreover the value group of F is 2-divisible. For, if $0 < t \in \pi^{-1}(0)$, then the rational function $x(x + t^{-1})(x - 2)^{-1}$ is defined everywhere on the interval $[-t^{-1}, 0]$ and is zero on the endpoints. Thus the derivative must have a zero, and hence (by the quadratic formula), $16 + 8t^{-1}$ is a square in F. But then

$$v(t) = -v(16 + 8t^{-1}) \in 2\Gamma.$$

It follows that F is real closed [4, §4].

QUESTION 3.2. Suppose F satisfies Rolle's theorem for polynomials. Must it then satisfy Rolle's theorem for all algebraic functions which are integral over $F[x]$? Some cases are easily checked; for example, if $f(x) = g(x)^{1/n}$ for an integer $n > 0$ and polynomial $g(x) \in F[x]$, then an easy application of the chain rule shows that F satisfies Rolle's theorem for f.

4. General fields

Suppose a polynomial $f(x)$ in our ordered field F has distinct roots a and b in F. What can we say about the behavior of f' between a and b? We can write

$$f(x) = (x - a)(x - b) \sum_{0 \le i \le t} s_i \left(\frac{x - a}{b - a}\right)^i$$

for some $s_i \in F$. We will show that we can find $c \in F$ between a and b such that $f'(c)$ is "small", at least in comparison with

$$\left(|b-a| \max_{0 \le i \le t} |s_i|\right)^{-1}.$$

Moreover, a single such choice of c can be made to work simultaneously for a whole family of orderings of F. Let us now be more specific.

Let Y be a family of orderings on F, possibly infinite, such that only a finite number of places into $\mathbf{R}$, say $\pi_1, \pi_2, \dots, \pi_n$, are induced by the orderings of Y. For each $i \le n$, let v_i be the valuation associated with the valuation ring $V_i := \pi_i^{-1}(\mathbf{R})$.

LEMMA 4.1. *There exists $s \in F$ with*

$$v_i(s) = \min\{v_i((b-a)s_k) : 0 \le k \le t\} \quad \textit{for all } i \le n.$$

Proof. For each $i \le n$, pick $t_i \in \{s_0, \dots, s_t\}$ with

$$v_i(t_i) = \min\{v_i(s_k) : 0 \le k \le t\}.$$

Then for any $i \ne j$, $v_i(t_i) \le v_i(t_j)$ and $v_j(t_j) \le v_j(t_i)$. Hence

$$t_iV_iV_j \supseteq t_jV_iV_j = t_jV_jV_i \supseteq t_iV_jV_i = t_iV_iV_j.$$

Now apply Theorem 2.1B of [3].

THEOREM 4.2. *Let f and s be as above. Let $0 < q \in \mathbf{Q}$. Then there exists $c \in F$ such that with respect to every ordering in Y, $|f'(c)| < q|s|$ and c is between a and b.*

Proof. Let $g(x) = s^{-1}(b-a)^{-1}f((b-a)x + a)$, so that

$$g(x) = x(x-1) \sum_{0 \le i \le t} s^{-1}(b-a)s_i x^i.$$

Let $i \le n$. By the choice of s, $\pi_i(g(x))$ is a nonzero polynomial in $\mathbf{R}[x]$ with roots 0 and 1. Hence $\pi_i(g'(x)) = \pi_i(g(x))'$ has a root between 0 and 1. Thus there exist nonzero rational numbers β_i in $(0,1)$ with $|\pi_i(g'(\beta_i))| < q$. Applying Theorem 2.1A of [3] to the map assigning to each π_i the element β_i, we can find $\beta \in F$ with $0 < \pi_i(\beta) < 1$ and $|\pi_i(g'(\beta))| < q$ for all $i \le n$. Now set $c = (b-a)\beta + a$. Then for any ordering in Y, the element c lies between a and b since $0 < \pi_i(\beta) < 1$, where π_i denotes the $\mathbf{R}$-place induced by the ordering. Moreover

$$q > |s^{-1}f'(c)|$$

since

$$q > |\pi_i(g'(\beta))| = \pi_i(|s^{-1}f'((b-a)\beta + a)|).$$

Therefore $|f'(c)| < q|s|$.

Example 4.3. If Y consists of a single Archimedean ordering, then we may take $s = 1$ and Theorem 4.2 says we can make $f'(c)$ as small as we like; this is clearly the best we can hope for since f' need not have a root in F. Next suppose Y consists of the unique ordering on the field $F := \mathbf{Q}(t)(t^{2^{-n}}\colon n = 1, 2, 3, \dots)$ obtained by joining all the 2^n-th roots of t to the field of Laurent series in the indeterminant t over $\mathbf{Q}$. Let $f(x) = x^4 + tx - 1$. This polynomial has two roots in F, near -1 and $+1$. Its derivative $f'(x) = 4x^3 + t$ has no root in F, but $f'(t) = 4t^3 + t$ is less than every rational number; this is a somewhat stronger statement than can be made in Theorem 4.2. On the other hand, the root of $f'(x)$ in the real closure of F cannot be approximated arbitrarily closely by elements of F since it lies in a "gap" with no elements of F near it.

5. Quadratic Forms

By Corollary 2.4, any field satisfying Rolle's theorem for polynomials is hereditarily pythagorean and superpythagorean. Such fields have been intensively studied; one can compute the absolute Galois groups, the Brauer groups, and the Witt rings of such fields very explicitly in terms of their value groups (e.g., see [1], [6]). The next proposition (specifically, part (d)) shows that fields which satisfy Rolle's theorem for polynomials can be used as "local objects" in a theory of quadratic forms over formally real fields.

THEOREM 5.1. *Let F be an ordered field. Let π be the induced place into $\mathbf{R}$ and let Γ be the value group. Let K be an algebraic extension of F. The following statements are equivalent, and they are satisfied by a unique (up to F-isomorphism) algebraic extension of F:*

(a) *K is a minimal ordered extension of F satisfying Rolle's theorem for polynomials.*

(b) *K is a Henselian ordered extension of F with real closed residue class field and with value group $\{\gamma/n\colon \gamma \in \Gamma,\ n \text{ is an odd integer}\}$.*

(c) *K is a maximal extension of F admitting a place π_K into $\mathbf{R}$ such that every ordering of F inducing π extends to an ordering of K inducing π_K.*

(d) *K is a maximal extension of F with respect to being Henselian with real closed residue class field and having the natural map $W(F) \to W(K)$ induce an isomorphism $W(F/T_\pi) \to W(K)$, where $T_\pi = F^{\cdot 2}\pi^{-1}(\mathbf{R}^{\cdot 2})$.*

In (d) above, $W(F)$ denotes the Witt ring [13] of F and $W(F/T_\pi)$ denotes $W(F)$ modulo the ideal generated by all $\langle 1, -t\rangle$ where $t \in T_\pi$ (cf., [8]). For the role of the ring $W(F/T_\pi)$ in the study of the reduced Witt ring of F (i.e. $W(F)$ modulo its nil radical) see [8, Theorem 3.2], which describes the reduced Witt ring as a particular subdirect product of the rings $W(F/T_\pi)$, where π ranges over all places from F into $\mathbf{R}$. Many conditions equivalent to the four conditions of Theorem 5.1 are easily formulated. For example in (c) we could require that all the orderings of 2-power level extend to K (cf. [1]), or that the orderings extend uniquely.

Proof. We can construct a field extension E of F which is Henselian with real closed residue class field and which has value group $\Delta := \{\gamma/n: \gamma \in \Gamma,\ n$ is odd$\}$ (use the standard techniques of value group and residue class adjunction to the Henselization). All the orderings of F associated with π extend uniquely to E (essentially because $\Delta/2\Delta \cong \Gamma/2\Gamma$, cf. [2]). If K is another field satisfying (b), then K and E are F-isomorphic by the isomorphism theorem of [7]. We next show that E also satisfies conditions (a), (c) and (d). For (a), we see that E satisfies Rolle's theorem by Theorem 2.1. The field E is minimal since any subfield of E satisfying Rolle's theorem will have the same residue class field and value group (namely, Δ) as E, and so will equal E [16, p. 236]. For (c), we know that the orderings of F associated with π extend to E. If L is a minimal proper algebraic extension of E which admits any orderings at all, then L is obtained by adjoining the square root of an element of odd value [5, proof of Theorem 1.2]. There is an ordering P of E making this element negative [2]. Then $P \cap F$ cannot possibly extend to L. Thus E is maximal in the sense of (c). Finally for (d), the isomorphism of $W(E)$ with $W(F/T_\pi)$ follows from the natural isomorphism $F^{\cdot}/\pm T^{\cdot} \to E^{\cdot}/\pm E^{\cdot 2}$ (i.e., of $\Gamma/2\Gamma$ and $\Delta/2\Delta$) [8, Proposition 3.1]. Since E is Henselian and Δ is odd divisible, any proper extension of E will kill some ordering. This ordering corresponds uniquely to a minimal prime ideal of $W(F/T_\pi)$, so the induced Witt ring homomorphism cannot be an isomorphism.

Now we show that any field satisfying (a), (c) or (d) must be isomorphic to E. This will complete the proof. Let K satisfy (a). Its value group must contain that of E. Hence some (ordered) totally ramified extension of E has the same value group as K, and hence is isomorphic to K [7]. Thus E is F-isomorphic to K by minimality. Next assume K satisfies (c). The maximality allows us to immediately conclude that K is Henselian with real closed residue class field and odd divisible value group, since otherwise these could be extended as in the construction of E above. Assume K has an element $\alpha \neq 0$ with $2v(\alpha) \in \Gamma \setminus 2\Gamma$, where v denotes the valuation associated with π_K. Then there exists

$b \in F$ with $0 < \pi_K(\alpha^2 b) < \infty$, and so $\alpha^2 b \in K^{\cdot 2}$ by Hensel's lemma. Now b has odd value in Γ and hence is not positive in some ordering P associated with π. But then P cannot possibly extend to an ordering of K since $b \in K^2$. Therefore no such α can exist. This implies $v(K^\cdot) = \Delta$, and so K is isomorphic to E. Finally, assume that K satisfies (d). The hypothesis on K says that its value group Φ is as large as possible, subject to the requirement that $\Gamma/2\Gamma \cong \Phi/2\Phi$ [8, Proposition 3.1]. But this means $\Phi = \Delta$ and again K is isomorphic to E.

6. Fields with finitely many minimal extensions

Let F^{alg} be an algebraic closure of F. By a *minimal extension* of F, we mean a minimal proper field extension of F in F^{alg}. For example, all quadratic extensions of F in F^{alg} are minimal. Thus if F has only finitely many minimal extensions, it must have only finitely many orderings (otherwise the square factor group, and hence the number of quadratic extensions, is infinite). In the theorem below we characterize a class of fields with finitely many orderings which admit only quadratic minimal extensions.

THEOREM 6.1. *Let F be a field with n orderings, where $0 < n < \infty$. The following are equivalent:*

(a) *F has exactly $2n - 1$ minimal extensions.*

(b) *F is superpythagorean and all minimal extensions are quadratic.*

(c) *F is Henselian with respect to a valuation with odd divisible value group and with a residue class field k such that for s equal to 1 or 2, k has exactly s places into $\mathbf{R}$ and k is maximal with respect to exactly s orderings.*

COROLLARY 6.2. *Suppose F has n orderings, $0 < n < \infty$. Then F satisfies Rolle's theorem for polynomials if and only if F has exactly $2n - 1$ minimal extensions and exactly one place into $\mathbf{R}$.*

Proof of 6.2. By Theorem 2.1, F satisfies Rolle's theorem for polynomials if and only if it satisfies condition (c) of 6.1 and F admits a unique place in $\mathbf{R}$ (i.e., "s" = 1).

Remark 6.3. (A) The hypothesis (appearing in 6.1(b)) that F is a field all of whose minimal extensions are quadratic is equivalent to the hypothesis that F has no extensions of odd degree. After all, if F has no extensions of odd degree and E is some minimal extension (say contained in a finite Galois extension K of F), then the Galois group $G(K/F)$ has 2-power order [12, Theorem 57, p. 67] and hence by Sylow's theorem admits a subgroup $H \supseteq G(K/E)$ of index 2; clearly minimality says E is the fixed field of H, so E/F is quadratic.

(B) The condition on the residue class field k in 6.1(c) really says that either k is real closed (if $s = 1$) or k is 3-maximal with two places into **R** (if $s = 2$). For this fact, and many characterizations of 3-maximal fields, see [11], especially Theorem 2.

(C) Suppose F satisfies the conditions of 6.1(c). Each ordering of F induces an ordering on k, so $n = s|\Gamma/2\Gamma|$, where Γ is the value group of F [2]. For example, if $F = \mathbf{R}$, we have $\Gamma = 0$ and $s = 1$, so **R** has a unique minimal extension. If $F = \mathbf{R}((t))$ (field of Laurent series), then Γ is infinite cyclic and $s = 1$, so F has two minimal extensions.

Proof of 6.1. (a) ⇒ (b). Let $\Sigma F^{\cdot 2}$ denote the set of nonzero sums of squares in F. Then

$$2n \leq |F^{\cdot}/\Sigma F^{\cdot 2}| \leq |F^{\cdot}/F^{\cdot 2}| \leq 2n,$$

where the first inequality follows from the fact that each ordering of F corresponds to a character of $F^{\cdot}/\Sigma F^{\cdot 2}$, and the last inequality follows from the hypothesis since the quadratic extensions of F correspond bijectively to the nontrivial square classes of F. It follows that F is superpythagorean [6, Theorem 1] and that all of the minimal extensions are quadratic.

(b) ⇒ (c). Every quadratic extension of F, and thus every proper algebraic extension, must kill an ordering of F. Therefore F has no immediate extensions with respect to τ, the maximal place through which all places from F to **R** factor, and thus F is Henselian with respect to τ. Clearly the value group of τ is odd divisible. The residue class field k of τ has at most two orderings [6, Theorem 1], and hence the number s of places into **R** is at most two. The field k is also pythagorean with no extension of odd degree. If $s = 1$, then k has a unique ordering, and hence every element is a square or its negative is a square and every polynomial of odd degree has a root. It follows that k is real closed, so it is maximal with respect to its ordering. If $s = 2$, then k contains an element a such that ± 1, $\pm a$ represent the four square classes and a is in only one ordering of k. Any proper extension of k must contain one of the quadratic extensions $k(\sqrt{-1})$, $k(\sqrt{a})$ or $k(\sqrt{-a})$, so the orderings of k cannot both extend. Thus k is maximal with respect to its two orderings.

(c) ⇒ (a). By maximality, k has no extensions of odd degree. Since F is also Henselian with odd divisible value group, it follows that F has no extensions of odd degree [16, p. 236]. Hence the minimal extensions of F are quadratic, and correspond bijectively to nontrivial elements of $F^{\cdot}/F^{\cdot 2}$. But

$$|F^{\cdot}/F^{\cdot 2}| = |k^{\cdot}/k^{\cdot 2}|\,|\Gamma/2\Gamma|$$

where Γ is the value group of F. Now since k has at most two orderings, it is superpythagorean (by maximality and [6, Theorem 1]) and has $2s$ square classes. Also $n/s = |\Gamma/2\Gamma|$ by Remark 6.3C.

Thus the number of minimal extensions of F is

$$|F^{\cdot}/F^{\cdot 2}| - 1 = (2s)(n/s) - 1 = 2n - 1.$$

This completes the proof of Theorem 6.1.

The last implication of (6.1) (and "necessity" in Corollary 6.2) can also be deduced from the following general property of Henselian fields.

PROPOSITION 6.4. *Let F be a Henselian field with value group Γ and residue class field k. Suppose k has characteristic zero and exactly n minimal extensions. For each prime p, let $s(p)$ be the dimension of $\Gamma/p\Gamma$ as a vector space over $\mathbf{Z}/p\mathbf{Z}$ and let $t(p)$ denote the number of p-th roots of unity in k. Then F has exactly*

$$n + \sum_{p \text{ prime}} |k^{\cdot}/k^{\cdot p}|(p^{s(p)} - 1)p(p-1)^{-1}t(p)^{-1}$$

minimal extensions if this number is finite, and infinitely many minimal extensions otherwise.

We leave the details of the proof of 6.4 to the interested reader. It is a straightforward counting argument using the computation in [5, §1] of all tamely ramified extensions of F.

We end this section with the observation that F has n minimal extensions if and only if F is "n-maximal" [14].

PROPOSITION 6.5. *Let F be any field (not necessarily formally real). Then F has exactly n minimal proper extensions in some algebraic closure F^{alg} if and only if n is the least integer such that there exists a set A of cardinality n in F^{alg} which is disjoint from F but intersects every proper extension of F in F^{alg} nontrivially (i.e., F is "n-maximal").*

Proof. If F is $\{b_1, \ldots, b_n\}$-maximal with n minimal, then $F(b_1), \ldots, F(b_n)$ are clearly minimal extensions of F. On the other hand, any minimal extension of F must contain some b_i, hence must be the extension $F(b_i)$.

Conversely, if $F(a_1), \ldots, F(a_n)$ is an exhaustive list of minimal extensions of F, then F must be $\{a_1, \ldots, a_n\}$-maximal. If F were s-maximal for $s < n$, then F would have only s minimal extensions as in the first half of the proof. Thus F is n-maximal.

REFERENCES

1. E. BECKER, *Hereditarily-pythagorean fields and orderings of higher level*, Monografias de Matemática, no. 29, Instituto de Matemática Pura e Aplicada, Rio de Janeiro, 1978.
2. R. BROWN, *Real places and ordered fields*, Rocky Mountain J. Math., vol. 1 (1971), pp. 633–636.

3. ______, *An approximation theorem for extended prime spots*, Canadian J. Math., vol. 24 (1972), pp. 167–184.
4. ______, *Extended prime spots and quadratic forms*, Pacific J. Math., vol. 51 (1974), pp. 379–395.
5. ______, *Tamely ramified extensions of Henselian fields*, Rocky Mountain J. Math., vol. 5 (1975), pp. 543–557.
6. ______, *Superpythagorean fields*, J. Algebra, vol. 42 (1976), pp. 483–494.
7. ______, *An isomorphism theorem for Henselian fields with real closed residue class fields*, in preparation.
8. R. Brown and M. Marshall, *The reduced theory of quadratic forms*, Rocky Mountain J. Math., vol. 11 (1981), pp. 161–175.
9. T. Craven and G. Csordas, *Location of zeros. Part II: Ordered fields*, Illinois J. Math., vol. 27 (1983), pp. 279–299.
10. O. Endler, *Valuation theory*, Universitext, Springer-Verlag, New York, 1972.
11. A. Engler and T.M. Viswanathan, *Digging holes in algebraic closures a la Artin-II*, Relatório Interno No. 192, IMECC Universidade Estadual de Campinas, Brazil.
12. I. Kaplansky, *Fields and rings*, Chicago Lectures in Math., The University of Chicago Press, Chicago, 1969.
13. T.Y. Lam, *Algebraic theory of quadratic forms*, Benjamin/Cummings, Reading, Mass., 1973.
14. P.J. McCarthy, *Maximal fields disjoint from certain sets*, Amer. Math. Soc. Proc., vol. 18 (1967), pp. 347–351.
15. M.J. Pelling, *Solution of advanced problem No.* 5861, Amer. Math. Monthly, vol. 88 (1981), pp. 150–152.
16. P. Ribenboim, *Théorie des Valuations*, Les Presses de l'Université de Montréal, Montreal, Quebec, 1964.
17. A. Tarski, *A decision method for elementary algebra and geometry*, 2nd edition, University of California Press, Berkeley, Calif., 1951.

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