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TWO UNIFORM INTRINSIC CONSTRUCTIONS FOR THE LOCAL TIME OF A CLASS OF LÉVY PROCESSES

BY

MARTIN T. BARLOW, EDWIN A. PERKINS AND S. JAMES TAYLOR

Summary

We show that if X is a Lévy process with a regularly varying exponent function and a local time, L_t^x, that satisfies a mild continuity condition, then for an appropriate function ϕ,

$$\phi\text{-}m\{s \leq t | X_s = x\} = L_t^x \quad \forall t \geq 0,\ x \in \mathbf{R} \quad \text{a.s.}$$

Here ϕ-$m(E)$ denotes the Hausdorff ϕ-measure of the set E. In particular if X is a stable process of index $\alpha > 1$, this solves a problem of Taylor and Wendel. We also prove that under essentially the same conditions, a construction of L_t^0 due to Kingman, in fact holds uniformly over all levels.

1. Introduction and statement of results

We study the local time and level sets of a broad class of Lévy processes, i.e., real-valued processes with stationary, independent increments that are continuous in probability. Given such a process, we may select a version, X_t, having right-continuous paths with left limits. Such a version is strong Markov. More precisely, by passing to the canonical space of such paths, we may assume $(\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ is a Hunt process in the sense of Blumenthal and Getoor [5, p. 45]. Here $\{\mathscr{F}_t\}$ denotes the canonical filtration for the Lévy process X, augmented in the usual way (see [5, I.5]). Write P and E for P^0 and E^0, respectively.

The characteristic measure of X_t is given by

$$E(e^{i\lambda X_t}) = e^{-t\psi(\lambda)}, \tag{1.1}$$

where

$$\psi(\lambda) = -ia\lambda + \sigma^2\lambda^2/2 - \int_{-\infty}^{\infty}\left(e^{i\lambda y} - 1 - \frac{i\lambda y}{1+y^2}\right)\mu(dy), \tag{1.2}$$

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and μ is a measure on $\mathbf{R}$ such that $\int_{\mathbf{R}}(1 \wedge y^2)\mu(dy) < \infty$ and $\mu\{0\} = 0$. We assume throughout that

(1.3) 0 is regular for $\{0\}$, and either $\sigma^2 > 0$ or $\mu(\mathbf{R}) = \infty$

or, equivalently (see [6] or [14]),

$$\text{(1.4)} \qquad \begin{aligned} &\int_{-\infty}^{\infty} \operatorname{Re}\left(\frac{1}{s+\psi(\lambda)}\right) d\lambda < \infty\ \forall s > 0, \\ &\qquad\qquad \text{and either } \sigma^2 > 0 \quad \text{or} \quad \int_{-\infty}^{\infty} |x| \wedge 1\, d\mu(x) = \infty. \end{aligned}$$

Then for each x there is a local time, L_t^x, that is continuous in t, jointly measurable in (t, x, ω), and is normalized so that

$$\text{(1.5)} \qquad \int_0^t f(X_s)\, ds = \int_{-\infty}^{\infty} f(x) L_t^x\, dx \quad \forall t \geq 0 \text{ and measurable } f \geq 0 \quad \text{a.s.}$$

(see Getoor and Kesten [11, Thm. 4]).

It is easy to check that if X is a stable process of index $\alpha > 1$, then (1.4) holds and a local time, L_t^x, exists. In this setting, Taylor and Wendel [20] showed that if $\phi(h) = h^{\beta}(\log|\log h|)^{1-\beta}$, where $\beta = 1 - 1/\alpha$, then for some constant $c > 0$,

$$\text{(1.6)} \qquad \phi\text{-}m\{s \leq t | X_s = 0\} = cL_t^0, \quad \forall t \geq 0 \quad \text{a.s.}$$

Here ϕ-$m(E)$ denotes the Hausdorff ϕ-measure of the set E.

Fristedt and Pruitt [9] extended (1.6) to general Lévy processes (in fact, to any strong Markov process with a local time at 0). In order to describe their results in our setting we need some notation. Let

$$\tau^x(t) = \inf\{s | L_s^x > t\}$$

and write τ for τ^0. The strong Markov property of X shows that τ is a subordinator (Blumental and Getoor [5, V, Thm. 3.21]) and therefore has Laplace transform

$$\text{(1.7)} \qquad E(e^{-s\tau(t)}) = e^{-tg(s)},$$

where

$$\text{(1.8)} \qquad g(s) = bs + \int_0^{\infty} (1 - e^{-sr})\nu(dr).$$

ν is the Lévy measure of τ and satisfies

$$(1.9)\qquad \int_0^\infty r \wedge 1\nu(dr) < \infty, \quad \nu\{0\} = 0.$$

As $\int_0^\infty I(X_s = 0)\, ds = 0$ a.s. (by (1.3)), b must be zero and the continuity of $L^0_\cdot$ implies $\nu(0, \infty) = \infty$. (Throughout this work $I(A)$ denotes the indicator function of the set A.) Using (1.8) and (4.9) of Getoor and Kesten [11], one can obtain g from ψ as follows:

$$(1.10)\qquad g(s) = 2\pi\left[\int_{-\infty}^\infty \mathrm{Re}\frac{1}{s + \psi(\lambda)}\, d\lambda\right]^{-1}.$$

(1.7) shows that $g(s)$ is strictly increasing to ∞ as $s \uparrow \infty$ and so g has a well-defined inverse, η. For t small enough,

$$(1.11)\qquad h(t) = \frac{\log|\log t|}{\eta(t^{-1}\log|\log t|)} \quad (\eta = g^{-1})$$

is increasing, and approaches zero as $t \downarrow 0$ (see (9) in Fristedt and Pruitt [9]). Let

$$(1.12)\qquad f = h^{-1},$$

i.e., f is the continuous increasing inverse of h $(f(0) = 0)$. As we will only be interested in the asymptotic behavior of $f(t)$ as $t \downarrow 0$, the fact that $f(t)$ is only defined for small enough t will not concern us. Fristedt and Pruitt [9, Theorem 3] showed that there is a constant $c > 0$ such that for each x,

$$(1.13)\qquad f\text{-}m\{s \le t | X_s = x\} = cL_t^x, \quad t \ge 0 \quad \text{a.s.}$$

Hawkes [13, Thm. 1, 2] proved $1/2 \le c \le 1$ and noted that in the α-stable case,

$$c = \beta^\beta(1 - \beta)^{1-\beta} \quad (\beta = 1 - 1/\alpha),$$

thus showing that this range is best possible.

It is then natural to ask if

$$(1.14)\qquad f\text{-}m\{s \le t | X_s = x\} = cL_t^x,\ t \ge 0, \quad x \in \mathbf{R} \quad \text{a.s.}$$

It is not hard to see that the left side of (1.14) defines a measurable $(in(t, x))$ version of cL_t^x, and in general this is as much as one can say. If, however, the local time of X has a jointly continuous version L_t^x, then (1.14) becomes an interesting question concerning the "continuity" of the level sets in the space variable. This problem was posed by Taylor and Wendel [20] in the stable case and solved by Perkins [18] if X is a Brownian motion. In the Brownian case we have [18, Theorem 1]

$$\phi\text{-}m\{s \le t | X_s = x\} = L_t^x \quad \forall t \ge 0,\ x \in \mathbf{R}, \tag{1.15}$$

where $\phi(t) = (2t \log\log 1/t)^{1/2}$. We will extend this result to a broad class of Lévy processes, including the stable processes of index $\alpha > 1$, and thus give a complete answer to the question posed in [20]. Before stating the theorem, we recall that sufficient conditions for the existence of a jointly continuous local time given by Getoor and Kesten [11, Thm. 4] and improved by Barlow [1].

Notation.

$$\delta_0(x) = \pi^{-1}\int_{-\infty}^{\infty}(1 - \cos\lambda x)\,Re\left(\frac{1}{1+\psi(\lambda)}\right)d\lambda$$
$$(< \infty \text{ by } (1.4)),$$

$$\delta(x) = \sup_{|u|\le x} \delta_0(x) \tag{1.16}$$

$$\rho(y) = \int_0^y (\log u^{-1})^{1/2}\, d(\delta^{1/2}(u)) \quad (0 \le y < 1). \tag{1.17}$$

THEOREM 1.1 (Barlow [1, Thm. 1.1]). *If ρ is finite, then X has a jointly continuous local time, L_t^x. Moreover, there is a constant $c > 0$, and for each $t > 0$, an $\varepsilon_t(\omega) > 0$* a.s. *such that*

$$\sup_{s\le t}|L_s^a - L_s^b| \le c\left(\sup_x L_t^x\right)^{1/2}\rho(|b-a|) \quad \textit{for all } |b-a| \le \varepsilon_t(\omega). \tag{1.18}$$

Notation. If $\beta \in [0,1]$, let $c(\beta) = \beta^\beta(1-\beta)^{1-\beta}$, where $0^0 \equiv 1$.

Recall that ϕ: $[0,\infty) \to [0,\infty)$ varies regularly at ∞ with exponent γ if

$$\lim_{t\to\infty}\phi(ct)/\phi(t) = c^\gamma \quad \forall c > 0.$$

We are ready to state our first theorem.

THEOREM 1.2. *Let X be a Lévy process with characteristic function given by* (1.1) *and let g and δ be given by* (1.10) *and* (1.16), *respectively. Assume* (1.4),

(R_β) *g varies regularly at* ∞ *with exponent* β,
and

(H) $\delta(x) \le (\log 1/x)^{-3-\varepsilon}$ *for small* $x > 0$ *and some* $\varepsilon > 0$.
Then $0 \le \beta \le 1/2$ *and X has a jointly continuous local time,* L_t^x, *which we normalize by* (1.5). *If f is given by* (1.11), (1.12), *then*

$$f\text{-}m\{s \le t \mid X_s = x\} = c(\beta)L_t^x \quad \forall t \ge 0,\ x \in \mathbf{R} \quad \text{a.s.}$$

The proof is given in Sections 4 and 5. The existence of a jointly continuous local time is immediate from Theorem 1.1. Indeed, that result shows that if $\delta(x) \le (\log 1/x)^{-1-\varepsilon}$ for small x and some $\varepsilon > 0$, then a jointly continuous local time exists. On the other hand, there are examples of Lévy processes for which $\delta(x) \sim (\log(1/x))^{-1}$ as $x \downarrow 0$ (this means $\lim_{x\downarrow 0}\delta(x)\log 1/x = 1$) and no jointly continuous version of local time exists (Getoor and Kesten [11, Section 4, e.g., (c)], Millar and Tran [16]). Hence condition (H) is slightly stronger than that needed to ensure the joint continuity of local time. Nonetheless in [3] (where a preliminary version of Theorem 1.2 is stated without proof) we conjectured that if X is a Lévy process with a jointly continuous local time, then for some $c > 0$,

$$f\text{-}m\{s \le t \mid X_s = x\} = cL_t^x \quad \forall t \ge 0,\ x \in \mathbf{R} \quad \text{a.s.} \tag{1.19}$$

It is not hard to show that (R_β) is implied by the regular variation of ψ in the following sense:

PROPOSITION 1.3. *Assume* (1.4). *If*

$$\lim_{|\lambda|\to\infty} \frac{\operatorname{Re}\psi(c\lambda)}{c^\gamma \operatorname{Re}\psi(\lambda)} = \lim_{|\lambda|\to\infty} \frac{\psi(c\lambda)}{c^\gamma\psi(\lambda)} = 1$$

for all $c > 0$ *and some* $\gamma > 0$ *then* ($R_{1-1/\gamma}$) *holds.*

The proof is given in Section 3.

The proof of Theorem 1.2 uses some of the techniques employed in [18] to prove (1.15) (many of which originated in [20]). The arguments in [18], however, were simplified by the continuity of Brownian sample paths and the use of the Ray-Knight theorems for Brownian local time, two results which do not hold for a general Lévy process. We will have to rely more heavily on the spatial continuity of local time. Moreover, the L^p estimates obtained in [18, Lemma 1] by using Burkholder's inequalities will no longer be strong enough when $\beta = 0$ in Theorem 1.2. Instead, the following exponential inequalities, taken from Freedman [8, Thm. 4.1] will be used extensively.

Notation. If $\{(M_n, \mathcal{F}_n)|n = 0, 1, \dots\}$ is a sub- or supermartingale, let

$$\langle M \rangle_n = \sum_{k=1}^{n} E\left((M_k - M_{k-1})^2 | \mathcal{F}_{k-1}\right)$$
$$\langle M \rangle = \langle M \rangle_\infty$$

THEOREM 1.4 (Freedman). *Suppose that* $\{(M_n, \mathcal{F}_n)|n = 0, 1, \dots\}$ *is a supermartingale with* $M_0 = 0$ *and* $M_n - M_{n-1} \le c$, $\forall n \ge 1$ a.s. *Then for any* $\varepsilon, \sigma^2 > 0$,

$$P\left(\sup_n M_n \ge \varepsilon, \langle M \rangle \le \sigma^2\right) \le \exp\left\{\frac{-\varepsilon^2}{2(\sigma^2 + \varepsilon c)}\right\}.$$

Our second result (a preliminary version was also stated without proof in [3]) extends a construction of L_t^0 due to Kingman [15], to a global construction of L_t^x for all $x \in \mathbf{R}$.

Notation. $a(t, x, \varepsilon) = \{s \in \mathbf{R} | \exists u \le t \text{ such that } |u - s| \le \varepsilon/2 \text{ and } X_u = x\}$.
m denotes Lebesgue measure.
Write $p(x) \ll q(x)$ as $x \downarrow 0$ if $\lim_{x \downarrow 0} p(x)/q(x) = 0$.

THEOREM 1.5. *Let X be a Lévy process whose characteristic function is given by* (1.1) *and satisfies* (1.4). *Assume g and* δ *(given by* (1.10) *and* (1.16)) *satisfy* (R_β) *and*

$$\delta(x) \ll (\log 1/x)^{-2} \quad \text{as } x \downarrow 0, \tag{1.20}$$

respectively. Then $0 \le \beta \le 1/2$, *X has a jointly continuous local time* L_t^x *which we normalize by* (1.5), *and*

$$\lim_{\varepsilon \downarrow 0} \sup_{\substack{x \in R \\ t \le T}} \left| \frac{\Gamma(2 - \beta) m(a(t, x, \varepsilon))}{\varepsilon g(1/\varepsilon)} - L_t^x \right| = 0 \quad \forall T > 0 \quad \text{a.s.}$$

The proof is given in Sections 6 and 7. The result is known for x fixed when X is a standard Markov process having a local time at x (see Kingman [15] or Fristedt and Taylor [10, Corollary 7.2]). In the case where X is a Brownian motion, the theorem is proved in Perkins [17, Thm. 4.11]. Again the continuity of Brownian paths, simplifies the arguments.

Although many of the techniques used in the proofs of Theorems 1.2 and 1.5 are similar (e.g., Theorem 1.4), each has its own source of complications. The fundamental problem encountered in Theorem 1.2 (especially in the "upper bound" argument in Section 5) is that the optimal coverings of the level sets, which give the Hausdorff measure, involve random times that are not stopping times. In the proof of Theorem 1.5 we deal only with stopping times. On the other hand, a deterministic result that gives a lower bound for the Hausdorff measure of a set (Taylor and Wendel [20, Lemma 4]) simplifies the "lower bound" portion of the proof of Theorem 1.2. This part of the proof (Section 4) is even simpler than that given in [18] for the Brownian case. It is the absence of such a result that makes the "lower bound" portion of the proof of Theorem 1.5 (in Section 6) more difficult.

Hence our main results show that two particular constructions for L_t^x, known to be valid for each x a.s., in fact hold uniformly in x a.s. It is important to point out that this is not always true even in the case of Brownian motion. In [3] (see also [2]), it is shown that for any Lévy process with a jointly continuous local time, L_t^x, there are constructions of L_t^x that are valid for each x a.s. but fail a.s. at some $x(\omega)$, in fact for an uncountable dense set of levels.

Here is an outline of the paper. In Section 2 we gather together some preliminary results concerning subordinators in general. In particular if the subordinator has a regularly varying exponent function, then exact constants are found in the lim sup behaviour at 0 of the first passage time process (Theorem 2.9). In Section 3 we return to the setting described above and present some preliminary lemmas. Theorems 1.2 and 1.5 are proved in Sections 4, 5 and 6, 7, respectively. Finally, in Section 8 we consider some examples of Lévy processes to which Theorems 1.2 and 1.5 apply. These include Lévy processes with a Brownian component (i.e., $\sigma^2 > 0$ in (1.2)), stable processes of index $\alpha > 1$, and some "critical" processes which are close to Cauchy and illustrate the gap between our hypotheses and those needed to ensure the existence of a jointly continuous local time.

We use c to denote several positive constants whose exact values are unimportant. Hence the value of c may change from line to line. The complement of a set A is denoted by A^c.

2. Some preliminaries about subordinators

In this section our point of view is more general than that in the introduction, as $\tau(t)$ is assumed only to be a driftless subordinator with an infinite Lévy measure, ν, and not necessarily the inverse local time of a Lévy process. $g(u)$ is still given by (1.7) and (1.8) (with $b = 0$), f is still defined by (1.11) and (1.12) and the first passage time to a is denoted by

$$P(a) = \inf\{t \mid \tau(t) > a\}.$$

The following simple, but fundamental, lemma is taken from Fristedt and Pruitt [9, Lemma 1].

LEMMA 2.1. *For any* $w > 0$,

$$\frac{e^{-tg(w)} - e^{-wa}}{1 - e^{-wa}} \le P(\tau(t) < a) \le e^{-tg(w)+wa}, \tag{2.1}$$

$$P(\tau(t) \ge a) \le \frac{tg(w)}{1 - e^{-wa}}. \tag{2.2}$$

■

LEMMA 2.2. (*a*) $\frac{1}{3}(g(1/a))^{-1} \le E(P(a)) \le E(P(2a)) \le e^2 g(1/a)^{-1}$.
(b) *If* $y \ge (2g(1/a))^{-1}$, *then*

$$E(P(a/2) \wedge y) \ge \left(1 - \tfrac{3}{2}e^{-1/2}\right) g(1/a)^{-1}.$$

(c) $E(P(2a)^2) \le 2e^2 g(1/a)^{-2}$.

Proof. (a) and (b) are slight modifications of [9, Lemma 6].
(c)

$$\begin{aligned} E\left(P(2a)^2\right) &= \int_0^\infty 2vP(\tau(v) \le 2a)\, dv \\ &\le \int_0^\infty 2ve^{-vg(1/a)+2}\, dv \quad \text{(by (2.1))} \\ &= 2e^2 g(1/a)^{-2}. \end{aligned}$$

■

Although Lemma 2.1 will suffice for most purposes in order to find the exact constant c $(= c(\beta))$ in Theorem 1.2 we must refine the lower bound in (2.1).

LEMMA 2.3. *For any* $\varepsilon \in [0,1]$, $x > u > w > 0$,

$$\begin{aligned} P(\tau(t) < a) \ge e^{ua(1-\varepsilon)}\Big[& e^{-tg(u)}(1 - e^{-ua}) \\ & - g(u)(g(u) - g(w))^{-1} \exp\{-a(u - w) - tg(w)\} \\ & - u(x - u)^{-1} \exp\{-tg(x) + a(1 - \varepsilon)(x - u)\}\Big] \end{aligned}$$

Proof. Let

$$\mathcal{G}_t = \bigcap_{s > t} \sigma(\tau(u) | u \le s).$$

Use the strong Markov property at the $\{\mathscr{G}_t\}$-stopping time $P(a)$ to get

$$\begin{aligned}
&E\big(e^{-u\tau(t)}I(\tau(t) \geq a)\big)\\
&\quad = E\big(I(P(a) \leq t)e^{-u\tau(P(a))}E\big(e^{-u(\tau(t)-\tau(P(a)))}|\mathscr{G}_{P(a)}\big)\big)\\
&\quad \leq e^{-ua}E\big(I(P(a) \leq t)e^{-(t-P(a))g(u)}\big)\\
&\quad = e^{-ua-tg(u)}\left[\int_0^t e^{sg(u)}P(s < P(a))\,ds\,g(u) + 1\right]\\
&\quad \leq e^{-ua-tg(u)}\left[\int_0^t \exp\{sg(u) - sg(w) + wa\}\,ds\,g(u) + 1\right]
\end{aligned}$$

for any $w \in [0, u)$ by (2.1). Evaluating this integral, we find

$$(2.3)\quad E\big(e^{-u\tau(t)}I(\tau(t) \geq a)\big) \leq e^{-a(u-w)-tg(w)}g(u)(g(u) - g(w))^{-1} + e^{-ua-tg(u)} \quad \forall w \in (0, u).$$

On the other hand for each $\varepsilon \in [0,1]$, one has

$$\begin{aligned}
(2.4)\qquad E\big(e^{-u\tau(t)}I(\tau(t) < a)\big) &= E\left(\int_0^\infty e^{-us}I(\tau(t) < s \wedge a)\,ds\,u\right)\\
&\leq \int_0^{a(1-\varepsilon)} e^{-us}P(\tau(t) < s)\,ds\,u\\
&\quad + e^{-ua(1-\varepsilon)}P(\tau(t) < a).
\end{aligned}$$

Use (2.1) again to show that for any $x > u$, the first term in (2.4) is bounded by

$$\int_0^{a(1-\varepsilon)} e^{-tg(x)}e^{s(x-u)}\,ds\,u \leq \exp\{-tg(x) + a(1-\varepsilon)(x-u)\}u(x-u)^{-1}.$$

Substitute this into (2.4) and then add the resulting inequality to (2.3) to obtain (for $x > u > w > 0$)

$$\begin{aligned}
e^{-tg(u)} \leq{}& \exp\{-a(u-w) - tg(w)\}g(u)(g(u) - g(w))\\
&+\exp\{-ua - tg(u)\}\\
&+\exp\{-tg(x) + a(1-\varepsilon)(x-u)\}u(x-u)^{-1}\\
&+\exp\{-ua(1-\varepsilon)\}P(\tau(t) < a).
\end{aligned}$$

The result follows. ∎

We now consider the hypothesis (R_β). An easy computation using (1.8) (with $b = 0$) and (1.9) shows that $\lim_{s\to\infty} g(s)/s = 0$. As $g(s)$ increases to ∞ as $s \uparrow \infty$, it follows that if (R_β) holds, then $\beta \in [0,1]$ (see Lemma 2.4(a) below).

We recall some classical results on regular variation (due to Karamata) from Feller [7].

LEMMA 2.4. (a) [7, p. 282] *Suppose ϕ: $[0,\infty) \to [0,\infty)$ varies regularly at ∞ with exponent γ. There are functions $a(s)$, $\varepsilon(s)$ such that $a(s) \to c > 0$, $\varepsilon(s) \to 0$ as $s \to \infty$ and*

$$\phi(s) = a(s)s^\gamma \exp\left\{\int_1^s \varepsilon(y)/y\,dy\right\}.$$

(b) [7, p. 279] *Suppose ϕ: $[0,\varepsilon) \to [0,\infty)$ is regularly varying at 0 with exponent $\gamma > -1$. Then*

$$\int_0^\delta \phi(s)\,ds \sim \delta\phi(\delta)(\gamma+1)^{-1} \quad \text{as } \delta \downarrow 0.$$

(c) [7, *p*. 446] *If $\phi(s) \geq 0$ is monotone and $\omega(\lambda) = \int_0^\infty e^{-\lambda s}\phi(s)\,ds$ varies regularly at ∞ with exponent $-\gamma < 0$, then $\phi(s) \sim s^{-1}\omega(1/s)\Gamma(\gamma)^{-1}$ as $s \downarrow 0$.* ■

In (c) we have used Theorem 4 in [7, p. 446] with $\lambda \to \infty$ and $x \to 0$ instead of $\lambda \to 0$ and $x \to \infty$ (see [7, Thm. 3, p. 445]).

We state some simple consequences of (R_β) for future reference. We assume (R_β) throughout the rest of this section.

LEMMA 2.5. (a) *If $\beta < 1$, then $\lim_{\varepsilon\downarrow 0}\nu(\epsilon,\infty)g(1/\varepsilon)^{-1}\Gamma(1-\beta) = 1$.*
(b) *f varies regularly at 0 with exponent β.*
(c) $\lim_{a\downarrow 0} E(P(a))g(1/a)\Gamma(\beta+1) = 1$.

Proof. (a) This is an easy consequence of Lemma 2.4(c) and an integration by parts.
(b) This is a routine computation.
(c)

$$E\left(\int_0^\infty e^{-\lambda a}\,dP(a)\right) = E\left(\int_0^\infty e^{-\lambda\tau(u)}\,du\right) = \int_0^\infty e^{-ug(\lambda)}\,du = g(\lambda)^{-1} \quad (\lambda > 0).$$

Integrate by parts on the left to see that

$$\begin{aligned}\int_0^\infty e^{-\lambda a}\,dP(a) &= \lim_{N\to\infty} e^{-\lambda N}P(N) + \lambda\int_0^N P(a)e^{-\lambda a}\,da \\ &= \lambda\int_0^\infty P(a)e^{-\lambda a}\,da\end{aligned}$$

(as the latter integral must converge, $\lim_{N\to\infty}\inf e^{-\lambda N}P(N) = 0$). It follows that

$$\int_0^\infty E(P(a))e^{-\lambda a}\,da = (\lambda g(\lambda))^{-1}.$$

Lemma 2.4(c) now gives (c). ∎

LEMMA 2.6. *For any $\varepsilon > 0$ there is a $u_0 > 0$ such that if $u \in (0, u_0)$ then*

$$P\big(P(u) > (1+\varepsilon)c(\beta)^{-1}f(u)\big) \le |\log f(u)|^{-1-\varepsilon/2}.$$

Proof. Fix $\varepsilon > 0$. Apply (2.1) with

$$w = w(u) = su^{-1}\log|\log f(u)|$$

(s is a positive constant whose exact value will be chosen later) to get

$$(2.5)\qquad \begin{aligned} &P\big(P(u) > (1+\varepsilon)c(\beta)^{-1}f(u)\big)\\ &\quad\le \exp\big\{-(1+\varepsilon)c(\beta)^{-1}f(u)\\ &\qquad\quad \times g\big(su^{-1}\log|\log f(u)|\big) + s\log|\log f(u)|\big\}. \end{aligned}$$

The definition of f implies (see (16) of [9])

$$(2.6)\qquad g\big(u^{-1}\log|\log f(u)|\big) = \log|\log f(u)|f(u)^{-1}.$$

As $f(u)\downarrow 0$, as $u\downarrow 0$, $w(u)$ must increase to ∞ as $u\downarrow 0$. The regular variation of g at ∞ and (2.5) imply that for some $u_0 > 0$ and all $u \in (0, u_0)$,

$$\begin{aligned} &P\big(P(u) > (1+\varepsilon)c(\beta)^{-1}f(u)\big)\\ &\quad\le \exp\big\{-(1+3\varepsilon/4)c(\beta)^{-1}s^{\beta}f(u)g\big(u^{-1}\log|\log f(u)|\big)\\ &\qquad\qquad + s\log|\log f(u)|\big\}\\ &\quad= |\log f(u)|^{-(1+3\varepsilon/4)c(\beta)^{-1}s^{\beta}+s}\quad (\text{by } (2.6)). \end{aligned}$$

The result follows by setting

$$s = \begin{cases} \varepsilon/4 & \text{if } \beta = 0\\ \beta(1-\beta)^{-1} & \text{if } \beta \in (0,1)\\ (1+\varepsilon/2)(3\varepsilon/4)^{-1} & \text{if } \beta = 1. \end{cases}$$

∎

LEMMA 2.7. *If $\theta > c(\beta)$ there are constants ρ, γ in $(0,1)$ and $\alpha > 1$ such that if $t_k = e^{-k^\alpha}$, then for large enough k,*

$$P(\tau(t_k) < \rho h(\theta t_k)) \geq k^{-\gamma}.$$

Proof. Let r_k denote the above probability. If $\beta = 0$ or 1, the result is easily obtained from (2.1) (see the derivation of (6) in Hawkes [13]). Now assume $1 > \beta > 0$. Choose $\alpha > 1$, ε, $\rho \in (0,1)$, and $\theta > c(\beta)$. Let $t_k = e^{-k^\alpha}$, and, recalling that η is the inverse function to g, set

$$\begin{aligned}
f_k &= \log|\log \theta t_k| \sim \alpha \log k \quad \text{as } k \to \infty,\\
a_k &= \rho h(\theta t_k) = \rho f_k \eta(f_k/\theta t_k)^{-1},\\
w_k &= b\eta(f_k/\theta t_k),\\
u_k &= c\eta(f_k/\theta t_k),\\
x_k &= d\eta(f_k/\theta t_k),
\end{aligned}$$

where $d > c > b > 0$. Lemma 2.3 and (R_β) show that for any $\varepsilon' > 0$ and for large enough k,

$$\begin{aligned}
r_k \geq e^{(1-\varepsilon)c\rho f_k} \Big[&\exp\{-c^\beta(1+\varepsilon')f_k/\theta\}(1 - e^{-c\rho f_k})\\
&-(1+\varepsilon')c^\beta(c^\beta - b^\beta)^{-1}\exp\{-\rho f_k(c-b) - (1-\varepsilon')b^\beta f_k/\theta\}\\
&-c(d-c)^{-1}\exp\{-(1-\varepsilon')d^\beta f_k/\theta + (1-\varepsilon)\rho f_k(d-c)\}\Big]\\
\geq k^{(1-2\varepsilon)c\rho\alpha}\Big[&k^{-(1+2\varepsilon')c^\beta\alpha/\theta} - (1+\varepsilon')c^\beta(c^\beta - b^\beta)^{-1}k^{-\alpha(1-\varepsilon')(\rho(c-b)+b^\beta/\theta)}\\
&-c(d-c)^{-1}k^{-(1-2\varepsilon')\alpha d^\beta/\theta + (1-\varepsilon)\rho\alpha(d-c)}\Big].
\end{aligned}$$

Choose $c > 0$ such that

$$\beta c^{\beta-1} = (1 - \varepsilon/2)\rho\theta \tag{2.7}$$

and then choose $b < c < d$ such that

$$\frac{c^\beta - b^\beta}{c - b} < \rho\theta, \qquad \frac{d^\beta - c^\beta}{d - c} > (1-\varepsilon)\rho\theta.$$

Now we may select $\varepsilon' > 0$ such that

$$\begin{aligned}
(1-\varepsilon')(\rho(c-b) + b^\beta/\theta) &> (1+2\varepsilon')c^\beta/\theta,\\
(1-2\varepsilon')d^\beta/\theta - (1-\varepsilon)\rho(d-c) &> (1+2\varepsilon')c^\beta/\theta,
\end{aligned}$$

and so for large enough k,

$$r_k \geq k^{(1-3\varepsilon)c\rho\alpha-(1+2\varepsilon')c^{\beta}\alpha/\theta}.$$

First substitute for c (use (2.7)), and then note that the exponent on k may be made greater than -1 by taking ε, ε' small enough, and α and ρ close enough to 1 (use $\theta > c(\beta)$ here).

We are now ready to state a key lemma for the derivation of the upper bound for the Hausdorff measure of the level sets in Theorem 1.2.

LEMMA 2.8. *If $\theta > c(\beta)$, there are constants γ_0, $\lambda > 0$ and $\varepsilon \in (0,1)$ such that*

$$P\left(\sup_{\gamma \leq u \leq \delta} \frac{P(u)}{f(u)} < \theta^{-1} \right) \leq \exp\{-|\log f(\gamma)|^{-\lambda}\},$$

whenever $0 < \gamma < \gamma_0$ and $f(\delta) \geq f(\gamma)\exp\{|\log f(\gamma)|^{\varepsilon}\}$.

Proof. As there are already three detailed proofs of various versions of this result in the literature we are not going to clutter things up with a fourth. The proof is similar to that of Lemma 3 in Hawkes [13]. The only significant difference is that Lemma 2.7 is used to prove the appropriate version of (6) in [13]. Although our condition on $f(\delta)$ and $f(\gamma)$ appears weaker than that in [13], it is obtained by making some trivial modifications to the argument given there. ∎

The above result, originally due to Taylor and Wendel [20, Lemma 3] for stable subordinators, was generalized by Fristedt and Pruitt [9, Lemma 7] for general subordinators (although they did not attempt to find the smallest possible value of θ), and then refined by Hawkes [13, Lemma 3] (who did). The novelty of our version of this result is that, under (R_β), it applies for all $\theta > c(\beta)$ and not just $\theta > 1$. Theorem 2.9 shows this is the smallest possible value of θ.

Fristedt and Pruitt [9, Theorem 2] showed that

$$\limsup_{u \to 0+} \frac{P(u)}{f(u)} = c \quad \text{a.s.}$$

for some constant $c \in [1,2]$. If τ is a stable subordinator of index β then it is known that (see the proof of Theorem 2 in Hawkes [13]) the above limit is $c(\beta)^{-1}$. Using Lemmas 2.5 and 2.8, one can easily extend this latter result:

THEOREM 2.9. *Assume (R_β). Then*

$$\limsup_{u \to 0+} \frac{P(u)}{f(u)} = c(\beta)^{-1} \quad \text{a.s.}$$ ∎

Remark. Lemma 2.7 (and hence Lemma 2.8 and Theorem 2.9) could also have been derived from a recent result of Jain and Pruitt [21, Theorem 5.3] which gives precise estimates on $P(\tau_t < \varepsilon_t)$ as $t \downarrow 0$.

3. Some preliminaries about Lévy processes

In the rest of this paper our setting is that described in the introduction. That is, $\tau^x(t)$ now denotes the inverse of the local time (at x), L_t^x, of the Lévy process X. We state the following elementary result without proof. Recall that μ is the Lévy measure of X.

LEMMA 3.1.

$$\lim_{|\lambda| \to \infty} \lambda^{-2}\left[\int_{-\infty}^{\infty} e^{ix\lambda} - 1 - ix\lambda(1+x^2)^{-1}\mu(dx)\right] = 0. \qquad \blacksquare$$

LEMMA 3.2. *If (R_β) holds, then $\beta \in [0,1/2]$.*

Proof. It suffices to show $g(s)s^{-1/2}$ is bounded as $s \to \infty$ (use Lemma 2.4(a)). If $\psi = \psi_1 + i\psi_2$ where ψ_j is real-valued then $\psi_1 \geq 0$ and $|\psi(\lambda)| \leq K(\lambda^4+1)^{1/2}$ for all λ and some $K \geq 1$, by Lemma 3.1. Therefore for $s \geq K$,

$$\begin{aligned} \mathrm{Re}\left(\frac{1}{s+\psi(\lambda)}\right) &= \frac{s+\psi_1(\lambda)}{(s+\psi_1(\lambda))^2+\psi_2(\lambda)^2} \\ &\geq \frac{s}{2}\left(\frac{1}{s^2+K(\lambda^4+1)}\right) \\ &\geq \frac{s}{4k}\frac{1}{(s^2+\lambda^4)}. \end{aligned}$$

Therefore (1.10) implies for $s \geq K$,

$$\begin{aligned} g(s) &\leq 2\pi\frac{(4K)}{s}\left[\int_{-\infty}^{\infty}(s^2+\lambda^4)\,d\lambda\right]^{-1} \\ &= 2\pi\frac{(4K)}{s}\frac{\sqrt{2}}{\pi}s^{3/2} \leq cs^{1/2} \end{aligned}$$

by a contour integration.

To prove Proposition 1.3 we need the following:

LEMMA 3.3. *Assume* $\psi_1, \psi_2\colon \mathbf{R} \to \mathbf{C}$ *are continuous and satisfy*
(i) $\operatorname{Re}\psi_i \geq 0$, $i = 1, 2$,
(ii) $h_i(s) = \int_{-\infty}^{\infty} \operatorname{Re}\left(\frac{1}{s + \psi_i(\lambda)}\right) d\lambda < \infty \quad \forall s > 0$, $i = 1, 2$.

If

$$\lim_{|\lambda| \to \infty} \psi_1(\lambda)/\psi_2(\lambda) = \lim_{|\lambda| \to \infty} \operatorname{Re}\psi_1(\lambda)/\operatorname{Re}\psi_2(\lambda) = 1,$$

then

$$\lim_{s \to \infty} h_1(s)/h_2(s) = 1.$$

Proof.

$$\frac{\operatorname{Re}\left(\dfrac{1}{s + \psi_1(\lambda)}\right)}{\operatorname{Re}\left(\dfrac{1}{s + \psi_2(\lambda)}\right)} = \frac{s + \operatorname{Re}\psi_1(\lambda)}{s + \operatorname{Re}\psi_2(\lambda)} \; \frac{|s + \psi_2(\lambda)|^2}{|s + \psi_1(\lambda)|^2}.$$

Now for $s > 0$,

$$\left|\frac{s + \operatorname{Re}\psi_1(\lambda)}{s + \operatorname{Re}\psi_2(\lambda)} - 1\right| = \frac{\left|\dfrac{\operatorname{Re}\psi_1(\lambda)}{\operatorname{Re}\psi_2(\lambda)} - 1\right|}{|s/\operatorname{Re}\psi_2(\lambda) + 1|} \leq \left|\frac{\operatorname{Re}\psi_1(\lambda)}{\operatorname{Re}\psi_2(\lambda)} - 1\right| \quad \text{(by (i))}$$

and

$$\left|\frac{|s + \psi_2(\lambda)|}{|s + \psi_1(\lambda)|} - 1\right| \leq \frac{|\psi_2(\lambda)/\psi_1(\lambda) - 1|}{|s/\psi_1(\lambda) + 1|} \leq |\psi_2(\lambda)/\psi_1(\lambda) - 1| \quad \text{(by (i))}.$$

The above results show that for $\varepsilon > 0$ there is an M such that

$$\left|\frac{\operatorname{Re}\left(\dfrac{1}{s + \psi_1(\lambda)}\right)}{\operatorname{Re}\left(\dfrac{1}{s + \psi_2(\lambda)}\right)} - 1\right| < \varepsilon \quad \text{for all } s > 0 \text{ and } |\lambda| > M.$$

It is now clear that there is an s_0 such that

$$\left| \frac{\operatorname{Re}\left(\dfrac{1}{s + \psi_1(\lambda)} \right)}{\operatorname{Re}\left(\dfrac{1}{s + \psi_2(\lambda)} \right)} - 1 \right| < \varepsilon \quad \text{for all } \lambda \text{ real and } s > s_0,$$

and therefore $1 - \varepsilon < h_1(s) h_2(s)^{-1} < 1 + \varepsilon$ for $s > s_0$. ■

Proof of Proposition 1.3. Fix $c > 0$ and apply the above lemma with $\psi_1(\lambda) = \psi(c\lambda)$ and $\psi_2(\lambda) = c^\gamma \psi(\lambda)$ ((ii) holds by (1.4)) to get

$$\begin{aligned} 1 &= \lim_{s \to \infty} \int_{-\infty}^{\infty} \operatorname{Re}\left(\frac{1}{s + \psi(c\lambda)} \right) d\lambda \left(\int_{-\infty}^{\infty} \left(\frac{1}{s + c^\gamma \psi(\lambda)} \right) d\lambda \right)^{-1} \\ &= \lim_{s \to \infty} c^{-1+\gamma} g(s)^{-1} g(sc^{-\gamma}). \end{aligned}$$

A bit of algebra completes the proof. ■

Notation. For $t \geq 0$ and each real interval I, let

$$T(t, I) = \inf\{ s \geq t | X_s \in I \} \quad (\inf \phi = \infty).$$

Write $T(t, x)$ for $T(t, \{x\})$ and $T(x)$ for $T(0, x)$. Let

$$p(u, y) = \sup_{|x| \leq y} P^x(T(0) > u) = \sup_{|x| \leq y} P(T(x) > u),$$

$$L_t^* = \sup_x L_t^x, \quad \tau_u^* = \inf\{ t | L_t^* > u \}.$$

LEMMA 3.4. *There is a constant c such that*

(a)

$$p(u, y) \leq c\delta(y) g(u^{-1}) \left(\left(\log \frac{1}{g(u^{-1})\delta(y)} \right) \vee 1 \right)$$

for all $y, u > 0$,

(b)

$$\inf_{|x| \leq \delta_0} E^x(L_u^0 \wedge y) \geq c(1 - p(u/2, \delta_0)) g(1/u)^{-1}$$

for all $y > (2g(1/u))^{-1}$ *and* $u, \delta_0 > 0$.

Proof. (a) For each $\eta > 0$,

$$P(T(x) > u) \leq P\left(L_u^0 \leq \eta\right) + P\left(L_{u \wedge \tau(\eta)}^0 - L_{u \wedge \tau(\eta)}^x > \eta, \tau(\eta) \leq u\right)$$
$$\leq (1 - e^{-1})^{-1} \eta g(u^{-1}) + \exp\{-\eta/16\delta(|x|)\}.$$

We have used (2.2), and Proposition 2.7 and Lemma 2.4 of Barlow [1]. To minimize this last expression let

$$\eta = 16\delta(|x|)\log\left(\frac{(1 - e^{-1})}{16g(u^{-1})\delta(|x|)}\right).$$

The desired inequality now follows easily.

(b) If $y \geq (2g(1/u))^{-1}$ and $|x| \leq \delta_0$, then

$$E^x\left(L_u^0 \wedge y\right) \geq P^x(T(0) \leq u/2) E\left(L_{u/2}^0 \wedge y\right)$$
$$\geq (1 - p(u/2, \delta_0))(1 - \tfrac{3}{2}e^{-1/2}) g(1/u)^{-1},$$

the last by Lemma 2.2 (b). ∎

We now prove a result that will play an important role in the proof of both Theorems 1.2 and 1.5.

Notation. For each interval I with finite end points $x_1 < x_2$, and each y, $u > 0$, let

$$N(y, u, I) = \sum_{j=0}^{\infty} I\left(ju \leq \tau^{x_1}(y), \exists t \in [ju, (j+1)u] \ni X_t \in I\right).$$

That is, $N(y, u, I)$ is the number of time intervals of length u before $\tau^x(y)$ during which X visits I.

LEMMA 3.5. *There is a positive constant c_0 such that if I is an interval with finite end points $x_1 < x_2$, and if*

$$P^x(T(x_1) < u) \geq 1/2 \quad \forall x \in I, \tag{3.1}$$

$$n \geq 36yg(u^{-1}), \tag{3.2}$$

then for all $n \in \mathbf{N}$, and $x \in \mathbf{R}$,

$$P^x(N(y, u, I) > n) \leq e^{-c_0 n}.$$

Proof. By spatial homogeneity we may assume $x_1 = 0$. Define stopping times $\{S_i\}$ by

$$S_0 = T(0, I)$$
$$S_i = \inf\{t \geq S_{i-1} + 2u | X(t) \in I\}.$$

Let

$$H(y, u, I) = \sum_{i=0}^{\infty} I(S_i \leq \tau_y),$$

so that

$$N(y, u, I) \leq 3H(y, u, I). \tag{3.3}$$

If

$$U_j = L^0_{S_j} - L^0_{S_{j-1}}, \quad m(x) = E^x\left(L^0_{2u}\right), \quad d_j = m\left(X_{S_{j-1}}\right) - U_j,$$

then $\{(d_j, \mathscr{F}_{S_j}) | j \geq 1\}$ is a martingale difference sequence satisfying

$$E^x\left(d_j^2 | \mathscr{F}_{S_j}\right) \leq E^0\left(L^{0\,2}_{2u}\right) \leq 2e^2 g(1/u)^{-2} \quad \text{(Lemma 2.2(c))} \tag{3.4}$$

$$d_j \leq E^0\left(L^0_{2u}\right) \leq e^2 g(1/u)^{-1} \quad \text{(Lemma 2.2(a))}. \tag{3.5}$$

Moreover, the strong Markov property shows that for $x \in I$,

$$m(x) \geq P^x(T(0) < u) E\left(L^0_u\right) \geq \tfrac{1}{6} g(1/u)^{-1}$$

(by Lemma 2.2(a) and (3.1)). Hence by (3.3), for n satisfying (3.2) we have

$$\begin{aligned} P(N(y, u, I) > n) &\leq P(H(y, u, I) > n/3) \\ &\leq P\Bigg(\sum_{1 \leq j \leq n/3} U_j < y\Bigg) \\ &\leq P\Bigg(\sum_{1 \leq j \leq n/3} d_j > \sum_{1 \leq j \leq n/3} m\left(X_{S_j}\right) - y\Bigg) \\ &\leq P\Bigg(\sum_{1 \leq j \leq n/3} d_j > n(36 g(1/u))^{-1}\Bigg). \end{aligned}$$

(3.4) and (3.5) allow us to apply Theorem 1.4 and bound the above by

$$\begin{aligned} &\exp\left\{\frac{-n^2(36 g(1/u))^{-2}}{2\left((n/3) 2e^2 g(1/u)^{-2} + n(36 g(1/u))^{-1} e^2 g(1/u)^{-1}\right)}\right\} \\ &= \exp\left\{\frac{-n 36^{-2}}{\frac{4}{3}e^2 + e^2/36}\right\}. \end{aligned}$$

■

4. Lower Bound for the Hausdorff measure

THEOREM 4.1. *Assume* (R_β) *and* (H). *Then*

$$f\text{-}m\{s \le t | X(s) = x\} \ge c(\beta) L_t^x \quad \forall x \in \mathbf{R},\ t \ge 0 \quad \text{a.s.}$$

Proof. An integration by parts shows that (H) implies that for some $\varepsilon' > 0$,

$$\rho(u) \ll (\log 1/u)^{-1-\varepsilon'} \equiv \tilde{\rho}(u) \quad \text{as } u \downarrow 0. \tag{4.1}$$

Fix $M \in \{5, 6, \dots\}$ and let $\varepsilon = M^{-1}$. Define a random set $A(\omega)$ by

$$A = \left\{ t | \limsup_{h \downarrow 0} \left(L_{t+h}^{X_t} - L_t^{X_t} \right) f(h)^{-1} > (1 + 4\varepsilon) c(\beta)^{-1} \right\}.$$

The method of Taylor and Wendel [20] shows that the result will follow from

$$\int_0^\infty I_A(s)\, d_s L_s^x = 0 \quad \forall x \in \mathbf{R} \quad \text{a.s.} \tag{4.2}$$

Indeed, Lemma 4 of [20] would then imply for a.a. ω and all $t \ge 0$, $x \in \mathbf{R}$,

$$\begin{aligned}
&(1 + 4\varepsilon) c(\beta)^{-1} f\text{-}m\{s \le t | X_s = x\} \\
&\quad \ge (1 + 4\varepsilon) c(\beta)^{-1} f\text{-}m\{s \le t | X_s = x,\ s \in A^c\} \\
&\quad \ge \int_0^t I(X_s = x,\ s \in A^c)\, d_s L_s^x \\
&\quad = L_t^x.
\end{aligned}$$

Let $\varepsilon \to 0$ to complete the proof.

Note we have used the fact that

$$\int_0^t I(X_s = x) d_s L_s^x = L_t^x \quad \forall x \in \mathbf{R},\ t \ge 0 \quad \text{a.s.}$$

This follows easily from the joint continuity of L_t^x.

Fix $u \in ((1 + 3\varepsilon)(1 + 4\varepsilon)^{-1}, 1)$, choose $u_n \downarrow 0$, $\delta_n \downarrow 0$ such that $c(\beta)^{-1} f(u_n) = u^n$ and $2\tilde{\rho}(\delta_n) = \varepsilon u^n$. (4.1) shows that

$$\delta_n^{-1} = \exp\left\{ (\varepsilon/2)^{-(1/1+\varepsilon')} u^{-n/(1+\varepsilon')} \right\}. \tag{4.3}$$

If

$$A_n = \left\{ t \le \tau^*(1) \wedge 1 | L_{t+u_n}^{X_t} - L_t^{X_t} > (1 + 4\varepsilon) c(\beta)^{-1} f(u_{n+1}) \right\},$$

the Borel-Cantelli Lemma shows that (4.2) would follow from

$$\sum_{n=1}^{\infty} \int_0^{\infty} I_{A_n}(s)\, d_s L_s^x < \infty \quad \forall x \in \mathbf{R} \quad \text{a.s.} \tag{4.4}$$

By (1.18) and (4.1) there is an $N(\omega) < \infty$ a.s. such that

(4.5) $\qquad u_N < 1$, and $\sup_{t \le 2} |L_t^x - L_t^y| < \tilde{\rho}(|x - y|)$ for all $|x - y| \le \delta_N(\omega)$.

Fix $x \in \mathbf{R}$, let $\tau_{-1} = 0$ and write τ_k for $\tau^x(k\varepsilon u^n)$, $k \ge 0$. Suppressing dependence on n, inductively define

$$\begin{aligned} k_1 &= \min\{ k \ge 0 | \tau_{k+M+1} - \tau_k < u_n \}, \\ k_{i+1} &= \min\{ k \ge k_i + M + 2 | \tau_{k+M+1} - \tau_k < u_n \}, \\ I_i &= \left[\tau_{k_i - 1}, \tau_{k_i + M + 1} \right] \cap \left[\tau_{k_i + M + 1} - u_n, \tau_{k_i + M + 1} \right], \quad i \ge 1. \end{aligned}$$

We claim that if $M_n = [M u^{-n}] + 1$ then for $n \ge N(\omega)$,

$$A_n \cap X^{-1}([x, x + \delta_n]) \subset \bigcup_{k_i \le M_n} I_i. \tag{4.6}$$

Assume $v \in A_n$ and $X_v = y \in [x, x + \delta_n]$. Then $v \in [\tau_{k-1}, \tau_k)$ for some $k \in \{0, \dots, M_n\}$, and for this value of k,

$$\begin{aligned} L_{v+u_n}^x - L_{\tau_k}^x &\ge L_{v+u_n}^x - L_v^x - \varepsilon u^n \\ &\ge L_{v+u_n}^y - L_v^y - \varepsilon u^n - \sup_{v \le 1, |x' - y'| \le \delta_n} \left| L_{v+u_n}^{x'} - L_v^{x'} - \left(L_{v+u_n}^{y'} - L_v^{y'} \right) \right| \\ &\ge (1 + 4\varepsilon) u^{n+1} - \varepsilon u^n - 2\tilde{\rho}(\delta_n) \quad \text{(by (4.5))} \\ &> (1 + \varepsilon) u^n. \end{aligned} \tag{4.7}$$

Therefore

$$\tau_{k+M+1} - \tau_k = \tau_{k\varepsilon u^n + (1+\varepsilon)}^x - \tau_{k\varepsilon u^n}^x < v + u_n - \tau_k < u_n$$

and $v > \tau_{k+M+1} - u_n$. It follows that $k \in [k_i, k_i + M + 1]$ for some $k_i \le M_n$ and hence that $v \in I_i$ for this value of i. This proves the claim (4.6), which in turn shows that for $n \geq N(\omega)$,

$$\begin{aligned}
&\sup_{y\in[x,x+\delta_n]}\int_0^\infty I_{A_n}(s)\,d_sL_s^y \\
&\quad\leq \sup_{y\in[x,x+\delta_n]}\sum_{i=1}^\infty I(k_i\leq M_n)\Big(L^y_{(\tau_{k_i+M+1})\wedge 1} \\
&\qquad\qquad - L^y_{((\tau_{k_i+M+1}-u_n)\vee\tau k_i-1)\wedge 1}\Big)
\end{aligned}$$

$$\begin{aligned}
(4.8)\qquad &\leq \sum_{i=1}^\infty I(k_i\leq M_n)\Big[\Big(L^x_{\tau_{k_i+M+1}} - L^x_{\tau_{k_i-1}}\Big) \\
&\qquad + \sup_{v\leq 1,\,|y'-x'|\leq\delta_n}\Big|L^{x'}_{v+u_n} - L^{x'}_v\Big(L^{y'}_{v+u_n} - L^{y'}_v\Big)\Big|\Big] \\
&\leq \sum_{i=1}^\infty I(k_i\leq M_n)\big[(1+2\varepsilon)u^n + 2\tilde\rho(\delta_n)\big].
\end{aligned}$$

Therefore

$$(4.9)\qquad \sup_{y\in[x,x+\delta_n]}\int_0^\infty I_{A_n}(s)\,d_sL_s^y \leq (1+4\varepsilon)u^n\sum_{j=0}^M X(n,x,j),$$

where

$$X(n,x,j) = \sum_{0\leq k\leq M_n(M+1)^{-1}} I\big(\tau_{(k+1)(M+1)+j} - \tau_{k(M+1)+j} < u_n\big).$$

$X(n,x,j)$ has a binomial distribution with parameters

$$N_n = \big[M_n(M+1)^{-1}\big] + 1 \leq u^{-n} \quad \text{(for large } n\text{)}$$

and

$$\begin{aligned}
p_n &= P\big(\tau((1+\varepsilon)u^n) < u_n\big) \\
&= P\Big(L^0_{u_n} > (1+\varepsilon)c(\beta)^{-1}f(u_n)\Big) \\
&\leq |\log f(u_n)|^{-1-\varepsilon/2} \quad (\text{for } n\geq n_0 \text{ by Lemma 2.6}).
\end{aligned}$$

Therefore

$$p_n \leq cn^{-1-\varepsilon/2}.$$

Let $0 < \delta' < \varepsilon/2$. Then

$$P\left((1 + 4\varepsilon)u^n \sum_{j=0}^{M} X(n, x, j) > (M + 1)n^{-1-\delta'}\right)$$
$$\leq (M + 1)P((1 + 4\varepsilon)u^n X(n, x, 0) > n^{-1-\delta'})$$
$$\leq (M + 1)P\left(X(n, x, 0) - N_n p_n > (1 + 4\varepsilon)^{-1}n^{-1-\delta'}u^{-n} - cu^{-n}n^{-1-\varepsilon/2}\right)$$
$$\leq (M + 1)P\left(X(n, x, 0) - N_n p_n > (1/2)n^{-1-\delta'}u^{-n}\right)$$

for large enough n, independent of x. Apply Theorem 1.4 to bound the above by

$$(M + 1)\exp\left\{\frac{-\frac{1}{4}n^{-2-2\delta'}u^{-2n}}{2\left(N_n p_n + \frac{1}{2}n^{-1-\delta'}u^{-n}\right)}\right\} \leq (M + 1)\exp\left\{-\frac{1}{8}n^{-1-\delta'}u^{-n}\right\}$$

for large n, independent of x. Therefore, if $S_n = \{k\delta_n | k \in \mathbf{Z}\}$, then for large enough n,

$$P\left(\max_{x \in S_n, |x| \leq n} (1 + 4\varepsilon)u^n \sum_{j=0}^{M} X(n, x, j) > (M + 1)n^{-1-\delta'}\right)$$
$$\leq 3n\delta_n^{-1}(M + 1)\exp\{-(1/8)n^{-1-\delta'}u^{-n}\}$$

which is summable over n by (4.3). The Borel-Cantelli Lemma and (4.9) imply that for a.a. ω and large enough n,

$$\sup_y \int_0^\infty I_{A_n}(s)\, d_s L_s^y \leq (M + 1)n^{-1-\delta'}.$$

This implies (4.4) and completes the proof. ∎

Remark. 4.2. An examination of the above proof (see especially (4.7) and (4.8)) shows that the above theorem remains true if (H) is weakened to:

(H″) For all $u \in (0,1)$, $\varepsilon > 0$, there exists $\delta_n \downarrow 0$ such that if $c(\beta)^{-1}f(u_n) \sim u^n$ then

(a) $\sup_{t \leq 1,\ |x-y| \leq \delta_n} |(L_{t+u_n}^x - L_t^x) - (L_{t+u_n}^y - L_t^y)| \leq \varepsilon u^n$ for $n \geq N(\omega)$, where $N(\omega) < \infty$ a.s. and

(b) $\sum_{n=1}^{\infty} \delta_n^{-1} \exp\{-u^{-n}u^{-1-\delta}\} < \infty$ for some $\delta > 0$.

Note that if (H) holds then (4.5) shows that (H″) holds with

$$\delta_n = \exp\left\{-(2/\varepsilon)^{1/(1+\varepsilon')}u^{-n/(1+\varepsilon')}\right\}$$

for some $\varepsilon' > 0$. We will give an example in Section 8 where it is easy to verify (H″) but (H) fails.

5. Upper bound for the Hausdorff measure

The proof of the opposite inequality to that in Theorem 4.1 is more involved, although we are able to weaken (H) slightly.

THEOREM 5.1. *Assume* (R_β) *and for some* $\varepsilon > 0$,

$$\text{(H}'\text{)} \qquad \delta(x) \leq \left(\log\frac{1}{x}\right)^{-2-\varepsilon} \quad \textit{for small } x.$$

Then

$$f\text{-}m\{s \leq t | X(s) = x\} \leq c(\beta) L_t^x \quad \forall x \in \mathbf{R},\ t \geq 0 \quad \text{a.s.}$$

Proof. Fix $\theta_0 > \theta > c(\beta)$ and $p > 0$. Choose $u_n \downarrow 0$, $\tilde{u}_n \downarrow 0$, $\delta_n \downarrow 0$ such that

$$f(u_n) = e^{-n}, \qquad f(\tilde{u}_n) = e^{-n/2}, \qquad \delta(\delta_n) = n^{-p}e^{-n},$$

and let $y_n = p \log n\, g(1/\tilde{u}_n)^{-1}$. It follows from (2.6) that

$$(5.1) \quad g\big((\log n) u_n^{-1}\big) = e^n \log n, \quad g\big(\log(n/2)\tilde{u}_n^{-1}\big) = e^{n/2}\log(n/2).$$

Moreover, an easy application of Lemma 2.4(a) (recall also that $\beta \leq 1/2$ by Lemma 3.2) shows that

$$(5.2) \qquad e^{n/2} \leq g\big(\tilde{u}_n^{-1}\big) \leq e^{n/2}\log(n/2) \quad \text{for large enough } n$$

(the upper bound is of course immediate from (5.1)). (H′) implies that

$$(5.3) \qquad \delta_n^{-1} \leq \exp\{n^{p/(2+\varepsilon)}e^{n/(2+\varepsilon)}\} \quad \text{for large } n.$$

Let $S_n = \{k\delta_n | k \in \mathbf{Z}\}$, and for each x let $x_n(x)$ denote the unique element in S_n such that $x \in [x_n, x_n + \delta_n)$.

For each $x \in \mathbf{R}$ and $n \in \mathbf{N}$ inductively define stopping times by

$$U_0^n(x) = 0, \qquad T_i^n(x) = T\big(U_{i-1}^n(x), [x_n, x_n + \delta_n)\big),$$

$$V_i^n(x) = \inf\left\{ t > T_i^n(x) + u_n \,\middle|\, \frac{\left(L_t^{x_n} - L_{T_i^n}^{x_n}\right)}{f(t - T_i^n(x))} \geq \theta^{-1} \right\} \quad (\inf \phi = \infty),$$

$$U_i^n(x) = \begin{cases} \min\left\{ k u_n (\log n)^{-1} \geq V_i^n(x) | k = 0,1,\dots \right\} & (5.4) \\ \quad \text{if } V_i^n - T_i^n \leq \tilde{u}_n \text{ and } L_{V_i^n}^{x_n} - L_{T_i^n}^{x_n} \leq y_n \\ \min\left\{ k u_n (\log n)^{-1} \geq T_i^n(x) + u_n | k = 0,1,\dots \right\} & (5.5) \\ \quad \text{otherwise.} \end{cases}$$

Call $[T_i^n(x), U_i^n(x)]$ a good interval if it is defined by (5.4) and a bad interval if it is defined by (5.5).

The following facts are immediate:

$$\{s \leq t | X_s = x\} \subset \bigcup_{i=1}^{\infty} \left([T_i^n(x), U_i^n(x)] \cap [0, t] \right), \tag{5.6}$$

$$u_n \leq U_i^n(x) - T_i^n(x) \leq \tilde{u}_n + u_n/\log n, \tag{5.7}$$

$$\{U_i^n(x) | i = 0,1,\dots\} \subset \left\{ k u_n (\log n)^{-1} | k = 0,1,\dots \right\}. \tag{5.8}$$

As f is regularly varying at 0 by Lemma 2.5(b), (5.7) shows that for large enough n (independent of (i, x)) and any good interval $[T_i^n(x), U_i^n(x)]$,

$$\begin{aligned} f\big(U_i^n(x) - T_i^n(x)\big) &\leq \theta_0 \theta^{-1} f\big(V_i^n(x) - T_i^n(x)\big) \\ &\leq \theta_0 \left(L_{V_i^n}^{x_n} - L_{T_i^n}^{x_n} \right) \\ &\leq \theta_0 \left(L_{U_i^n}^{x_n} - L_{T_i^n}^{x_n} \right). \end{aligned}$$

Therefore, if Σ_i^g and Σ_i^b denote summations over good and bad intervals, respectively, then

$$\begin{aligned} \limsup_{n \to \infty} \sum_{i=1}^{\infty}{}^{g} f\big(U_i^n(x) - T_i^n(x)\big) I\big(T_i^n(x) \leq t\big) &\leq \lim_{n \to \infty} \theta_0 L_{t + \tilde{u}_n + u_n (\log n)^{-1}}^{x_n} \\ &= \theta_0 L_t^x \quad \forall x \in \mathbf{R},\ t \geq 0 \quad \text{a.s.} \end{aligned}$$

As $\theta_0 > c(\beta)$ is arbitrary, (5.6) and (5.7) show that to complete the proof it suffices to establish

$$\lim_{n \to \infty} \max_{\substack{x \in S_n \\ |x| \leq n}} \sum_{i=1}^{\infty}{}^{b} f(u_n) I\big(T_i^n(x) \leq t\big) = 0 \quad \text{a.s.} \quad \forall t > 0. \tag{5.9}$$

Note that we have used the regular variation of f at 0 and the fact that $T_i^n(\cdot)$ is constant on $[k\delta_n, (k+1)\delta_n)$.

Let

$$A_n(x) = \left\{\omega \middle| \sup_{u_n \le t \le \tilde{u}_n \wedge \tau^x(y_n)} \frac{L_t^x}{f(t)} < \theta^{-1}\right\}.$$

Use (5.8) to see that (5.9) would follow from

$$(5.10) \qquad \lim_{n\to\infty} \max_{\substack{x \in S_n,\\ |x| \le n}} e^{-n} \sum_{i=1}^{\infty} I\Big(T\big(iu_n(\log n)^{-1}, [x, x+\delta_n)\big) < (i+1)u_n(\log n)^{-1} \wedge \tau_1^*, \; \theta_{T(iu_n(\log n)^{-1}, [x, x+\delta_n))}(\omega) \in A_n(x)\Big) = 0 \quad \text{a.s.}$$

To show this we will need the following upper bound on $P^y(A_n(x))$.

LEMMA 5.2. *There is a positive c such that*

$$\sup_{|y| \le \delta_n} P^y(A_n(0)) \le cn^{-p/2} \quad \forall n \in \mathbf{N}.$$

Proof. Fix $|y| \le \delta_n$ and choose $\theta_1 \in (c(\beta), \theta)$. Then

$$\begin{aligned}
P^y(A_n(0)) &\le P^y(\tau(y_n) \le \tilde{u}_n) + P^y\big(T(0) > u_n(\log n)^{-1}\big) \\
&\quad + P^y\Big(T(0) \le u_n(\log n)^{-1}, \sup_{u_n \le t \le \tilde{u}_n} L_t^0/f(t) < \theta^{-1}\Big) \\
&\le e\exp\{-y_n g(\tilde{u}_n^{-1})\} \\
&\quad + c\delta(\delta_n) g(u_n^{-1}\log n)\Big(\big(\log\big(g(u_n^{-1}\log n)\delta(\delta_n)\big)^{-1}\big) \vee 1\Big) \\
&\quad + P^0\Big(\sup_{u_n \le t \le \tilde{u}_n - u_n(\log n)^{-1}} L_t^0\big(f\big(t + u_n(\log n)^{-1}\big)\big)^{-1} < \theta^{-1}\Big) \\
&\qquad\qquad \text{(Lemmas 2.1 and 3.4)}
\end{aligned}$$

$$(5.11) \qquad \le en^{-p} + cn^{-p/2} + P^0\Big(\sup_{u_n \le t \le \tilde{u}_n - u_n(\log n)^{-1}} L_t^0/f(t) < \theta_1^{-1}\Big),$$

where the last line holds for large enough n by the regular variation of f and by (5.1). Checking the hypotheses of Lemma 2.8, we have

$$f\left(\tilde{u}_n - u_n(\log n)^{-1}\right) \geq f\left(\tilde{u}_n\left(1 - (\log n)^{-1}\right)\right) \geq (1/2)e^{-n/2}$$
$$\text{(large } n\text{, by Lemma 2.5(b))}$$
$$\geq f(u_n)^{3/4} \quad \text{(large } n\text{)}.$$

Hence we may apply Lemma 2.8 to bound the last term in (5.11) for large enough n, by $e^{-cn^{\lambda}}$ for some $\lambda > 0$. This completes the proof. ∎

Returning to the derivation of (5.10), let $m_n(y) = P^y(A_n(0))$ and

$$T(n, i, x) = T\left(iu_n(\log n)^{-1}, [x, x + \delta_n)\right).$$

Then

$$\begin{aligned}\sup_{x \in S_n, |x| \leq n} \; & e^{-n} \sum_{i=1}^{\infty} I(T(n, i, x) < \left((i+1)u_n(\log n)^{-1} \wedge \tau_1^*\right) \\ & \times m_n(X(T(n, i, x)) - x) \\ (5.12) \qquad & \leq cn^{-p/2}e^{-n} \sup_{x \in S_n, |x| \leq n} N\left(1, u_n(\log n)^{-1}, [x, x + \delta_n)\right)\end{aligned}$$

(see Section 3 for the definition of N). As $g((\log n)u_n^{-1}) = e^n \log n$ (by (5.1)), an application of Lemma 3.5 ((3.1) holds by Lemma 3.4) shows that for large n,

$$\begin{aligned}P\Bigg(& \sup_{x \in S_n, |x| \leq n} N\left(1, u_n(\log n)^{-1}, [x, x + \delta_n)\right) \geq 36e^n \log n\Bigg) \\ & \leq 3n\delta_n^{-1}e^{-c_0 36 e^n \log n} \\ & \leq 3n \exp\left\{n^{p/(2+\varepsilon)}e^{n/(2+\varepsilon)} - c_0 36 e^n \log n\right\} \quad \text{(by (5.3))}.\end{aligned}$$

As this is summable, the Borel-Cantelli Lemma implies that (5.12) approaches 0 a.s. as $n \to \infty$. Hence to prove (5.10) it suffices to show

$$\begin{aligned}(5.13) \quad \lim_{n \to \infty} \sup_{x \in S_n, |x| \leq n} \; & e^{-n} \sum_{i=1}^{\infty} I\left(T(n, x, i) < \left((i+1)u_n(\log n)^{-1}\right) \wedge \tau_1^*\right) \\ & \times \left[I\left(\theta_{T(n,x,i)}(\omega) \in A_n(x)\right) - m_n(X(T(n, x, i)))\right] \\ & \leq 0 \quad \text{a.s.}\end{aligned}$$

For each (n, x), we divide the random collection of times

$$R(n, x) = \left\{T(n, x, i) | T(n, x, i) < \left((i+1)u_n(\log n)^{-1}\right) \wedge \tau_1^*\right\}$$

into a finite number of random blocks and then bound each of the summations contributing to (5.13) formed by selecting one time from each block. This effectively spaces out the times so that each of the summations obtained in this way will be a martingale and we will be able to apply the exponential bounds in Theorem 1.4. The blocks are given by

$$B(n, x, i) = R(n, x) \cap \left[W_{i-1}^n(x), W_i^n(x)\right),$$

where

$$W_0^n(x) = 0,$$
$$W_{i+1}^n(x) = \inf\left\{t \geq T\left(W_i^n(x), [x, x+\delta_n)\right) | L_t^x - L_{W_i^n(x)}^x \geq y_n \quad \text{or} \quad t - W_i^n(x) \geq \tilde{u}_n\right\}.$$

We first bound the number of blocks,

$$N_n(x) = \inf\{i | W_i^n(x) \geq \tau_1^*\}.$$

Noting that either $L_{W_i^n}^x - L_{W_{i-1}^n}^x \geq y_n$ or $W_i^n - W_{i-1}^n \geq \tilde{u}_n$, and that

$$X\left([W_{i-1}^n, W_i^n]\right) \cap [x, x+\delta_n) \neq \varnothing,$$

one sees that

$$N_n(x) \leq y_n^{-1} + 1 + N\left(1, \tilde{u}_n, [x, x+\delta_n)\right).$$

Let $H_n = (\log n)^{5/2} e^{n/2}$. As $H_n/2 \geq 36 g(\tilde{u}_n^{-1})$, for large n by (5.2), Lemma 3.5 implies ((3.1) holds by Lemma 3.4) that for large n

$$P\left(N\left(1, \tilde{u}_n, [x, x+\delta_n)\right) \geq H_n/2\right) \leq \exp\{-(c_0/2) H_n\}.$$

Therefore, because $y_n^{-1} + 1 \leq H_n/2$ for large n (use (5.2)), one gets that

$$(5.14) \quad 3n\delta_n^{-1} P\left(N_n(x) > H_n\right) \leq 3n \exp\left\{n^{p(2+\varepsilon)} e^{n/(2+\varepsilon)} - (c_0/2) H_n\right\},$$

for large enough n.

Next use Lemma 3.5 once more to bound the cardinality of each block, $|B(n, x, i)|$. (5.1) and (5.2) show that for large n

$$36 y_n g\left((\log n)/u_n\right) \leq H_n.$$

Therefore Lemma 3.5 (and (5.3)) now gives us, for large n,

$$\begin{aligned}
&3n\delta_n^{-1}P\big(|B(n, x, i)| > H_n\big)\\
&\quad\le 3n\delta_n^{-1}H_nP\big(N\big(y_n, u_n/\log n, [x, x+\delta_n)\big) > H_n\big)\\
&\quad\le 3nH_n\exp\big\{n^{p/(2+\varepsilon)}e^{n/(2+\varepsilon)} - c_0H_n\big\}.
\end{aligned}$$

It follows from (5.14) and the above that

$$P\Big(\sup_{x\in S_n, |x|\le n, 1\le i\le N_n(x)} |B(n, x, i)| > H_n\Big)$$

is summable and so by a Borel-Cantelli argument

$$\sup_{x\in S_n, |x|\le n, 1\le i\le N_n(x)} |B(n, x, i)| \le H_n \quad \text{for large enough } n \quad \text{a.s.} \tag{5.15}$$

Let $S(n, x, i, j)$ denote the j-th time in the block $B(n, x, i)$, where

$$S(n, x, i, j) = W_i^n(x)$$

if there is no such time. For $l = 0$ or 1, define

$$\begin{aligned}
M(n, x, j, l)(k) = \sum_{i=1}^{k} &e^{-n}I\big(S(n, x, 2i+l, j) < W_i^n(x) \wedge \tau_1^*\big)\\
&\times\Big[I\big(\theta_{S(n,x,2i+l,j)}(\omega) \in A_n(x)\big)\\
&\qquad -m_n\big(X(S(n, x, 2i+l, j)) - x\big)\Big].
\end{aligned}$$

That is, $M(n, x, j, l)(\infty)$ is obtained by summing only those terms in (5.13) that are the j-th times in each of the even ($l = 0$) or odd ($l = 1$) blocks. Note that each $S(n, x, i, j)$ is a stopping time, as $B(n, x, i)$ is an optional set. It is now an easy exercise to check that

$$\Big\{\big(M(n, x, j, l)(k), \mathscr{F}_{S(n, x, 2(k+1)+l, j)}\big) | k = 0, 1, \dots\Big\}$$

is a martingale. Moreover, Lemma 5.2 implies

$$\langle M(n, x, j, l)\rangle_\infty \le cn^{-p/2}e^{-2n}N_n(x).$$

Use Theorem 1.4 to show

$$
\begin{aligned}
P&\Big(\sup_{k\le\infty} M(n,x,j,l)(k) \ge e^{-n/2}(\log n)^{-3}\Big)\\
&\le P(N_n(x)\ge H_n) + P\Big(\sup_{k\le\infty} M(n,x,j,l)(k) \ge e^{-n/2}(\log n)^{-3},\\
&\qquad\qquad \langle M(n,x,j,l)\rangle_\infty \le cn^{-p/2}(\log n)^{5/2}e^{-(3/2)n}\Big)\\
&\le P(N_n(x)\ge H_n)\\
&\quad + \exp\left\{\frac{-e^{-n}(\log n)^{-6}}{2\big(cn^{-p/2}(\log n)^{5/2}e^{-(3/2)n} + e^{-(3/2)n}(\log n)^{-3}\big)}\right\}\\
&\le P(N_n(x)\ge H_n) + \exp\{-e^{n/2}(\log n)^{-3}/4\}
\end{aligned}
$$

for large enough n. Therefore for large n we have

$$
\begin{aligned}
P&\Big(\sup_{x\in S_n,\,|x|\le n,\, j\le H_n,\, l=0,1} M(n,x,j,l)(\infty) \ge e^{-n/2}(\log n)^{-3}\Big)\\
&\le 2H_n(3n\delta_n^{-1}) \sup_{x\in S_n,\,|x|\le n} P(N_n(x)\ge H_n)\\
&\quad + 2H_n(3n\delta_n^{-1})\exp\{-e^{n/2}(\log n)^{-3}/4\}.
\end{aligned}
$$

This is summable over n by (5.14) and (5.3) and hence

$$
\sup_{\substack{x\in S_n,\, j\le H_n,\, l=0,1\\ |x|\le n}} M(n,x,j,l)(\infty) < e^{-n/2}(\log n)^{-3}
$$

for large enough n a.s.

Combine this with (5.15) to get for a.a. ω and large enough n,

$$
\begin{aligned}
&\max_{\substack{x\in S_n\\|x|\le n}} e^{-n}\sum_{i=1}^{\infty} I\Big(T(n,x,i) < \big((i+1)u_n(\log n)^{-1}\big)\wedge\tau_1^*\Big)\\
&\qquad\times\big[I\big(\theta_{T(n,x,i)}(\omega)\in A_n(x)\big) - m_n\big(X(T(n,x,i)) - x\big)\big]\\
&\quad\le \max_{x\in S_n,\,|x|\le n}\sum_{j=1}^{\infty} I\Big(j\le \max_{i\le N_n(x)}|B(n,x,i)|\Big)\\
&\qquad\times\big(M(n,x,j,0)(\infty) + M(n,x,j,1)(\infty)\big)\\
&\quad\le 2H_ne^{-n/2}(\log n)^{-3}\\
&\quad= 2(\log n)^{-1/2}\to 0 \quad\text{as } n\to\infty.
\end{aligned}
$$

This completes the proof of (5.13) and hence the theorem. ■

Theorem 1.2 is an immediate corollary to Theorems 4.1 and 5.1, and Lemma 3.2.

6. Lower bound for the Kingman construction

As mentioned in §1, the lower bound for $m(a(t, x, \delta))$ in Theorem 1.5 is harder to derive than the analogous bound for f-$m\{s \le t | X(s) = x\}$ because of the lack of a result like Lemma 4 in Taylor and Wendel [20]. We require several preliminary lemmas.

Notation. For each interval I and u, y, $\delta > 0$, define

$$A(I, u, y) = \{\omega | I \subset X([0, u \wedge \tau(y)])\},$$
$$q(u, \delta, y) = \sup_{x \in [0, \delta]} P^x\big(A([0, \delta], u, y)^c\big).$$

LEMMA 6.1. *If $u_n, \delta_n, y_n \to 0$ satisfy*

$$\lim_{n \to \infty} \rho(\delta_n) g\big(u_n^{-1}\big) y_n^{1/2} = \lim_{n \to \infty} \rho(\delta_n) y_n^{-1/2} = 0, \tag{6.1}$$

then $\lim_{n \to \infty} q(u_n, \delta_n, y_n) = 0$.

Proof. (3.7) and (3.4) of Barlow [1] (see also Lemma 2.4 of [1]) show there are ε, $c > 0$ and random variables $\{\Gamma_\lambda | \lambda \in Q^{>0}\}$ such that

(6.2)
$$\sup_{t \ge 0} |\lambda \wedge L_t^a - \lambda \wedge L_t^b| \le c\lambda^{1/2}\Big(\rho(|b-a|) + \big(\delta(|b-a|)\log(\Gamma_\lambda \vee 1)\big)^{1/2}\Big)$$
$$\forall \lambda \in Q^{>0}, a, b \in [-\varepsilon, \varepsilon],$$

$$E(\Gamma_\lambda) \le 3 \quad \forall \lambda \in Q^{>0}. \tag{6.3}$$

Choose $y_n' \in Q^{>0}$ such that (6.1) holds with y_n' in place of y_n, and $y_n' < y_n$. If

$$\eta_n = 2c\rho(\delta_n) g(1/u_n)(y_n')^{1/2},$$

then (6.1) implies $\eta_n \to 0$ and $y_n' \ge \eta_n g(1/u_n)^{-1}$ for large n. Therefore for large n we have

$$\begin{aligned}
&q(u_n, \delta_n, y_n)\\
&\quad \le P([-\delta_n, \delta_n] \not\subset X([0, u_n \wedge \tau(y_n')]))\\
&\quad \le P\Big(\inf_{|y| \le \delta_n} L^y_{u_n \wedge \tau(y_n')} = 0\Big)\\
&\quad \le P\Big(L^0_{u_n} \le \eta_n (g(1/u_n))^{-1}\Big)\\
&\qquad + P\Big(\sup_{|y| \le \delta_n,\, t \le u_n} L^0_t \wedge y_n' - L^y_t \wedge y_n' > \eta_n g(1/u_n)^{-1}\Big)\\
&\quad \le P(u_n \le \tau(\eta_n/g(1/u_n)))\\
&\qquad + P\Big(c y_n^{1/2}\Big(\rho(\delta_n) + \big(\delta(\delta_n)\log(\Gamma_{y_n'} \wedge 1)\big)^{1/2}\Big) > 2c\rho(\delta_n)(y_n')^{1/2}\Big)\\
&\qquad\qquad \text{(by (6.2))}\\
&\quad \le (1 - e^{-1})^{-1}\eta_n + P\Big(\big(\log(\Gamma_{y_n'} \vee 1)\big)^{1/2} > c\rho(\delta_n)\delta(\delta_n)^{-1/2}\Big)\\
&\quad \to 0 \quad \text{as } n \to \infty
\end{aligned}$$

(the last by (2.2) and (6.3), since $\rho(\delta_n)\delta(\delta_n)^{-1/2} \to \infty$ as $n \to \infty$). ■

For the next lemma, recall that

$$p(u, \delta) = \sup_{|x| \le \delta} P^x(T(0) > u).$$

We assume (R_β) throughout the rest of this section.

LEMMA 6.2. *There is a $c > 0$ such that for any $\gamma > 0$ there is an $\varepsilon_0 > 0$ for which*

$$\begin{aligned}
&\big(1 - \gamma - cp(u, \delta)g(1/u)g(1/\varepsilon)^{-1}\big)\Gamma(2 - \beta)^{-1}g(1/\varepsilon)E^x\big(L^0_u \wedge y\big)\\
&\quad \le E^x\big((T(u \wedge \tau(y), [0, \delta])\varepsilon^{-1}) \wedge (1 - \eta)\big)\\
&\quad \le E^x\big((T(u, [0, \delta])\varepsilon^{-1}) \wedge (1 + \eta)\big)\\
&\quad \le \big(1 + \gamma + cp(u, \delta)g(1/u)g(1/\varepsilon)^{-1}\big)\Gamma(2 - \beta)^{-1}g(1/\varepsilon)E^x\big(L^0_u\big)\\
&\quad \forall x \in [0, \delta),\ y \ge g(1/u)^{-1},
\end{aligned}$$

providing ε, u/ε, $\eta \in (0, \varepsilon_0)$, and $p(u/2, \delta) \le 1/2$.

Proof. Recall that $(A(t) = t, Q(x, dy) = \nu(d(y - x))$ is a Lévy system for τ (see Benveniste and Jacod [4]). Therefore for $v \geq u$,

$$
\begin{aligned}
&P^x(T(u,0) \geq v, \tau(y) > u) \\
&\quad = E^x\Big(\sum_{0<s\leq y} I(\tau(s-) \leq u < v \leq \tau(s))\Big) \\
&\qquad + P^x(T(u,0) \geq v, \tau(y) > u, T(0) \geq u) \\
&\quad = E^x\Big(I(T(0,\omega) < u)E^0\Big(\sum_{0<s\leq y} I(\tau(s-) \leq u - T(0,\omega)i \\
&\qquad\qquad < v - T(0,\omega) \leq \tau(s)\Big)\Big) + P^x(T(0) \geq v) \\
&\quad = E^x\Big(I(T(0,\omega) < u)E^0\Big(\int_0^y \int_0^\infty I(\tau(s-) \leq u - T(0,\omega) \\
&\qquad\qquad < v - T(0,\omega) \leq y)\nu(d(y - \tau(s-)))\, ds\Big)\Big) + P^x(T(0) \geq v) \\
&\quad = E^x\Big(I(T(0,\omega) < u) \\
&\qquad \times E^0\Big(\int_0^y I(\tau(s) \leq u - T(0,\omega))\nu \\
&\qquad \times [v - T(0,\omega) - \tau(s), \infty)\, ds\Big)\Big) \\
&\qquad + P^x(T(0) \geq v).
\end{aligned}
$$

As $\int_0^y I(\tau(s) \leq u - T(0,\omega))\, ds = L^0_{u-T(0,\omega)} \wedge y$, the strong Markov property implies

$$
\begin{aligned}
(6.4)\quad E^x(L_u^0 \wedge y)\nu[v,\infty) &\leq P^x(T(u,0) \geq v, \tau(y) > u) \\
&\leq E^x(L_u^0 \wedge y)\nu[v - u, \infty) + P^x(T(0) \geq v).
\end{aligned}
$$

Next we establish a relationship between the law of $T(u, [0,\delta)]$ and that of $T(u,0)$. We have

$$
\begin{aligned}
&P^x(T(u,0) \geq v + u, \tau(y) > u) \\
&\quad \leq P^x(T(u,[0,\delta]) \geq v, \tau(y) > u) \\
&\qquad + P^x(T(T(u,[0,\delta]),0) > u, \tau(y) > u, T(u,[0,\delta]) < v) \\
&\quad \leq P^x(T(u,[0,\delta]) \geq v, \tau(y) > u) + p(u,\delta)P^x(\tau(y) > u).
\end{aligned}
$$

Rearranging we get

$$
\begin{aligned}
(6.5)\qquad & P^x(T(u,[0,\delta]) \geq v, \tau(y) > u) \\
&\quad \geq [P^x(T(u,0) \geq v + u, \tau(y) > u) - p(u,\delta)]
\end{aligned}
$$

We are ready to consider the lower bound in the statement of the lemma. We have

$$
\begin{aligned}
E^x\big((T(u \wedge \tau(y), [0,\delta])\varepsilon^{-1}) \wedge (1-\eta)\big) \\
&\geq \varepsilon^{-1} \int_u^{\varepsilon(1-\eta)} P^x(T(u,[0,\delta]) > v, \tau(y) > u)\, dv \\
&\geq \varepsilon^{-1} \int_u^{\varepsilon(1-\eta)} P^x(T(u,0) \geq v + u, \tau(y) > u)\, dv \\
&\quad -(1-\eta-u\varepsilon^{-1})p(u,\delta) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (6.5))}
\end{aligned}
$$

$$
\geq \varepsilon^{-1} \int_{2u}^{u+\varepsilon(1-\eta)} \nu[v,\infty)\, dv\, E^x(L_u^0 \wedge y) - (1-\eta-u\varepsilon^{-1})p(u,\delta) \tag{6.6}
$$

$$\text{(by (6.4))}.$$

Fix $\gamma > 0$. Use Lemmas 2.5(a) and 2.4(b) and the fact that $tg(1/t)$ varies regularly at 0 with exponent $1 - \beta > 0$, to see that there is an $\varepsilon_0 > 0$ such that if ε, $u\varepsilon^{-1}$, $\eta \in (0, \varepsilon_0)$, then (6.6) is bounded below by

$$
(1-\gamma)\Gamma(2-\beta)^{-1} g(1/\varepsilon) E^x(L_u^0 \wedge y) - p(u,\delta).
$$

Finally use Lemma 3.4(b) in the above to see that for $y \geq g(1/u)^{-1}$ and $x \in [0, \delta]$,

$$
\begin{aligned}
E^x\big((T(u \wedge \tau(y), [0,\delta])\varepsilon^{-1}) \wedge (1-\eta)\big) \\
&\geq \left[1 - \gamma - p(u,\delta)\Gamma(2-\beta)c^{-1}(1-p(u/2,\delta))^{-1} g(1/u) g(1/\varepsilon)^{-1}\right] \\
&\quad \times \Gamma(2-\beta)^{-1} g(1/\varepsilon) E^x(L_u^0 \wedge y).
\end{aligned}
$$

The lower bound is now immediate from $p(u/2, \delta) \leq 1/2$.

The proof of the upper bound is similar. Use the upper bound in (6.4) (with $y = \infty$) and, instead of (6.5), use the obvious fact

$$
P^x(T(u,[0,\delta]) \geq v) \leq P^x(T(u,0) \geq v). \qquad \blacksquare
$$

LEMMA 6.3. *There are constants c, $\varepsilon_0 > 0$ such that*

$$
E^x\Big(\big((T(u,[0,\delta])\varepsilon^{-1}) \wedge 2\big)^2\Big) \leq c g(\varepsilon^{-1}) g(u^{-1})^{-1} \quad \forall x \in [0,\delta],
$$

whenever $0 < u < \varepsilon < \varepsilon_0$.

Proof. Fix $\delta > 0$ and let $x \in [0, \delta]$. Use the upper bound in (6.4) with $y = \infty$ and $x = 0$, to see that

$$\begin{aligned}
&E^x\Big(\big((T(u,[0,\delta])\varepsilon^{-1}) \wedge 2\big)^2\Big)\\
&\quad \le E^x\Big(\big((T(u,x)\varepsilon^{-1}) \wedge 2\big)^2\Big)\\
&\quad = E^0\Big(\big((T(u,0)\varepsilon^{-1}) \wedge 2\big)^2\Big)\\
&\quad = \varepsilon^{-2}\int_0^{2\varepsilon} 2vP(T(u,0) \ge v)\,dv\\
&\quad \le \varepsilon^{-2}\left[u^2 + 2\int_0^{2\varepsilon}(v+u)\nu[v,\infty)\,dv\,E^0(L_u^0)\right].
\end{aligned}$$

Use Lemmas 2.4(b) and 2.5(a), (c) to conclude there are c, $\varepsilon_0 > 0$ such that for $0 < u < \varepsilon < \varepsilon_0$, the above is bounded by

$$u^2\varepsilon^{-2} + cg(\varepsilon^{-1})g(u^{-1})^{-1}.$$

By Lemmas 2.4(a) and 3.2, we may choose ε_0 so that for u, ε as above

$$g(\varepsilon^{-1})g(u^{-1})^{-1} \ge (u/\varepsilon)^{\beta+1} \ge (u/\varepsilon)^2.$$

The result is now immediate.

THEOREM 6.4. *Assume* (R_β) *and* $\delta(x) \ll (\log 1/x)^{-2}$ *as* $x \downarrow 0$. *Then for each positive* T,

$$\limsup_{\varepsilon\downarrow 0}\ \sup_{\substack{x\in\mathbf{R},\\ 0\le t\le T}} L_t^x - \Gamma(2-\beta)\frac{m(a(t,x,\varepsilon))}{\varepsilon g(1/\varepsilon)} \le 0 \quad \text{a.s.}$$

Proof. Fix $u \in (0,1)$. An integration by parts shows that

$$\rho(x) \ll (\log 1/x)^{-1/2} \quad \text{as } x \downarrow 0$$

(ρ is given by (1.17)). Let $\varepsilon(x) = \rho(x)(\log 1/x)^{1/2}$ and choose $\delta_n \to 0$ such that

$$\varepsilon(\delta_n)^{1/2}(\log 1/\delta_n)^{-1/2} = u^{n/2}.$$

Then

$$\rho(\delta_n) = \varepsilon(\delta_n)(\log 1/\delta_n)^{-1/2} \ll u^{n/2} \ll (\log 1/\delta_n)^{-1/2}$$

and so

$$\rho(\delta_n) \ll u^{n/2},\ \delta_n \gg e^{-u^{-n}\varepsilon} \quad \forall \varepsilon > 0. \tag{6.7}$$

Choose $\varepsilon_n \to 0$ such that $g(1/\varepsilon_n) = u^{-n}$. It is clear from (6.7) that there are $u'_n \to 0$ such that

$$g(1/u'_n)\rho(\delta_n)u^{n/2} \to 0 \quad \text{and} \quad u'_n \ll \varepsilon_{n+1}. \tag{6.8}$$

Lemma 6.1 implies that

$$\lim_{n\to\infty} q(u'_n, \delta_n, u^n) = 0.$$

Now choose $u_n \to 0$ such that $u'_n \le u_n \ll \varepsilon_{n+1}$ and

$$\lim_{n\to\infty} g(1/u_n)g(1/\varepsilon_n)^{-1}q(u'_n, \delta_n, u^n) = 0.$$

As $q(u, \delta, y)$ is decreasing in u, we have

$$\lim_{n\to\infty} g(1/u_n)g(1/\varepsilon_n)^{-1}q(u_n, \delta_n, u^n) = 0, \quad u_n \ll \varepsilon_{n+1}. \tag{6.9}$$

Let $x_n = x_n(x)$ denote the unique element in $S_n = \{k\delta_n | k \in \mathbf{Z}]$ such that

$$x \in [x_n, x_n + \delta_n)$$

and let $I^n(x) = [x_n, x_n + \delta_n]$. For $n \in \mathbf{N}$, $x \in \mathbf{R}$, inductively define stopping times as follows:

$$\begin{aligned}
T_0^n(x) &= T(0, I^n(x)),\\
V_i^n(x) &= \inf\left\{t > T_i^n(x) | L_t^{x_n} - L_{T_i(x)}^{x_n} \ge u^n\right\} \wedge \left(T_i^n(x) + u_n\right),\\
T_{i+1}^n(x) &= T\left(V_i^n(x), I^n(x)\right).
\end{aligned}$$

Write $A_n(x)$ for $A(I_n(x), u_n, u^n)$. Note that

$$a(t, x, \varepsilon) \supset \bigcup_{i\ge 0}{}' \left[T_i^n(x) - \varepsilon/2 + u_n, T_i^n(x) + \varepsilon/2\right], \tag{6.10}$$

where $\cup'_{i\ge 0}$ indicates the union is over those indices $i \ge 0$ for which $T_i^n(x) \le t - u_n$ and $\theta_{T_i^n(x)}\omega \in A_n(x)$. The significant fact about (6.10) is that the right side is constant on $I_n(x)$.

The required result follows easily from

(6.11)

$$\limsup_{\varepsilon\downarrow 0}\ \sup_x L_{t\wedge\tau^*(y)}^x - \Gamma(2-\beta)m\left(a(t\wedge\tau^*(y), x, \varepsilon)\right)\varepsilon^{-1}g(1/\varepsilon)^{-1} \le 0$$

a.s. for each t, $y > 0$.

Indeed, letting $y \to \infty$ in (6.11) gives the result for each fixed t, and then use the continuity of $L^x_\cdot$ (uniformly in x) and the monotonicity of $t \to m(a(t, x, \varepsilon))$ to obtain the uniform convergence in t. Therefore we fix t, $y > 0$ and consider (6.11).

(6.10) shows that

(6.12)
$$\begin{aligned} m(a(t \wedge \tau^*(y), x, \varepsilon)) &\geq \sum_{i=0}^{\infty} I\Big(T_i^n(x) \leq t \wedge \tau^*(y) - u_n, \theta_{T_i^n(x)}\omega \in A_n(x), \theta_{T_{i+1}^n(x)}\omega \in A_n(x)\Big) \\ &\quad \times (T_{i+1}^n(x) - T_i^n(x)) \wedge (\varepsilon - u_n) \\ &\equiv m_n(x, \varepsilon). \end{aligned}$$

The continuity of local time and the fact that $m_n(\cdot, \varepsilon)$ is constant on $[k\delta_n, (k+1)\delta_n)$ shows that (6.11) would follow from

(6.13)
$$\limsup_{n \to \infty} \sup_{\substack{x \in S_n,\, \varepsilon \in [\varepsilon_{n+1}, \varepsilon_n], \\ |x| \leq n}} L^x_{t \wedge \tau^*(y)} - \Gamma(2-\beta) m_n(x, \varepsilon)\varepsilon^{-1} g(1/\varepsilon)^{-1} \leq 0 \quad \text{a.s.}$$

Use (6.12) to see that for $x \in S_n$ and $\varepsilon \in [\varepsilon_{n+1}, \varepsilon_n]$,

(6.14)
$$\begin{aligned} &u^2 L^x_{t \wedge \tau^*(y) - u_n} - \Gamma(2-\beta) m_n(x, \varepsilon)\varepsilon^{-1} g(1/\varepsilon)^{-1} \\ &\quad \leq \sum_{i=0}^{\infty} I(T_i^n(x) \leq t \wedge \tau^*(y) - u_n) \\ &\qquad \times \Bigg[u^2\Big(L^x_{V_i^n(x)} - L^x_{T_i^n(x)}\Big) - \Bigg(\Bigg(\frac{T_{i+1}^n(x) - T_i^n(x)}{\varepsilon_n}\Bigg) \\ &\qquad\qquad \wedge \Bigg(1 - \frac{u_n}{\varepsilon_{n+1}}\Bigg)\Bigg)\Gamma(2-\beta) g(1/\varepsilon_{n+1})^{-1}\Bigg] \\ &\qquad + 2\sum_{i=0}^{\infty} I(T_i^n(x) \leq t \wedge \tau^*(y) - u_n) \\ &\qquad \times I\Big(\theta_{T_i^n(x)}\omega \in A_n(x)^c\Big) g(1/\varepsilon_n)^{-1} + g(1/\varepsilon_n)^{-1} \\ &\quad \leq \sum_{i=0} I(T_i^n(x) \leq t \wedge \tau^*(y) - u_n) d_i(x, n) \\ &\qquad + \sum_{i=0}^{\infty} I(T_i^n(x) \leq t \wedge \tau^*(y) - u_n) 2q(u_n, \delta_n, u^n) u^n + u^n, \end{aligned}$$

where

$$\begin{aligned} d_i(x,n) &= u^2\Big(L^x_{V_i^n(x)} - L^x_{T_i^n(x)}\Big) \\ &\quad -\left(\left(\frac{T^n_{i+1}(x) - T_i^n(x)}{\varepsilon_n}\right) \wedge (1 - u_n/\varepsilon_{n+1})\right) \\ &\quad \times \Gamma(2-\beta) g(1/\varepsilon_{n+1})^{-1} \\ &\quad + \Big[2I\Big(\theta_{T_i^n(x)}\omega \in A_n(x)^c\Big) - 2m_n\big(X(T_i^n(x))\big)\Big] g(1/\varepsilon_n)^{-1}, \end{aligned}$$

and

$$m_n(x) = P^x\big(A_n(x)^c\big).$$

Note that $d_i(x,n) \in \mathscr{F}_{T^n_{i+1}(x)}$ and

$$\begin{aligned} E\Big(d_i(x,n)|\mathscr{F}_{T_i^n(x)}\Big) &= u^2 E^{X(T_i^n(x))}\big(L^x_{u_n} \wedge u^n\big) \\ &\quad -\Gamma(2-\beta)u^{n+1}E^{X(T_i^n(x))}\left(\left(\frac{\big(T(u_n \wedge \tau^x(u^n), I_n(x)\big)}{\varepsilon_n}\right) \wedge (1 - u_n/\varepsilon_{n+1})\right). \end{aligned}$$

Now note that we may apply Lemma 6.2 to bound the second term. Indeed, (6.8), $2/u_n \le 2/u'_n$, and Lemma 6.1 show that

$$p(u_n/2, \delta_n) \le q(u_n/2, \delta_n, u^n) \to 0 \quad \text{as } n \to \infty.$$

Therefore by Lemma 6.2 there are $\gamma_n \to 0$ such that

$$\begin{aligned} &E\Big(d_i(y,n)|\mathscr{F}_{T_i^n(x)}\Big) \\ &\quad \le u^2 E^{X(T_i^n(x))}\big(L^x_{u_n} \wedge u^n\big) \\ &\qquad -u\Big(1 - \gamma_n - cp(u_n,\delta_n)g\big(u_n^{-1}\big)g\big(\varepsilon_n^{-1}\big)^{-1}\Big)E^{X(T_i^n(x))}\big(L^x_{u_n} \wedge u^n\big) \\ &\quad \le 0 \end{aligned}$$

for $n \ge N_0$, say. We have used the fact that $p(u_n, \delta_n) \le q(u_n, \delta_n, u^n)$, and (6.9) in the last line. Therefore for $n \ge N_0$, if

$$M_k(x,n) = \sum_{i=0}^{k-1} d_i(x,n) I\big(T_i^n(x) \le t \wedge \tau^*(y) - u_n\big),$$

then

$$\Big\{\Big(M_k(x,n), \mathscr{F}_{T_k^n(x)}\Big)|k = 0,1,\dots,\infty\Big\}$$

is a supermartingale. Moreover (6.14) implies

$$(6.15)\qquad \sup_{\substack{x\in S_n,\ \varepsilon\in[\varepsilon_{n+1},\varepsilon_n],\\ |x|\le n}} u^2 L^x_{t\wedge\tau^*(y)-u_n} - \Gamma(2-\beta)m_n(x,\varepsilon)\varepsilon^{-1}g(1/\varepsilon)^{-1}$$
$$\le \sup_{x\in S_n,\,|x|\le n} M_\infty(x,n)$$
$$+\left[N(y,u_n,I_n(x)) + yu^{-n}+1\right]2q(u_n,\delta_n,u^n)u^n + u^n.$$

Clearly one has

$$(6.16)\qquad d_i(x,n)\le 3u^n.$$

In order to use Theorem 1.4 we now bound $\langle M(x,n)\rangle$. For large enough n, say $n\ge N_0$, we have

$$E\left(d_i(x,n)^2|\mathscr{F}_{T_i^n(x)}\right)$$
$$\le c\Big[E^{X(T_i^n(x))}\left(\left(L^x_{u_n}\wedge u^n\right)^2\right)$$
$$+E^{X(T_i^n(x))}\left(\left(T(u_n,I_n(x))/\varepsilon_n\right)\wedge 1\right)^2\Big)u^{2n} + u^{2n}q(u_n,\delta_n,u^n)\Big]$$
$$\le c\left(E^0\left(L^{0^2}_{u_n}\right) + u^{2n}g(\varepsilon_n^{-1})g(u_n^{-1})^{-1} + u^n g(u_n^{-1})^{-1}\right)$$
$$\text{(Lemma 6.3 and (6.9))}$$
$$\le c\left(g(1/u_n)^{-2} + u^n g(1/u_n)^{-1}\right),$$

where we have used Lemma 2.2(c) in the last. Therefore we have,

$$(6.17)\qquad \langle M(x,n)\rangle \le c\sum_{i=0}^{\infty} I(T_i^n(x)\le t\wedge\tau^*(y))u^n g(1/u_n)^{-1}$$
$$\le c\left(N(y,u_n,I_n(x)) + yu^{-n}+1\right)u^n g(1/u_n)^{-1}.$$

We now may use Theorem 1.4 to see that for $n\ge N_0$, $x\in S_n$, and $K>0$,

$$P(M_\infty(x,n)\ge\varepsilon)\le P(M_\infty(x,n)\ge\varepsilon,\langle M(x,n)\rangle\le Ku^n)$$
$$+P(N(y,u_n,I_n(x)) + yu^{-n}+1 > (K/c)g(1/u_n))$$
$$\le \exp\left\{\frac{-\varepsilon^2}{2(Ku^n+3\varepsilon u^n)}\right\}$$
$$+P(N(y,u_n,I_n(x)) > (K/c-y-u^n)g(1/u_n))$$
$$\le \exp\left\{\frac{-\varepsilon^2u^{-n}}{2(K+3\varepsilon)}\right\} + \exp\{-c_0 36yg(1/u_n)\},$$

where we may choose K large enough so that $K/c - y - 1 \geq 36y$ and hence apply Lemma 3.5. Therefore

$$P\left(\sup_{\substack{x \in S_n, \\ |x| \leq n}} M_\infty(x, n) \geq \varepsilon\right) \leq 3n\delta_n^{-1}\left[\exp\left\{\frac{-\varepsilon^2 u^{-n}}{2(K + 3\varepsilon)}\right\} + \exp\{-c_0 36 y u^{-n}\}\right],$$

which is summable by (6.7). The Borel-Cantelli Lemma shows

$$\lim_{n \to \infty} \sup_{x \in S_n, |x| \leq n} M_\infty(x, n)^+ = 0 \quad \text{a.s.} \tag{6.18}$$

Another application of Lemma 3.5 and the Borel-Cantelli Lemma (just as above) shows

$$\lim_{n \to \infty} \sup_{x \in s_n |x| \leq n} N(y, u_n, I_n(x)) q(u_n, \delta_n, u^n) u^n = 0 \quad \text{a.s.} \tag{6.19}$$

Applying (6.18) and (6.19) to (6.15) one obtains

$$\limsup_{n \to \infty} \sup_{\substack{x \in S_n, \varepsilon \in [\varepsilon_{n+1}, \varepsilon_n], \\ |x| \leq n}} u^2 L^x_{t \wedge T^*(y)} - \Gamma(2 - \beta) m_n(x, \varepsilon) \varepsilon^{-1} g(1/\varepsilon)^{-1} \leq 0.$$

Let $u \uparrow 1$ to obtain (6.13) and hence complete the proof. ∎

7. Upper bound for the Kingman construction

Fortunately, the easiest proof has been left to the last. Moreover the continuity condition we will need on $\delta(x)$ is so weak that it does not even guarantee the existence of a jointly continuous local time. Hence we must add this as a hypothesis in the following result.

THEOREM 7.1. *Assume* (R_β), *X has a jointly continuous local time*, L^x_t, *and*

$$\delta(x) \ll (\log 1/x)^{-1} \quad \text{as } x \downarrow 0.$$

Then for each positive T,

$$\limsup_{\varepsilon \downarrow 0} \sup_{\substack{x \in \mathbf{R}, \\ 0 \leq t \leq T}} \frac{\Gamma(2 - \beta) m(a(t, x, \varepsilon))}{\varepsilon g(1/\varepsilon)} - L^x_t \leq 0 \quad \text{a.s.}$$

Proof. Fix $u \in (0, 1)$. Choose $\delta_n \downarrow 0$ such that $\eta_n = \delta(\delta_n) u^{-n} \to 0$ as $n \to \infty$ but $\delta_n \gg e^{-\varepsilon u^{-n}}$ as $n \to \infty$ $\forall \varepsilon > 0$. Now choose $u_n \to 0$ such that $g(1/u_n) =$

$\eta_n^{-1/2}u^{-n}$, and $\varepsilon_n \to 0$ such that $g(1/\varepsilon_n) = u^{-n}$. Then we have

$$u_n \ll \varepsilon_{n+1}, \tag{7.1}$$

$$\delta_n \gg e^{-\varepsilon u^{-n}} \quad \forall \varepsilon > 0, \tag{7.2}$$

$$\delta(\delta_n) g(u_n^{-1}) = \eta_n^{1/2} \to 0 \quad \text{as } n \to \infty. \tag{7.3}$$

Lemma 3.4(a) therefore implies

$$p(u_n/2, \delta_n) \to 0 \quad \text{as } n \to \infty. \tag{7.4}$$

By making u_n larger if necessary we can maintain (7.1) and (7.4) and also have

$$p(u_n, \delta_n) g(1/u_n) g(1/\varepsilon_n)^{-1} \to 0 \quad \text{as } n \to \infty. \tag{7.5}$$

Fix t, $y > 0$. As in the proof of Theorem 6.4 it suffices to show

(7.6)

$$\limsup_{n\to\infty} \sup_{\substack{x\in\mathbf{R},\\ \varepsilon\in[\varepsilon_{n+1},\varepsilon_n]}} \frac{\Gamma(2-\beta) m(a(t\wedge\tau^*(y), x, \varepsilon)}{\varepsilon g(1/\varepsilon)} - L^x_{t\wedge\tau^*(y)} \le 0 \quad \text{a.s.}$$

As before, let $S_n = \{k\delta_n | k \in \mathbf{Z}\}$, and let $x_n = x_n(x) \in S_n$ be such that $x \in [x_n, x_n + \delta_n)$ and $I_n(x) = [x_n, x_n + \delta_n]$. Define stopping times by

$$T_0^n(x) = T(0, I_n(x)),$$
$$T_{i+1}^n(x) = T(T_i^n(x) + u_n, I^n(x)).$$

Then

$$\begin{aligned} &a(t\wedge\tau^*(y), x, \varepsilon) \\ &\quad\subset \bigcup_{i=0}^{\infty}{}' \left[T_i^n(x), T_{i+1}^n(x) \wedge (T_i^n(x) + u_n + \varepsilon)\right] \\ &\quad\cup \left[T_0^n(x) - \varepsilon/2, T_0^n(x)\right], \end{aligned}$$

where $\cup'$ indicates the union is over those indices i such that

$$T_i^n(x) \le t \wedge \tau^*(y) + u_n.$$

Therefore

$$\begin{aligned} &m(a(t\wedge\tau^*(y), x, \varepsilon)) \\ &\quad\le \sum_{i=0}^{\infty} I(T_i^n(x) \le t\wedge\tau^*(y) + u_n) \\ &\quad\quad\times ((T_{i+1}^n(x) - T_i^n(x)) \wedge (\varepsilon + u_n)) + \varepsilon/2 \\ &\quad\equiv \hat{m}_n(x, \varepsilon). \end{aligned}$$

As $\hat{m}_n(\cdot, \varepsilon)$ is constant on $[k\delta_n, (k+1)\delta_n)$ and $L^{\cdot}_{t \wedge \tau^*(y)}$ is continuous, (7.6) would be an immediate consequence of

$$\text{(7.7)} \quad \limsup_{n \to \infty} \sup_{\substack{x \in S_n, x \le n, \\ \varepsilon \in [\varepsilon_{n+1}, \varepsilon_n]}} \Gamma(2-\beta)\hat{m}_n(x, \varepsilon)\varepsilon^{-1}g(1/\varepsilon)^{-1} - L^x_{t \wedge \tau^*(y)} \le 0 \quad \text{a.s.}$$

If $x \in S_n$ and $\varepsilon \in [\varepsilon_{n+1}, \varepsilon_n]$, then

$$\text{(7.8)} \quad \begin{aligned} &u^2\Gamma(2-\beta)\hat{m}_n(x, \varepsilon)\varepsilon^{-1}g(1/\varepsilon)^{-1} - L^x_{t \wedge \tau^*(y) + 2u_n} \\ &\le \sum_{i=0}^{\infty} I(T_i^n(x) \le t \wedge \tau^*(y) + u_n) d_i(x, n), \end{aligned}$$

where

$$\begin{aligned} d_i(x, n) = \Bigg(&\left(\frac{T^n_{i+1}(x) - T_i^n(x)}{\varepsilon_{n+1}}\right) \wedge (1 + u_n/\varepsilon_{n+1})\Bigg) \\ &\times g(1/\varepsilon_n)^{-1}\Gamma(2-\beta)u^2 - \left(L^x_{T^n_{i+1}(x)} - L^x_{T^n_i(x)}\right). \end{aligned}$$

Clearly $d_i(x, n) \in \mathscr{F}_{T^n_{i+1}(x)}$. Moreover (7.1) and (7.4) show that we may apply Lemma 6.2 to find $\gamma_n \to 0$ such that

$$\begin{aligned} &E\left(d_i(x, n) | \mathscr{F}_{T_i^n(x)}\right) \\ &\quad = E^{X(T_i^n(x))}\left(\frac{T(u_n, I_n(x))}{\varepsilon_{n+1}} \wedge (1 + u_n/\varepsilon_{n+1})\right) u^{n+2}\Gamma(2-\beta) \\ &\qquad - E^{X(T_i^n(x))}\left(L^x_{u_n}\right) \\ &\quad \le \left(1 + \gamma_n + cp(u_n, \delta_n)g(1/u_n)g(1/\varepsilon_{n+1})^{-1}\right) \\ &\qquad \times uE^{X(T_i^n(x))}\left(L^x_{u_n}\right) \\ &\qquad - E^{X(T_i^n(x)}\left(L^x_{u_n}\right) \\ &\quad \le 0 \end{aligned}$$

for large enough n, say $n \ge N_0$, by (7.5). Therefore, if

$$M_k(x, n) = \sum_{i=0}^{k-1} d_i(x, n) I(T_i^n(x) \le t \wedge \tau^*(y) + u_n),$$

then for $x \in S_n$ and $n \ge N_0$,

$$\left\{\left(M_k(x, n), \mathscr{F}_{T_k^n(x)}\right) | k = 0, 1, \ldots, \infty\right\}$$

is a supermartingale. Note that for large n, $x \in S_n$ and all i,

$$(7.9) \qquad d_i(x,n) \le cu^n,$$

$$E\Big(d_i(x,n)^2 | \mathscr{F}_{T_i^n(x)}\Big)$$
$$\le c\left[u^{2n}E^{X(T_i^n(x))}\left(\left(\frac{T(u_n, I_n(x))}{\varepsilon_{n+1}} \wedge 2\right)^2\right) + E^0\left(L_{u_n}^{0\,2}\right)\right]$$
$$\le c\left[u^{2n}g(1/\varepsilon_n)g(1/u_n)^{-1} + g(1/u_n)^{-2}\right]$$
(Lemmas 2.2(c) and 6.3)
$$\le cu^n g(1/u_n)^{-1}.$$

Therefore for $n \ge N_0$ (increase N_0 if necessary)

$$(7.10) \quad \langle M(x,n)\rangle \le cu^n g(1/u_n)^{-1}\sum_{i=0}^{\infty} I(T_i^n(x) \le t \wedge \tau^*(y) + u_n)$$
$$\le cu^n g(1/u_n)^{-1}(N(y, u_n, I_n(x)) + 1).$$

Now proceed exactly as in the proof of Theorem 6.4. Use (7.9) and (7.10) in Theorem 1.4, as well as Lemma 3.5 and (7.2), to see that

$$\sum_n P\Big(\sup_{x\in S_n, |x|\le n} M_\infty(x,n) \ge \varepsilon\Big) < \infty \quad \forall \varepsilon > 0.$$

The Borel-Cantelli Lemma allows us to deduce from (7.8) that

$$\limsup_{n\to\infty} \sup_{\substack{x\in S_n, |x|\le n,\\ \varepsilon\in[\varepsilon_{n+1}, \varepsilon_n]}} u^2\Gamma(2-\beta)\hat{m}_n(x,\varepsilon)g(1/\varepsilon)^{-1} - L^x_{t\wedge\tau^*(y)+2u_n} \le 0 \quad \text{a.s.}$$

Let $u \uparrow 1$ to establish (7.7) and complete the proof. ∎

Theorem 1.5 follows immediately from Theorems 6.4 and 7.1.

8. Some examples

(a) Lévy processes with a Brownian component.

Assume X is a Lévy process such that $\sigma^2 > 0$ in (1.2). Lemma 3.1 shows that

$$\lim_{|\lambda|\to\infty} \frac{\psi(\lambda)}{(\sigma^2/2)\lambda^2} = 1,$$

and therefore by (1.10) and Lemma 3.3 we have

$$1 = \lim_{s\to\infty} g(s)\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(s - \frac{\sigma^2\lambda^2}{2}\right)^{-1} d\lambda$$
$$= \lim_{s\to\infty}\frac{g(s)}{2\pi}\frac{\pi}{\sqrt{\sigma^2 s/2}}.$$

Therefore

$$\lim_{s\to\infty}\frac{g(s)}{\sqrt{2}\,\sigma\sqrt{s}} = 1. \tag{8.1}$$

In particular $(\mathbf{R}_{1/2})$ holds. An easy computation shows that $\delta(x) = O(x)$ as $x \downarrow 0$. Therefore the hypotheses of both Theorems 1.2 and 1.5 are satisfied. Theorem 1.5 together with (8.1) implies for each $T > 0$,

$$\lim_{\varepsilon\downarrow 0}\sup_{x\in\mathbf{R},\, t\le T}\left|\frac{\sqrt{\pi}}{2\sqrt{2}\,\sigma}\frac{m(a(t,x,\varepsilon))}{\sqrt{\varepsilon}} - L_t^x\right| = 0 \quad \text{a.s.} \tag{8.2}$$

(8.1) implies that

$$f(t) \sim \frac{1}{\sqrt{2}\,\sigma}(t\log|\log t|)^{1/2} \quad \text{as } t\downarrow 0$$

and so Theorem 1.2 gives us

$$\phi\text{-}m\{s\le t \mid X_s = x\} = L_t^x \quad \forall x\in\mathbf{R},\ t\ge 0 \quad \text{a.s.}, \tag{8.3}$$

where

$$\phi(t) = \sigma^{-1}(2t\log|\log t|)^{1/2}.$$

(b) *Stable Processes.*

Let X_t be a stable process of index α. It is well known that X has a jointly continuous local time if and only if $\alpha > 1$. In this case the exponent function of X is given by

$$\psi(\lambda) = c_1|\lambda|^\alpha\left(1 - ih\,\mathrm{sgn}(\lambda)\tan\frac{\pi\alpha}{2}\right)$$

where $c_1 \in \mathbf{R}$, $-1 \le h \le 1$ and $1 < \alpha \le 2$. Then

$$g(s) = c_2 s^\beta, \tag{8.4}$$

where $\beta = 1 - 1/\alpha$ and

(8.5) $c_2^{-1} = \pi^{-1}\Gamma(1 + 1/\alpha)\Gamma(1 - 1/\alpha)c_1^{-1/\alpha}\mathrm{Re}\left[(1 - ih\tan(\pi\alpha/2))^{-1/\alpha}\right].$

(See Hawkes [12, Section 2]). In particular (R_β) holds. A routine calculation shows

$$\delta(x) \sim cx^{\alpha-1} \quad \text{as } x \downarrow 0, \text{ for some } c > 0.$$

Therefore the hypotheses of Theorems 1.2 and 1.5 are satisfied. Theorem 1.5 and (8.4) imply

$$\lim_{\varepsilon \downarrow 0} \sup_{\substack{x \in \mathbf{R},\\ t \le T}} \left| c_2^{-1}\Gamma(1 + 1/\alpha)\frac{m(a(t, x, \varepsilon))}{\varepsilon^{1/\alpha}} - L_t^x \right| = 0 \quad \text{a.s.,} \tag{8.6}$$

where c_2^{-1} is given by (8.5).

As $f(t) \sim c_2^{-1}t^\beta \log|\log t|^{1-\beta}$, Theorem 1.2 implies

$$\phi\text{-}m\{s \le t | X_s = x\} = L_t^x \quad \forall x \in \mathbf{R},\ t \ge 0 \quad a.s., \tag{8.7}$$

where

$$\phi(t) = c_2^{-1}(1 - 1/\alpha)^{-(1-1/\alpha)}\alpha^{1/\alpha}t^{1-1/\alpha}(\log|\log t|)^{1/\alpha}.$$

(c) *Critical symmetric processes.*

Let X be a Lévy process whose exponent function is of the form

$$\psi(\lambda) = -\int_{-\infty}^{\infty}\left(e^{i\lambda y} - 1 - \frac{i\lambda y}{1 + y^2}\right)\mu(dy), \tag{8.8}$$

where

$$\mu(dy) = |y|^{-2}\left(\log\frac{1}{|y|}\right)^\alpha I(|y| < 1).$$

Then X has a jointly continuous local time if and only if $\alpha > 2$ (see Barlow [1, Section 4, e.g., 3]), and

$$\psi(\lambda) \sim \pi|\lambda|(\log|\lambda|)^\alpha \quad \text{as } |\lambda| \to \infty \tag{8.9}$$

(see (4.21) in Getoor and Kesten [11]). Proposition 1.3 gives us (R_0). In fact a calculation (use Lemma 3.3 and (8.9)) shows that

$$g(s) \sim \pi^2(\alpha - 1)(\log s)^{\alpha-1} \quad \text{as } s \to \infty. \tag{8.10}$$

Moreover from Barlow [1, Section 4, e.g., 3] one obtains

$$\delta(x) \sim c(\log 1/x)^{1-\alpha} \quad \text{as } x \downarrow 0$$

for some $c > 0$. Therefore the hypotheses of Theorem 1.5 are satisfied if $\alpha > 3$ and in this case we have

$$(8.11)\quad \limsup_{\substack{\varepsilon \downarrow 0\ x \in \mathbf{R},\\ t \leq T}} \left| \frac{m(a(t,x,\varepsilon))\pi^{-2}(\alpha-1)^{-1}}{\varepsilon(\log 1/\varepsilon)^{\alpha-1}} - L_t^x \right| = 0 \quad \forall T > 0 \quad \text{a.s.}$$

(8.10) implies

$$f(t) \sim \pi^{-2}(\alpha-1)^{-1}(\log 1/t)^{1-\alpha}\log(\log|\log 1/t|) \equiv \phi(t)$$

and so for $\alpha > 4$ we may apply Theorem 1.2 to get

$$(8.12)\qquad \phi\text{-}m\{s \leq t | X_s = x\} = L_t^x \quad \forall x \in R,\ t \geq 0 \quad \text{a.s.}$$

In fact one can do slightly better by using Remark 4.2. Fix $\varepsilon > 0$ and $u \in (0,1)$. If $\varepsilon_0 > 0$ and $S_n = \{ie^{-n}n^{-1} | i = 0, 1, \dots, [e^n n]\}$, then Lemma 7 and Theorem 8 of Perkins [19] together show there is an $N(\omega) < \infty$ a.s. such that

$$\sup_{t \in S_n,\, 0 \leq s \leq e^{-n}} |(L_{s+t}^x - L_t^x) - (L_{s+t}^y - L_t^y)|$$
$$\leq cn^{(2-\alpha)/2+\varepsilon_0}(\log|x-y|^{-1})^{1-(\alpha/2)} \quad \forall |x-y| \leq e^{-n} \quad \text{and} \quad n \geq N(\omega).$$

An elementary interpolation argument now shows there is an $\eta(\omega) >$ a.s. such that for $0 < \delta < s < \eta(\omega)$,

$$(8.13)$$
$$\sup_{\substack{t \leq 1,\\ |x-y| \leq \delta}} |(L_{s+t}^x - L_t^x) - (L_{s+t}^y - L_t^y)| \leq c(\log 1/s)^{(2-\alpha)/2+\varepsilon_0}(\log 1/\delta)^{1-(\alpha/2)}.$$

If $f(u_n) = u^n$, then $u_n \ll e^{-u^{-n/(\alpha-1)}}$. Let $\delta_n = e^{-u^{-n\gamma}}$ where $\gamma \geq (\alpha-1)^{-1}$. Therefore (8.13) implies that for a.a. ω and large enough n,

$$\sup_{t \leq 1,\, |x-y| \leq \delta_n} \left| L_{t+u_n}^x - L_t^x - \left(L_{t+u_n}^y - L_t^y\right) \right| \leq u^{nk}$$

where

$$k = \frac{\alpha-2}{2(\alpha-1)} - \frac{\varepsilon_0}{\alpha-1} - \gamma(1-\alpha/2).$$

The above expression will be less than εu^n for some $\varepsilon_0 > 0$ and for large n, if

$$\gamma > \frac{2}{\alpha - 2} - \frac{1}{\alpha - 1}, \quad \gamma \ge (\alpha - 1)^{-1}.$$

Therefore (H″) will hold (see Remark 4.2) if $1 > 2/(\alpha - 2) - 1/(\alpha - 1)$ and $(\alpha - 1)^{-1} < 1$, or equivalently to $\alpha > 2 + \sqrt{2}$, and hence the conclusion of Theorem 4.1 will hold for $\alpha > 2 + \sqrt{2}$. Since Theorem 5.1 applies for $\alpha > 3$, we see that (8.12) is true whenever $\alpha > 2 + \sqrt{2}$.

(d) *Critical asymmetric processes.*

Finally we consider Lévy processes X whose exponent function ψ is of the form (8.8) where

$$\mu(dy) = y^{-2}\left(\log\frac{1}{|y|}\right)^{\alpha}(pI(0 < y < 1) + qI(-1 < y < 0)),$$
$$p, q > 0,\ p + q = 1,\ p \neq 1/2.$$

X has a jointly continuous local time if and only if $\alpha > 0$ (Barlow [1, Section 4, e.g., 3]). Some uninteresting calculations lead to

$$\left.\begin{aligned} \operatorname{Re}\psi(\lambda) &\sim \frac{\pi}{2}|\lambda|(\log|\lambda|)^{\alpha} \\ I_m\psi(\lambda) &\sim \frac{p-q}{\alpha+1}\lambda(\log|\lambda|)^{\alpha+1} \end{aligned}\right\} \quad \text{as } |\lambda| \to \infty$$

$$g(s) \sim \frac{2(p-q)^2}{\alpha+1}(\log s)^{\alpha+1} \qquad \text{as } s \to \infty$$

$$f(t) \sim \frac{(\alpha+1)}{2(p-q)^2}(\log 1/t)^{-1-\alpha}\log(\log(\log 1/t)) \qquad \text{as } t \downarrow 0$$

$$\delta(x) \sim c(\log 1/x)^{-1-\alpha} \qquad \text{as } x \downarrow 0.$$

Theorem 1.5 implies that for $\alpha > 1$,

$$\text{(8.14)} \quad \limsup_{\varepsilon\downarrow 0}\ \sup_{\substack{x\in\mathbf{R},\\ t\le T}}\left|\frac{(\alpha+1)}{2(p-q)^2}\frac{m(a(t,x,\varepsilon))}{\varepsilon(\log 1/\varepsilon)^{\alpha+1}} - L_t^x\right| \quad \text{for all T} > 0 \quad \text{a.s.}$$

Theorem 1.2 implies that for $\alpha > 2$,

$$\text{(8.15)} \qquad \phi\text{-}m\{s \le t | X_s = x\} = L_t^x \quad \forall x \in \mathbf{R},\ t \ge 0 \quad \text{a.s.},$$

where

$$\phi(t) = \frac{(\alpha+1)}{2(p-q)^2}(\log 1/t)^{-1-\alpha}\log(\log(\log 1/t)).$$

The energetic reader may use the techniques of [19] to show that (8.15) holds for a slightly larger class of α, as in (c), but unfortunately not for all $\alpha > 0$.

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STATISTICAL LABORATORY
16 MILL LANE, CAMBRIDGE, ENGLAND

UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER, BRITISH COLUMBIA, CANADA

UNIVERSITY OF VIRGINIA
CHARLOTTESVILLE, VIRGINIA