ILLINOIS JOURNAL OF MATHEMATICS
Volume 30, Number 2, Summer 1986

ON DEHN PRESENTATIONS AND DEHN ALGORITHMS

BY

ELENA GREENDLINGER AND MARTIN GREENDLINGER

Dedicated to the memory of William Werner Boone

The origin of this note is the observation that somewhat stronger results can be obtained from certain proofs of theorems yielding the applicability of Dehn's algorithm for the word problem to one or another class of groups. This has undoubtedly been noticed by many, but never stated, because the stronger results have less elegant formulations and no known additional applications. However, some of their analogues for groups in larger classes do have additional applications, allowing simplifications of proofs and algorithms. The reader is assumed to be familiar with Chapter V of [4]. The smallest number of pieces into which a word w can be decomposed will be denoted by $\|w\|$. Pieces will mean non-empty pieces. Diagrams of minimal R-sequences will be called minimal R-diagrams. Words will mean cyclic words whenever possible.

We now define various classes of Dehn presentations. Let $G = \langle \mathbf{X}; R \rangle$, where R is symmetrized.

DEFINITION 1. G is a weak, strict, metric Dehn presentation iff given any freely reduced word $w = 1$ in G, there exist a subword s of w and a word t such that $s\bar{t} \in R$ and $|s| > |t|$.

DEFINITION 2. G is a strong, strict, metric Dehn presentation iff given any freely reduced word $w = 1$ in G and any minimal R-diagram M for w, there exist boundary region D and words s, t such that $s\bar{t}$ is a label of ∂D, s is the label of $\partial D \cap \partial M$, a consecutive part of M, and $|s| > |t|$.

Definitions 3 and 4 are obtained from Definitions 1 and 2, respectively, by omitting "strict" and replacing " $>$ " by " $\geq$ ".

Definitions 5, 6, 7 and 8 are obtained from Definitions 1, 2, 3 and 4, respectively, by changing "metric" to "non-metric" and "$|\ \ |$" to "$\|\ \ \|$".

Conjecture. The finitely presented subclass of each of the above classes is non-recursive.

Received December 2, 1985.

It is clear that every strong Dehn presentation is also a corresponding weak Dehn presentation and that Dehn's algorithm solves the word problem for all weak, strict, metric Dehn presentations with finite R or finite X and recursive R. We now define three strong Dehn algorithms which can be applied to the classes of presentations given by Definitions 4, 6 and 8, but not to those given by Definitions 3, 5 and 7.

DEFINITION 9. The strong, metric Dehn algorithm consists in starting with a word w, repeatedly freely reducing and replacing in all possible ways subwords s by words t whenever $s\bar{t} \in R$ and $|s| \geq |t|$, concluding that $w = 1$ if we get the empty word and concluding that $w \neq 1$ if the set of freely reduced words so obtained is finite and closed under all such replacements, followed by free reductions, but does not contain the empty word.

DEFINITION 10. The strong, strict, non-metric Dehn algorithm consists in decomposing a given freely reduced word w into products of pieces in all possible ways. For each such decomposition α, we look for an integer n, a subword s of w and a word t such that $s\bar{t} \in R$, α decomposes s into n pieces, s begins and ends between pieces of α and t can be decomposed into fewer than n pieces. If there are no such n, s, t for any α, we conclude that $w \neq 1$. If there are, we replace s by t and freely reduce. Let $w = xsy$ and let x_1, t_1, y_1 be the uncancelled subwords of x, t, y, respectively. Starting with each decomposition β of $x_1t_1y_1$ which is compatible with α, i.e., which coincides with α on x_1, y_1 and decomposes t_1 into fewer than n pieces, repeat the procedure described for α, etc. If the empty word occurs, conclude that $w = 1$. If repetitions yield neither new words nor new decompositions of old words, but the empty word has not occurred, conclude that $w \neq 1$.

Definition 11 is obtained from Definition 10 by deleting "strict" and changing "fewer than n pieces" to "fewer than $n + 1$ pieces".

It is not difficult to show that unlike Dehn's algorithm, none of the strong Dehn algorithms just defined works for all the corresponding finite weak Dehn presentations. Consider, for example,

$$G = \langle a_i, b_i, c_i, d_i;\ \bar{a}_i b_i a_{i+1},\ \bar{b}_i c_i d_i \bar{b}_{i+1} \rangle,$$

where subscripts are taken mod 4 and $i = 0, 1, 2, 3$.

PROPOSITION. *G is a weak, metric Dehn presentation to which the strong, metric Dehn algorithm is not applicable.*

Proof. It is easily seen that $b_1 b_2 b_3 b_0 = 1$ in G. However, if we start with this word, the strong, metric Dehn algorithm yields the following set of freely reduced words:

$$S = \{b_1b_2b_3b_0,\, c_1d_1b_3b_0,\, b_1b_2c_3d_3,\, b_1c_2d_2b_0,\, c_1d_1c_3d_3,\, c_0d_0b_2b_3,\, c_0d_0c_2d_2\}.$$

S is closed under all replacements of at least half of a defining relator by the inverse of the remainder.

To complete the proof, let M be a minimal R-diagram for a word w. If ∂M is simple, consider cases based first on the number (either $0, 1$, or ≥ 2) of M's regions with defining relators of length 4, and then on the total number (either $1, 2, 3, 4$, or ≥ 5) of M's regions. A routine analysis suffices in each case. For arbitrary ∂M, we apply Lemma 4.2 in Chapter V of [4].

Problem. In each class of weak Dehn presentations, find one with the smallest possible number of generators or defining relators, whose word problem cannot be solved by the corresponding strong Dehn algorithm.

Question. Is the word problem solvable for all finite weak Dehn presentations?

It turns out that with an appropriate assumption on X or R, each strong Dehn algorithm solves the word problem for the corresponding class of strong Dehn presentations. We confine ourselves to a precise statement and proof of this fact for the largest class, that of strong, non-metric Dehn presentations. The others can be handled similarly.

Remark. No Dehn algorithm can ever falsely conclude that $w = 1$, since every word it yields is conjugate to the one with which it starts.

LEMMA. *When applied to a word w in a finitely related group G, the strong non-metric Dehn algorithm terminates in a finite number of steps.*

Proof. The number of pieces in any decomposition of any word obtained will be $\leq |w|$. The finiteness of R insures that the number of pieces in G is finite, say equal to P. Therefore, the number of decompositions obtained will be $\leq (P+1)|w|$.

THEOREM. *The strong, non-metric Dehn algorithm solves the word problem for all finitely related strong, non-metric Dehn presentations.*

Proof. In view of the remark and lemma, we need only prove tht the given algorithm never falsely concludes that $w \neq 1$. Assume that it does. Let M be a minimal R-diagram for w. Let α be a decomposition of w based on M, i.e., one obtained by decomposing the labels of M's boundary edges into products of minimal numbers of pieces. If we apply the algorithm to α, instead of to all possible decompositions of w, it will again fail to yield the empty word. Let $n(\alpha)$ be the number of regions of M. Among all the decompositions based on minimal R-diagrams, such that the algorithm does not yield the empty word when applied to them, consider one, say α, with the smallest possible $n(\alpha)$. By definition, there are words s, t and a boundary region D of M such that $s\bar{t}$ is a label of ∂D, S is a label of $\partial D \cap \partial M$, a consecutive part of M, and $\|s\| \geq \|t\|$. Therefore, the word w' obtained from w by substituting t for s and freely reducing has a minimal R-diagram M' which can be gotten from M by deleting D and possibly identifying parts of edges and then deleting pairs of other regions. Thus M' has at most $n(\alpha) - 1$ regions. Let β be a decomposition of w' based on M' which is compatible with α. If we apply the algorithm to β, it will not yield the empty word, since it yields β when applied to α. But this contradicts our choice of α.

COROLLARY. *The strong, non-metric Dehn algorithm solves the word problem for all finitely related Dehn presentations satisfying $C(6)$ or $C(4)$ & $T(4)$.*

Proof. A careful reading of proofs in [3] or Chapter V of [4] shows that every presentation satisfying $C(6)$ or $C(4)$ & $T(4)$ is a strong, non-metric Dehn presentation. This can also be extracted from [2] and Lemma 4.2 in Chapter V of [4]. For the $C(6)$ case, it can be found in [1], but in a combinatorial guise.

In conclusion, we note that the solvability of the word problem for all finitely related presentations satisfying $C(6)$ or $C(4)$ & $T(4)$ by means of a more complicated algorithm was first proven in [3].

LITERATURE

1. A.V. GLADKIĬ, *On groups with k-reducible bases*, Sibirsk. Math. J., vol. 2 (1961), pp. 366–383.
2. L. GREENDLINGER and M. GREENDLINGER, *On three of Lyndon's results about maps*, Contemp. Math., vol. 33 (1984), pp. 212–213.
3. R.C. LYNDON, *On Dehn's algorithm*, Math. Ann., vol. 166 (1966), pp. 208–228.
4. R.C. LYNDON and P.E. SCHUPP, *Combinatorial group theory*, Springer-Verlag, Berlin, 1977.

SHUYA STATE FURMANOV PEDAGOGICAL INSTITUTE
SHUYA, USSR