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ELEMENTARY GROUP EQUIVALENCE WITH THE INTEGRAL LENGTH FUNCTION

BY

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In Memoriam W.W. Boone

The paper determines criteria of elementary equivalence for some classes of free groups with operators and free products with the length function. The case of a group with operators admitting rational coordinatization with a finite basis is completely analyzed. They are polycyclic, solvable groups of finite rank without torsion, and Chernikov groups. The concept of ω-isomorphism of groups intermediate between elementary equivalence and isomorphism is important for the aspects of elementary equivalence of groups with operators and free product. It is proved that ω-isomorphism of arbitrary groups of operators is followed by the elementary equivalence of the respective free operator groups (free products) with length function.

1. Groups with the length function

This section presents some information on free groups, free groups with operators and free products which will be needed later.

1.1. Free groups and free products. The facts given below may be found, for example, in [1]. Let G be a group, $N = \{0, 1, 2, \dots\}$ be a set of positive integers. The function $|\ \ |\colon G \to N$ will be called a length function on the group G if it satisfies the following conditions (for the sake of brevity, we let $d(x, y) = \frac{1}{2}(|x| + |y| - |xy^{-1}|)$).

(A1) $|x| = 0 \Leftrightarrow x = 1$,
(A2) $|x^{-1}| = |x|$,
(A3) $d(x, y) \geq 0$,
(A4) $d(x, y) \geq d(x, z) \Rightarrow d(y, z) = d(x, z)$,
(A5) $d(x, y) + d(x^{-1}, y^{-1}) \geq |x| = |y| \Rightarrow x = y$.

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Let $F(X)$ be a free group with the basis X. If $y = x_1 \dots x_n$ is an irreducible expression for the element $y \in F(X)$ in the basis X, $x_i^{\pm 1} \in X$, then the number $n = |y|$ is called the length of y, and $d(x, y)$ is the length of the largest coinciding end of the words x and y. Apparently $y \to |y|$ is a length function on the group $F(X)$ since it satisfies axioms (A1)–(A5) and, in addition, satisfies:

(A6) $x \neq 1 \Rightarrow |x| < |x^2|$.

Similarly, the free product $A = A_1 * \cdots * A_m$ of arbitrary groups $A_1, \dots, A_m$ is a group with a length function, if one considers the length $|y|$ of the element $y \in A$ to be the number of syllables n in the canonical expression $y = a_1 \dots a_n$ where each $1 \neq a_i$ is in one of the groups $A_1, \dots, A_m$, with neighbouring elements in different groups. The following theorems show that free groups and free products are completely characterized by the presence of an integral length function.

THEOREM 1 [1]. *Let G be an arbitrary group with the length function satisfying axiom* (A6). *Then G is a free group; here G could be imbedded in the free group $F(X)$ with the basis X so that the length function on G is the restriction of the length function on $F(X)$ with respect to the basis X.*

THEOREM 2 [1]. *Let G be an arbitrary group with the length function. Then G can be imbedded in the free product $A = A_1 * \cdots * A_m$ such that the length function on G is the restriction of the length function on A with respect to the given decomposition.*

1.2. Free operator groups. For an arbitrary group A and a set of letters X we will construct a free A-operator group $\Gamma(X, A)$ with the basis X. Let X^A be a set of symbols $\{x^a | x \in X, a \in A\}$; then $F(X^A)$ is a free group with the basis X^A. We determine the action A on $F(X^a)$ in the following way: for each $a \in A$ the map $x^b \to x^{ba}$, $b, a \in A$ is extended to automorphism φ_a of the group $F(X^A)$; for the element $u \in F(X^A)$ we set $u^a = u^{\varphi_a}$. The group $F(X^A)$ along with the action of A on $F(X^A)$ becomes a free A-operator group with the basis X. The group will be denoted by $\Gamma(X, A)$.

For $u \in F(X^A)$ the inverse to the element u^a will be written in the form u^{-a}; $u^{\pm A} = \{u^{\pm a} | a \in A\}$. If U is a subset of $F(X^A)$, $U^{\pm A} = \bigcup_{u \in U} u^{\pm A}$; $gp(U)$ is the subgroup of $F(X^A)$ generated by U (non-operator).

DEFINITION. Elements $u, v \in F(X^A)$ will be called collinear if $u = v^{\pm a}$ for some $a \in A$, i.e., $u \in v^{\pm A}$.

The length function $|\ |$ defined on $F(X^A)$ with respect to the basis X^A will be called a length function on $\Gamma(X, A)$.

2. Model theory information

2.1. Multi-base models. Later we will constantly make use of the language of multisorted predicate calculus of the first order (for example, see [2]). As mentioned in Section 1, the presence of the length function is characteristic for free groups and free products. Therefore in studying these objects by model theory methods it seems natural to extend the language of group signature introducing length function there. In exactly the same way, in considering operator groups it is natural to take into account the structure of a group of operators. Thus, with a free product of groups $G = A_1 * \cdots * A_n$ we associate a two-base model $\Pi(G) = \langle G, N; |\ \ | \rangle$ where G is a group, N is a set of positive integers with selected element 1, operation of addition and predicate of order, and $|\ \ |$ is a predicate selecting length function $G \to N$.

With a group of $\Gamma(X, A)$ we associate a three-base model

$$\Pi(X, A) = \left\langle F(X^A), A, N; \delta_A, |\ \ | \right\rangle,$$

where $F(X^A)$, A are groups, N and $|\ \ |$ are the same as in $\Pi(G)$ and δ_A is a predicate selecting the action A on $F(X^A)$.

To each multi-base model $\mathfrak{M}$ there corresponds an ordinary model $\mathfrak{M}^*$ of the first-order predicate calculus obtained from $\mathfrak{M}$ by unifications of variables. Speaking of model theory properties of $\mathfrak{M}$ one can mean corresponding properties of $\mathfrak{M}^*$. And elementary equivalence of models $\mathfrak{M}$ and $\mathfrak{N}$ means elementary equivalence of their unifications $\mathfrak{M}^*$ and $\mathfrak{N}^*$.

2.2. Partial isomorphisms and elementary equivalence. Let $\mathfrak{A}$ be an algebraic system of signature Ω. A set of all the closed formulas of signature Ω true in $\mathfrak{A}$ is called the theory of the system $\mathfrak{A}$ and is denoted by $\mathrm{Th}(\mathfrak{A})$. Systems $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent (briefly $\mathfrak{A} \equiv \mathfrak{B}$) if $\mathrm{Th}(\mathfrak{A}) \equiv \mathrm{Th}(\mathfrak{B})$.

Let A and B be basic sets of systems $\mathfrak{A}$ and $\mathfrak{B}$ respectively. The injective map φ: $X \to B$, $X \subseteq A$ is called a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ if for any $a_1, \ldots, a_n \in X$ the following conditions are fulfilled:

(1) If f is a symbol of operation from Ω then

$$f(a_1, \ldots, a_{n-1}) = a_n \Leftrightarrow f(\varphi(a_1), \ldots, \varphi(a_{n-1})) = \varphi(a_n).$$

(2) If P is a predicate symbol from Ω then

$$P(a_1, \ldots, a_n) \Leftrightarrow P(\varphi(a_1), \ldots, \varphi(a_n)).$$

The domain of definition X of the map φ will be denoted by $\operatorname{dom}\varphi$ and the domain of values $\varphi(X)$ by $\operatorname{Im}\varphi$. If the set $\operatorname{dom}\varphi$ is finite, the partial isomorphism φ is called finite; if φ is a restriction of the partial isomorphism ψ, ψ will be called a covering of φ.

The following criterion of elementary equivalence is valid (for example, see [3]).

THEOREM 3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be systems of a finite signature. Then $\mathfrak{A} \equiv \mathfrak{B}$ iff for any positive integer m there exist non-empty sets $\phi_1(m) \subseteq \cdots \subseteq \phi_m(m)$ of partial isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ with the following property*:

(*) *If $\varphi \in \phi_i(m)$, $1 \leq i < m$, then for any $a \in A$, $b \in B$ there exist coverings $\psi_1, \psi_2 \in \phi_{i+1}(m)$ for φ with the property $a \in \operatorname{dom} \psi_1$, $b \in \operatorname{Im} \psi_2$.*

Note 1. Theorem 3 remains valid if the condition (*) is fulfilled only for some infinite subset of positive integers.

Note 2. Let $\phi_1(m) \subseteq \cdots \subseteq \phi_m(m)$ be a succession of sets of partial isomorphisms satisfying (*). Then for any $\varphi \in \phi_1(m)$ there exists a succession of non-empty sets of partial isomorphisms $\phi_1(m, \varphi) \subseteq \cdots \subseteq \phi_m(m, \varphi)$ satisfying the condition (*), where each isomorphism from $\phi_i(m, \varphi)$ is a covering of φ.

In fact, one may take the set $\phi_i(m, \varphi) = \{\psi \in \phi_i(m) | \psi \text{ covers } \varphi\}$.

Note 3. For any positive integer m, let the systems $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the condition (*), and let A_0 be an arbitrary finite subset of A. Then for any m there exists a succession $\phi_1(m, A_0) \subseteq \cdots \subseteq \phi_m(m, A_0)$ of non-empty sets of finite partial isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ satisfying (*), where, for any $\psi \in \phi_i(m, A_0)$ we have $A_0 \subseteq \operatorname{dom} \psi$, $1 \leq i \leq m$.

In fact, let $m_0 = |A_0|$; then for any m there exists $\varphi \in \phi_{m_0}(m + m_0)$ such that $A_0 \subseteq \operatorname{dom} \varphi$. Now, it is sufficient to make use of Note 2 for the succession

$$\phi'_1(m) \subseteq \cdots \subseteq \phi'_m(m) \quad \text{where } \phi'_i(m) = \phi_{m_0+i}(m_0 + m), 1 \leq i < m.$$

Note 4. Theorem 3 remains valid if $\mathfrak{A}$ and $\mathfrak{B}$ are multi-base models of many-sorted logics of the first order predicates. For the proof, it is sufficient to proceed to unified models $\mathfrak{A}^*$ and $\mathfrak{B}^*$ and make use of Theorem 3.

2.3. ω-isomorphism. The following conception is central for the formulation of basic results of the paper.

DEFINITION. Systems $\mathfrak{A}$ and $\mathfrak{B}$ will be called ω-isomorphic if for any positive integer m there exist non-empty sets $\phi_1(m) \subseteq \cdots \subseteq \phi_m(m)$ of finite partial isomorphisms of $\mathfrak{A}$ in $\mathfrak{B}$ with the following property:

(**) For any $\varphi \in \phi_i(m)$, $1 \leq i < m$, and for any finite subsets $X \subseteq A$, $Y \subseteq B$ there exist coverings $\psi_1, \psi_2 \in \phi_{i+1}(m)$ for φ such that $X \subseteq \operatorname{dom} \psi_1$, $Y \subseteq \operatorname{Im} \psi_2$.

It is clear that the notes following Theorem 3 hold if one replaces $(*)$ by $(**)$.

It is evident that ω-isomorphic systems are elementarily equivalent. To study the properties of ω-isomorphic systems we need the concept of a restricted type. Let m be a positive integer. The set of all formulas $\phi(x_1, \ldots, x_n)$ whose prenex normal form contains not more than m-blocks of quantifiers of the same type (the so-called Σ_m^0-, Π_m^0- formulas) and which are true in $\mathfrak{A}$ on $\bar{a}$ is called the m-restricted type of a procession of elements $\bar{a} = (a_1, \ldots, a_n)$ from A.

PROPOSITION 1. *Let systems $\mathfrak{A}$ and $\mathfrak{B}$ be ω-isomorphic. Then:*

(1) *The same restricted types are implemented in $\mathfrak{A}$ and $\mathfrak{B}$.*

(2) *If $\mathfrak{A}$ and $\mathfrak{B}$ are finitely generated, then $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic.*

Proof. Let the systems $\mathfrak{A}$ and $\mathfrak{B}$ be ω-isomorphic, ϕ a formula given in the prenex normal form, $r(\phi)$ the number of blocks of quantifiers of the same type as in ϕ. We prove by induction on $r(\phi)$ that for any $\varphi \in \phi_i(m)$, $i < m - r(\phi)$, and any procession $\bar{a} = (a_1, \ldots, a_n)$ from A, $a_k \in \operatorname{dom} \varphi$, the formula $\phi(\bar{a})$ is true in $\mathfrak{A}$ iff $\phi(\bar{b})$ is true in $\mathfrak{B}$ where $\bar{b} = (\varphi(a_1), \ldots, \varphi(a_n))$, $1 \leq k \leq n$.

Let $r(\phi) = 0$, $i < m$. Then ϕ is an atomic formula (i.e., ϕ does not contain quantifiers). Consider the set $F = \{f_j(\bar{a})\}$ of all the terms contained in $\phi(\bar{a})$. It is sufficient to prove that

$$f_j(\bar{a}) \equiv f_k(\bar{a}) \Leftrightarrow f_j(\bar{b}) = f_k(\bar{b}) \tag{1}$$

for any $f_j, f_k \in F$. From the condition we find the covering $\psi \in \phi_{i+1}(m)$ for φ such that $F \subseteq \operatorname{dom} \psi$. Due to the partial isomorphism of ψ equivalence (1) is valid, consequently $\phi(\bar{b})$ is true in $\mathfrak{B}$. Now assume that

$$\phi(\bar{a}) = \exists \bar{x} \phi_1(\bar{x}, \bar{a}), r(\phi_1) < r(\phi), i < m - r(\phi)$$

and $\phi(\bar{a})$ is true in $\mathfrak{A}$. Then $\phi_1(\bar{c}, \bar{a})$ is true in $\mathfrak{A}$ for some procession $\bar{c}$. Since $r(\phi) \geq 1$, $i < m - 1$, there exists a covering $\psi \in \phi_{i+1}(m)$ for φ such that ψ is defined on elements from $\bar{c}$. From the inequality $r(\phi_1) < r(\phi)$ we have $i + 1 < m - r(\phi_1)$; hence by induction we get the formula $\phi_1(\psi(\bar{c}), \bar{b})$, and the formula $\exists \bar{x} \phi_1(\bar{x}, \bar{b})$ is also true in $\mathfrak{B}$. The case

$$\phi(\bar{a}) = \forall \bar{x} \phi_1(\bar{x}, \bar{a})$$

is analyzed in the same way. Thus for any $\varphi \in \phi_i(m)$ and any procession $\bar{a} = (a_1, \ldots, a_n)$, $a_k \in \operatorname{dom} \varphi$, processions $\bar{a}$ and $\bar{b}$ where $\bar{b} = (\varphi(a_1), \ldots, \varphi(a_n))$ satisfy the same formulas with the condition $r(\phi) < m - i$. Consequently for any number t and procession $\bar{a}$ from A one can find the map $\psi \in \phi_2(t + 3)$ such that the elements from $\bar{a}$ belong to $\operatorname{dom} \psi$ and processions

$\bar{a}, \psi(\bar{a})$ produce the same t-restricted type in $\mathfrak{A}$ and $\mathfrak{B}$. Statement (1) is proved.

Let us prove statement (2). Let $A_0 = \{a_1, \ldots, a_k\}$, $B_0 = \{d_1, \ldots, d_n\}$ be finite sets of generators of the systems $\mathfrak{A}$ and $\mathfrak{B}$ respectively. By the ω-isomorphism of $\mathfrak{A}$ and $\mathfrak{B}$ there exists a partial isomorphism $\varphi \in \phi_2(4)$ such that $A_0 \in \operatorname{dom} \varphi$, and a covering $\psi \in \phi_3(4)$ for φ such that $B_0 \subseteq \operatorname{Im} \psi$. Let $\psi(a_i) = b_i$, $1 \leq i \leq k$. The map ψ can be extended to the map $\bar{\psi}$: $\mathfrak{A} \to \mathfrak{B}$ according to the rule $\bar{\psi}$: $f(\bar{a}) \to f(\bar{b})$ where f is an arbitrary term of the signature $\mathfrak{A}$, $\bar{b} = (b_1, \ldots, b_k)$. By (1) we have

$$f(\bar{a}) = g(\bar{a}) \Leftrightarrow f(\bar{b}) = g(\bar{b})$$

for the arbitrary terms f and g. Consequently, the map $\bar{\psi}$ is correctly defined and is a monomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. Moreover, according to construction, $B_0 \subseteq \operatorname{Im} \bar{\psi}$ so $\operatorname{Im} \bar{\psi} = \mathfrak{B}$, i.e., $\bar{\psi}$ is an isomorphism. The proposition is proved.

From Proposition 1 it is evident that the property of ω-isomorphism is stronger than elementary equivalence and weaker than isomorphism. Let us consider some examples. In the class of finitely generated abelian groups all the three conceptions coincide [4]. But it is quite different for non-finitely generated abelian groups. For example, groups $\mathbf{Z}$ and $\mathbf{Z} \oplus \mathbf{Q}$, the additive groups of the integers and rationals, are elementary-equivalent [4], however they are not ω-isomorphic, since in $\mathbf{Z}$, the 1-restricted type of any nonzero element from $\mathbf{Q}$ is not implemented. In the class of nilpotent groups there are examples of finitely generated groups which are elementarily equivalent but not isomorphic, and hence not ω-isomorphic [5], [6].

2.4. Regular definability. In studying elementary theories of algebraic systems the concept introduced in [7] of regular definability of one algebraic system in another appears useful. This is due to the fact that though regular definability is definability with constants, it possesses properties similar to definability without constants.

First of all, let us recall the concept of relative definability (i.e., with constants) of one algebraic system in another. Let $\mathfrak{A}$ be an algebraic system of signature Σ, $\mathfrak{B}$ a system of signature Δ. Proceeding to predicates representing signature operations in $\mathfrak{A}$, the signature Σ may be considered to contain only predicate symbols.

DEFINITION. The system $\mathfrak{A}$ is relatively defined in $\mathfrak{B}$ by means of a set of formulas

$$\psi = \left\{ A(\bar{x}, \bar{y}), \mathscr{E}(\bar{x}, \bar{y}^1, \bar{y}^2), \psi_\sigma(\bar{x}, \bar{y}^1, \ldots, \bar{y}^{t_\sigma}) | \sigma \in \Sigma \right\}$$

of the signature Δ where $\bar{x} = (x_1, \dots, x_n)$, $\bar{y}^i = (y_1^i, \dots, y_m^i)$, and a set of constants $\bar{b} = (b_1, \dots, b_n)$ from $\mathfrak{B}$ (symbolically, $\mathfrak{A} \simeq \psi(\mathfrak{B}, \bar{b})$) if the following conditions are fulfilled:

(1) The set $A = \{\bar{a} \in |\mathfrak{B}|^m | \mathfrak{B} \models A(\bar{b}, \bar{a})\}$ is non-empty;

(2) The formula $\mathscr{E}(\bar{b}, \bar{y}^1, \bar{y}^2)$ gives an equivalence relation on A (the equivalence class of the element $a \in A$ is denoted by $[a]$)

(3) If σ is a symbol of an s-ary predicate from Σ, then $t_\sigma = s$ and the formula $\psi_\sigma(\bar{b}, \bar{y}^1, \dots, \bar{y}^s)$ gives the predicate P_σ on the factor-set $A/\mathscr{E}$ according to the rule

$$P_\sigma([a^1], \dots, [a^s]) \overset{\text{df}}{\rightleftharpoons} \psi_\sigma(\bar{b}, \bar{a}^1, \dots, \bar{a}^s), \quad \text{where } \bar{a}^i \in A, 1 \le i \le s.$$

(4) The system $\psi(\mathfrak{B}, b) = \langle A/\mathscr{E}; P_\sigma | \sigma \in \Sigma \rangle$ is isomorphic to $\mathfrak{A}$.

The system $\psi(\mathfrak{B}, \bar{b})$ is called an interpretation of $\mathfrak{A}$ in $\mathfrak{B}$ and the isomorphism of the interpretation is the isomorphism μ: $\mathfrak{A} \to \psi(\mathfrak{B}, b)$.

DEFINITION. If $\mathfrak{A} \simeq \psi(\mathfrak{B}, \bar{b})$ and the set of constants $\bar{b}$ is empty, then we say $\mathfrak{A}$ is absolutely definable in $\mathfrak{B}$ or definable without constants.

For the formula ϕ of the signature Δ we set

$$\phi(\mathfrak{B}) = \{\bar{b} \in |\mathfrak{B}|^m | \mathfrak{B} \models \phi(\bar{b})\}$$

(such sets are called formula definable).

DEFINITION. The system $\mathfrak{A}$ is regularly definable in $\mathfrak{B}$ by a set of formulas ψ and formulas ϕ of the signature Δ (symbolically $\mathfrak{A} \simeq \psi(\mathfrak{B}, \phi)$) if $\mathfrak{A}$ is relatively definable in $\mathfrak{B}$ with the help of ψ and any set $\bar{b} \in \phi(\mathfrak{B})$.

It is evident that absolute definability implies regular definability and it, in turn, implies relative definability.

Let $\mathfrak{A} \simeq \psi(\mathfrak{B}, \phi)$. The interpretation $\psi(b_1\bar{b})$, $\bar{b} \in \phi(\mathfrak{B})$, will be denoted by $\mathfrak{A}(b)$ (for the sake of brevity).

DEFINITION. The arbitrary isomorphism $\mathfrak{A}(\bar{b}) \to \mathfrak{A}(\bar{c})$, $\bar{b}, \bar{c} \in \phi(\mathfrak{B})$, will be called the connecting isomorphism. Connecting isomorphisms of the interpretation of $\mathfrak{A}$ are formula definable in $\mathfrak{B}$ if there exists a formula $\mathrm{Is}(\bar{x}, \bar{y}, \bar{z}_1, \bar{z}_2)$ of the signature Δ such that for any $\bar{b}, \bar{c} \in \phi(\mathfrak{B})$ the formula $\psi(\bar{x}, \bar{y}, \bar{b}, \bar{a})$ defines in $\mathfrak{B}$ some connecting isomorphism $\mathfrak{A}(\bar{b}) \to \mathfrak{A}(\bar{c})$.

Let us note a simple but convenient characteristic of absolute definability $\mathfrak{A}$ in $\mathfrak{B}$ [7].

PROPOSITION 2. *Let $\mathfrak{A}$ be regularly definable in $\mathfrak{B}$ and let connecting isomorphisms of the interpretation of $\mathfrak{A}$ be formula definable in $\mathfrak{B}$. Then $\mathfrak{A}$ is absolutely definable in $\mathfrak{B}$.*

If $\mathfrak{A} \simeq \psi(\mathfrak{B}, \bar{b})$, define a complete formula enrichment $\mathfrak{A}^M$ of the system $\mathfrak{A}$ in $\mathfrak{B}$ (according to Morley's conception) by adding all the predicates on $\psi(\mathfrak{B}, \bar{b})$ which are formula definable in $\mathfrak{B}$ in the signature of $\mathfrak{A}$.

3. Formula predicates of the model $\Gamma(X, A)$

Let A be a group, X a set, $\Gamma(X, A) = \langle F(X^A), A, N \rangle$ a three-base model (see Section 2) corresponding to the free A-operator group $F(X^A)$ with the length function $| \ \ |: F(X^A) \to N$. In this section we will determine formula definability in $\Gamma(X, A)$ of some predicates on $F(X^A)$, A, N which are not formula definable in their own signature in these systems.

The language of the model $\Gamma(X, A)$ contains three types of variables; later we will denote variables belonging to the group $F(X^A)$ by lower case letters near the end of the Latin alphabet such as $u, v, w, \ldots,$ variables for A by $a, b, c, \ldots$ and variables for N by $k, l, m, \ldots$. We feel free to use the same symbols to denote elements in $F(X^A)$, N, A respectively and write u^a instead of $\delta_A(u, a)$.

3.1. Formula predicates on $F(X^A)$. Let us set

$$S_H(u, v) \overset{\text{df}}{\rightleftharpoons} \exists u_1(u = vu_1 \,\&\, |v| = 1 \,\&\, |u| = |u_1| + 1),$$

$$S_K(u, v) \overset{\text{df}}{\rightleftharpoons} \exists u_1(u = u_1 v \,\&\, |v| = 1 \,\&\, |u| = |u_1| + 1),$$

$$S_C(u, v_1, v_2) \overset{\text{df}}{\rightleftharpoons} \exists u_1, u_2(u = u_1 v_1 v_2 u_2 \,\&\, |v_1| = 1 \,\&\, |v_2| = 1 \\ \&|u| = |u_1| + |u_2| + 2).$$

The predicates S_H, S_K, S_C define in $\Gamma(X, A)$ the first letter, the last letter, two neighbouring letters, respectively, in the irreducible writing of the element $u \in F(X^A)$. Let us also set

$$\mathscr{D}(u, a) \overset{\text{df}}{\rightleftharpoons} \forall v_1, v_2(S_C(u, v_1, v_2) \to v_2 = v_1^a)$$

The validity of $\mathscr{D}(u, a)$ in $\Gamma(X, A)$ means that

$$u = v_0 v_0^a \ldots v_0^{a^{|u|-1}} \quad \text{for some } v_0 \in X^{\pm A}.$$

LEMMA 1. *The predicate*

$$K(u_1,\ldots,u_n) \overset{\text{df}}{\rightleftharpoons} \textit{elements are pairwise noncollinear}$$

is formula definable in $\Gamma(X, A)$.

The proof follows from the equivalence

$$K(u_1,\ldots,u_n) \Leftrightarrow \forall a\Big(\underset{1\le i<j\le n}{\&}\big(u_i \ne u_j^a \,\&\, u_i^{-1} \ne u_j^a\big)\Big).$$

LEMMA 2. *For any positive integer m there exists a closed formula* ϕ_m *of the signature of the model* $\Gamma(X, A)$ *such that* ϕ_m *is true on* $\Gamma(X, A)$ *iff the cardinality* $|X|$ *is m.*

Proof. With the help of the formula $K(u_1,\ldots,u_n)$ from Lemma 1 it is easy to write down by a formula that the model $\Gamma(X, A)$ possesses just m pairwise noncollinear elements of length 1, a condition equivalent to $|X| = m$.
The lemma is proved.

3.2. Formula definable predicates on A

LEMMA 3. *The predicates*

$$C(a,b,m) \overset{\text{df}}{\rightleftharpoons} b = a^m,$$

$$C_N(a,b) \overset{\text{df}}{\rightleftharpoons} b \in a^N = \{a^n | n \in N\},$$

$$C_{\mathbf{Z}}(a,b) \overset{\text{df}}{\rightleftharpoons} b \in a^{\mathbf{Z}} = gp(a)$$

are formula definable in $\Gamma(X, A)$.

The proof follows from the equivalences

$$C(a,b,m) \Leftrightarrow \exists u, v_0, v_1\big(|u| = m+1 \,\&\, S_H(u,v_0) \,\&\, S_K(u,v_1) \,\& \\ \&\mathscr{D}(u,a) \,\&\, v_0^b = v_1\big),$$

$$C_N(a,b) \Leftrightarrow \exists m C(a,b,m),$$

$$C_{\mathbf{Z}}(a,b) \Leftrightarrow C_N(a,b) \vee C_N(a,b^{-1}).$$

LEMMA 4. *The predicates*

$$P_\infty(a) \overset{\text{df}}{\rightleftharpoons} \textit{the order } a \textit{ is infinite},$$

$$P(a, m) \overset{\text{df}}{\rightleftharpoons} \textit{the order } a \textit{ is equal to } m$$

are formula definable in $\Gamma(X, A)$.

The proof follows from the equivalences

$$P_\infty(a) \Leftrightarrow \forall m \neq 0 \urcorner C(a, 1, m),$$
$$P(a, m) \Leftrightarrow C(a, 1, m) \,\&\, \forall n (C(a, 1, n) \to m \leq n).$$

Elements $a_1, \dots, a_n \in A$ are called linearly independent if

$$a_1^{m_1} \dots a_n^{m_n} = 1 \Rightarrow m_1 = \cdots = m_n = 0$$

for any positive integers $m_1, \dots, m_n$.

LEMMA 5. *The predicate*

$$I(a_1, \dots, a_n) \overset{\text{df}}{\rightleftharpoons} a_1, \dots, a_n \textit{ are linearly independent in } A$$

is formula definable in $\Gamma(X, A)$.

Proof. The formula sought is of the form

$$\forall b_1, \dots, b_n \left(\underset{i=1}{\overset{n}{\&}} C_{\mathbf{Z}}(a_i, b_i) \,\&\, \left(b_1 \dots b_n = 1 \to \underset{i=1}{\overset{n}{\&}} b_i = 1 \right) \right).$$

The lemma is proved.

Let $\sqrt{a}$ be the $\{x \in A \mid x^u \in gp(a) \text{ for some } n \in N\}$-isolater subgroup $gp(a)$ in A. It is evident that $\sqrt{a}$ is a set of all the elements from A dependent on a.

LEMMA 6. *The predicate*

$$C_Q(a, b) \overset{\text{df}}{\rightleftharpoons} b \in \sqrt{a}$$

is formula definable in $\Gamma(X, A)$.

The proof follows from the equivalence

$$C_Q(a, b) \Leftrightarrow I(a, b)$$

Let p be a prime number. The p-height $h_p(a)$ of an element $a \in A$ is ∞ if the equation $x^{p^m} = a$ is solvable in the group A for any positive integer m.

LEMMA 7. *Let the group A have an element of infinite order. Then the predicate*

$$H_\infty(a, p) \overset{\text{df}}{\rightleftharpoons} h_p(a) = \infty$$

is formula definable in $\Gamma(X, A)$.

Proof. In the forthcoming Proposition 3, it is proved that each recursively enumerable predicate is definable in the model $\Gamma(X, A)$, in particular the predicate $\deg(p, m)$ signifying that p is a prime number with degree m. Consequently,

$$H_\infty(a, p) \Leftrightarrow \forall m \deg(p, m) \to \exists x C(x, a, m),$$

which proves the lemma.

3.3. Formula predicates on N

LEMMA 8. *Let the group A have an element of infinite order. Then the predicate*

$$\text{prod}(k, l, m) \overset{\text{df}}{\rightleftharpoons} m \text{ equals the product of } k \text{ by } l$$

is formula definable in $\Gamma(X, A)$.

Proof. Let $k, l \geq 2$ be arbitrary positive integers, $x = X^{\pm A}$, a an element of infinite order in A. Let us set $v = x^{a^{k+1}} \dots x^{a^{k+l-1}}$ and consider an element in the group $F(X^A)$ of the form

$$w = x^a v x^{a^2} \dots x^{a^k} v \tag{1}$$

Then the condition $m = kl$ is equal to the condition $|w| = m$. In fact, $|v| = l - 1$, $|w| = k(l-1) + k = kl$. The existence of the element w, $|w| = m$, of form (1) in the group $F(X^A)$ is equivalent to the validity in $\Gamma(X, A)$ of some formula $\phi(k, l, m)$ which will be constructed in the proof of the lemma.

Let us first determine the formula definability of some auxiliary predicates. The formula

$$\phi_1(x, y, a, k) \overset{\text{df}}{\rightleftharpoons} \exists b(y = x^b \,\&\, C(a, b, k)),$$

where $C(a, b, k)$ is the predicate from Lemma 3, is valid in the model $\Gamma(X, A)$ iff $y = x^{a^k}$. Validity in $\Gamma(X, A)$ of the formula

$$\phi_2(w, v_1, v_2, x, a, i, j) \overset{\text{df}}{\rightleftharpoons} S_C(w_1 v_1, v_2) \,\&\, \phi_1(x_1 v_1, a, i) \,\&\, \phi_1(x, v_2, a, j),$$

where $S_C(w, w_1, w_2)$ is a formula from 3.1, means that v_1 and v_2 are neighbouring letters in the irreducible expression w and $v_1 = x^{a^i}$, $v_2 = x^{a^j}$. Validity in $\Gamma(X, A)$ of the formula

$$\begin{aligned}\phi_3(w, a, x, k, l) \overset{\text{df}}{\rightleftharpoons} \forall v_1, v_2, i, j(&\phi_2(w, v_1, v_2, x, a, i, j) \\ &\to (i \leq k \to j = k + 1) \,\&\, (k < i < k + l - 1 \to j = i + 1)) \\ &\&\exists w_1(w = x^a w_1 \,\&\, |w| = |w_1| + 1),\end{aligned}$$

where $S_H(w, x^a)$ is the predicate from 3.1, guarantees that the irreducible expression begins with x^a, that each entry x^{a^i}, $i \leq k$, is followed by $x^{a^{k+1}}$, and that each entry x^{a^i} is followed by $x^{a^{i+1}}$ if $k + 1 \leq i < k + l - 1$. Validity in $\Gamma(X, A)$ of the formula

$$\begin{aligned}\phi_4(v, a, x, k, l) \overset{\text{df}}{\rightleftharpoons} (|v| = l - 1 \,\&\, \mathscr{D}(v, a) \,\&\, \exists v_0(&S_H(v_0, v) \\ &\&\phi_1(x, v_0, a, k + 1))),\end{aligned}$$

where $\mathscr{D}(u, v)$ is a formula from 3.1, implies that $v = x^{a^{k+1}} \ldots x^{\dot{a}^{k+l-1}}$. Validity in $\Gamma(X, A)$ of the formula

$$\begin{aligned}\phi_5(w, v_1, v_2, x, a, k, l) \overset{\text{df}}{\rightleftharpoons} \exists w_1, w_2, v(|v_1| = 1 \,\&\, |v_2| = 1 \,\&\\ w = w_1 v_1 v v_2 w_2 \,\&|w| = |w_1| + |w_2| + 2 + |v| \,\&\, \phi_3(v, a, x, k, l))\end{aligned}$$

states that v_1 and v_2 are length 1 and the irreducible expression w contains a subword of the form $v_1 v v_2$. Similarly the formula $\phi_G(w, v_1, w_1, x, a, k, l)$ is constructed which states that $v_1 v w_1$ is a subword of irreducible expression w, and the length w_1 is arbitrary. Finally, the formula

$$\begin{aligned}\phi_7(w, x, a, k, l) \overset{\text{df}}{\rightleftharpoons} \forall v_1, v_2, i(\phi_5(w, v_1, v_2, x, a, k, l) \,\&\\ \phi_1(x, v_1, a, i) \,\&\, i < k \to \phi_1(x, v_2, a, i + 1) \,\&\, \forall v_1, w_1\\ (\phi_6(w, v_1, w_1, x, a, k, l) \,\&\, \phi_1(x, v_1, a, k) \to |w_1| = 0))\end{aligned}$$

implies that the irreducible expression w contains either the entries $x^{a^i} v x^{a^{i+1}}$, $i < k$, or $x^k v$ and this entry is the end of the word w.

Now it is evident that the desired formula is of the form

$$\phi(k, l, m) \overset{\text{df}}{\rightleftharpoons} \exists w, x, a(|w| = m \,\&\, |x| = 1 \,\&\, p_\infty(a) \,\&$$
$$\phi_3(w, a, x, k, l) \,\&\, \phi_7(w, x, a, k, l)).$$

The lemma is proved.

PROPOSITION 3. *Let the group A have an element of infinite order. Then each recursively enumerable predicate on N is formula definable in $\Gamma(X, A)$.*

Proof. According to Lemma 8 the system $\mathscr{N} = \langle N; 0, 1, +, \cdot, \leq \rangle$, where $\cdot$ is multiplication in N, is formula definable in $\Gamma(X, A)$. In its turn, the system $\mathscr{Z} = \langle \mathbf{Z}; 0, 1, +, \cdot \rangle$ is formula definable in $\mathscr{N}$. It is known that each recursive predicate on N is Diophantine (see [8]), i.e., formula definable in the system $\mathscr{Z}$ and, consequently, in the model $\mathscr{N}$.

The proposition is proved.

4. Groups of operators admitting rational coordinatization

4.1. Determination of rational coordinatization. Let A be a group, $a \in A$, $r = m/n \in \mathbf{Q}$. Let $a^{m/n}$ denote an arbitrary solution of the equation $x^n = a^m$ in the group A.

DEFINITION. A procession of elements $\bar{c} = (c_1, \ldots, c_n)$, $c_i \in A$, is called a pseudo-base of group A, if each element $a \in A$ admits a (not necessarily unique) decomposition of the form

$$(1) \qquad a = c_1^{t_1(a)} \ldots c_n^{t_n(a)}, \quad t_i(a) \in \mathbf{Q}$$

Elements $t_i(a)$ are called the coordinates of a with respect to the pseudo-base $\bar{c}$. In the general case, the procession of the coordinates $t(a) = (t_1(a), \ldots, t_n(a))$ for the element a is not uniquely defined; moreover, by virtue of the non-uniqueness of the solution of the equation $x^n = a^m$ in the group A, there may hypothetically occur a case where $a \neq b \in A$ with $t(a) = t(b)$.

Let $T(A, \overline{C}) \subseteq \mathbf{Q}^n$ be a set of all the processions of coordinates of all the elements A by the base $\bar{c}$, or a subset of this set satisfying some conditions of the form: $t_i(a)$ belongs to some fixed finite subset Q. On the set $T(A, \bar{c})$ we introduce the following relation $\mathscr{E}_c$:

$$(2) \qquad \mathscr{E}_c(s, r) \overset{\text{df}}{\rightleftharpoons} \exists a \in A(s = t(a) \,\&\, r = t(a))$$

The relation $\mathscr{E}_c$ is reflexive, symmetric but *a priori* non-transitive. Let us denote by $[t(a)]$ the set of all elements from $T(A, \bar{c})$ in the relation $\mathscr{E}_c$ with $t(a)$.

DEFINITION. The group A admits rational coordinatization if there exists a pseudo-base $\bar{c}$ of the group A such that $\mathscr{E}_c$ is a relation of equivalence and the map $b \to [t(b)]$, $b \in A$, is a bijection.

LEMMA 9. *A class of groups admitting rational coordinatization is closed with respect to extensions, finite direct products and homomorphic images.*

The proof is evident.

PROPOSITION. *The following classes of groups admit rational coordinatization.*

(1) *finite,*
(2) *quasi-cyclic,*
(3) *abelian torsion-free,*
(4) *polycyclic,*
(5) *torsion-free solvable groups of finite rank,*
(6) *Černikov groups, i.e., finite extensions of a direct product of quasi-cyclic groups.*

Proof. In cases (1)–(4) the coordinatization is evident. In case (5) (see [9]), the group A has a subnormal series of subgroups $A > A_1 > \cdots > A_n = 1$ such that A/A_1 is finite and the factors A_i/A_{i+1} are abelian torsion-free groups of rank 1. Then, by Lemma 9, (5) follows from (1) and (3). In case (6) the proof also follows by Lemma 9 from (2) and (1).

The proposition is proved.

COROLLARY. *Solvable Min- and Max-groups admit rational coordinatization.*

In fact, solvable Max-groups are polycyclic and Min-groups are Černikov groups.

4.2. Definability A in N^M. Let N^M be a complete formula enrichment of the model $\mathscr{N} = \langle N; +, 0, 1\rangle$ in $\Gamma(X, A)$. To each rational number $r = (-1)^\delta m/n$, $\delta \in \{0,1\}$, $m, n \in N$, $n \neq 0$, we associate the ordered triple of numbers $\lambda(r) = (\delta, m, n)$, and conversely, to each (δ, m, n) from the set

$$Q(N) = \{(\delta, m, n) | \delta \in \{0,1\}, m, n \in N, n \neq 0\}$$

there corresponds the rational number $r(\lambda) = (-1)^\delta m/n$.

The set $Q(N)$ is obviously definable in the model N^M.

LEMMA 10. *Let $a \in A$, $\bar{c} \in A^n$, $\bar{r}, \bar{s} \in Q(N)^n$. Then the predicates*

$$B(\bar{c}) \overset{\text{df}}{\rightleftharpoons} \bar{c} \text{ is a pseudo-base } A,$$

$$Cf(a, \bar{c}, \bar{r}) \overset{\text{df}}{\rightleftharpoons} \bar{r} \text{ is a procession of the coordinates of } a \text{ with respect to } c,$$

$$T(\bar{r}, \bar{c}) \overset{\text{df}}{\rightleftharpoons} r \in T(A, \bar{c}),$$

$$\mathscr{E}(\bar{r}, \bar{s}, \bar{c}) \overset{\text{df}}{\rightleftharpoons} \bar{r}, \bar{s} \in T(A, \bar{c}) \text{ and } \mathscr{E}_{\bar{c}}(\bar{r}, \bar{s}),$$

$$BR(\bar{c}) \overset{\text{df}}{\rightleftharpoons} \bar{c} \text{ is a pseudo-base of } A \text{ giving the rational coordinatization of } A$$

are formula definable in $\Gamma(X, A)$.

Proof. Let $C(c, b, m)$ be a formula from Lemma 3 defining the equality $b = c^m$. Then the formula

$$\psi_1(c, b, r) \overset{\text{df}}{\rightleftharpoons} \exists b_1, c_1 C(c, c_1, m) \,\&\, C(b, b_1, n) \,\&\, (\delta = 0 \to c_1 = b_1) \\ \&\big(\delta = 1 \to c_1 = b_1^{-1}\big),$$

where $r = (\delta, m, n) \in Q(N)$ indicates that $b = c^r$. Consequently, the formula

$$\phi_{cf}(a, \bar{c}, \bar{r}) \overset{\text{df}}{\rightleftharpoons} \exists b_1, \dots, b_n \Big(a = b_1 \dots b_n \,\&\, \mathop{\&}_{i=1}^{n} \psi_1(c_i, b_i, r_i) \Big)$$

defines the predicate $Cf(a, \bar{c}, \bar{r})$ in $\Gamma(X, A)$.

Therefore, the predicates $B(\bar{c})$, $T(r, \bar{c})$, $\mathscr{E}(\bar{r}, \bar{s}, \bar{c})$ are defined respectively by the formulas

$$\phi_B(\bar{c}) \overset{\text{df}}{\rightleftharpoons} \forall a \exists \bar{r} \phi_{cf}(a, \bar{r}, \bar{c}),$$

$$\phi_T(\bar{r}, \bar{c}) \overset{\text{df}}{\rightleftharpoons} \exists a \phi_{cf}(a, \bar{r}, \bar{c}),$$

$$\phi_{\mathscr{E}}(\bar{r}, \bar{s}, \bar{c}) \overset{\text{df}}{\rightleftharpoons} \exists a \big(\phi_{cf}(a, \bar{r}, \bar{c}) \,\&\, \phi_{cf}(a, \bar{c}, \bar{s}) \big)$$

Consequently, with the help of formulas $\phi_T(\bar{r}, \bar{c})$ and $\phi_{\mathscr{E}}(\bar{r}, \bar{s}, \bar{c})$, the fact that $\mathscr{E}_c$ is an equivalence relation on $T(A, \bar{c})$ is obviously expressible by a formula $\phi_T(\bar{r}, \bar{c})$ in the language of the model $\Gamma(X, A)$. In the same way, bijection of the map $\varphi_{\bar{c}}\colon a \to [t(a)]$ from A in $T(A, \bar{c})/\mathscr{E}_{\bar{c}}$ may be written by a formula

$\psi_4(\bar{c})$. Thus, the formula

$$\phi_{BR}(\bar{c}) \overset{\text{df}}{\rightleftharpoons} \psi_2(\bar{c}) \,\&\, \psi_3(\bar{c}) \,\&\, \psi_4(\bar{c})$$

defines the predicate $BR(\bar{c})$ in $\Gamma(X, A)$.

The lemma is proved.

PROPOSITION 5. *Let the group A admit rational coordinatization. Then A is formula definable in N^M.*

Proof. Let us use the notation from Lemma 10. Let $\bar{c}$ be an arbitrary pseudo-base, setting rational coordinatization in A. In the factor-set $T(A, \bar{c})/\mathscr{E}_{\bar{c}} = T_{\bar{c}}$ we define a multiplication $\circ$ by

$$[\bar{r}_1] \circ [\bar{r}_2] = [\bar{r}_3] \overset{\text{df}}{\rightleftharpoons} \exists a_1, a_2, a_3 \underset{i=1}{\overset{3}{\&}} \phi_{cf}(a_i, \bar{c}, \bar{r}_i) \,\&\, (a_1 a_2 = a_3)$$

It is evident that $\mathscr{T}_{\bar{c}} = \langle T_{\bar{c}}; \circ \rangle$ is a group, and the map $\varphi_{\bar{c}}: a \to [t(a)]$ is an isomorphism $A \to \mathscr{T}_{\bar{c}}$, of groups defined by the formula $\phi_{cf}(a, \bar{c}, \bar{r})$. Since the set B of all pseudo-bases setting rational coordinatization in A is formula definable in $\Gamma(X, A)$, the group A is regularly definable in N^M. Moreover, since the connecting isomorphisms

$$\varphi_{\bar{c}} \circ \varphi_{\bar{d}}^{-1}: \mathscr{T}_{\bar{d}} \to \mathscr{T}_{\bar{c}}, \quad \bar{c}, \bar{d} \in B$$

are also formula definable in $\Gamma(X, A)$, then by Proposition 2 from 2.4 the group A is formula definable in N^M without constants.

The proposition is proved.

5. Elementary equivalence of groups with the length function

5.1. Sufficient conditions.

THEOREM 4. *Let X, Y be sets, and let A, B be groups. If the cardinalities of X and Y are either equal or infinite and the groups A and B are ω-isomorphic, then $\Gamma(X, A) \equiv \Gamma(Y, B)$.*

Proof. Let the cardinalities $|X|$ and $|Y|$ be either equal or infinite, and let groups A and B be ω-isomorphic. In other words, for any positive integer M there exists a succession of non-empty sets

$$(4) \qquad \Delta_1(m) \subseteq \cdots \subseteq \Delta_m(m)$$

of finite partial isomorphisms A and B satisfying the condition $(**)$ from 2.3. Let us prove that $\Gamma(X, A) \equiv \Gamma(Y, B)$. To this end, by Theorem 3, it is sufficient to construct for any positive integer m a succession of non-empty sets

$$\phi_1(m) \subseteq \cdots \subseteq \phi_m(m)$$

of finite partial isomorphisms from $\Gamma(X, A)$ to $\Gamma(Y, B)$ satisfying the condition ($*$). As mentioned above, each partial isomorphism $\varphi\colon \Gamma(X, A) \to \Gamma(Y, B)$ represents the triple of partial isomorphisms $(\varphi_1, \varphi_2, \varphi_3)$ where

$$\varphi_1\colon F(X^A) \to F(Y^B), \varphi_2\colon A \to B, \varphi_3\colon N \to N$$

adjusted with respect to predicates of length and the action of operator groups. Let

$$\bar{x} = (x_1, \dots, x_s), \bar{y} = (y_1, \dots, y_s), \quad x_j \neq x_k, y_j \neq y_k \text{ for } k \neq j$$

be arbitrary processions (of the same length) of elements from X, Y respectively; let $\delta \in \Delta_i(m)$. Assume

$$X_{\bar{x}, \delta} = \{x_j^a | a \in \operatorname{dom} \delta, 1 \leq j \leq s\}$$

The map

$$x_j^a \to y_j^{\delta(a)}, \quad x_j^a \in X_{\bar{x}, \delta}$$

is extended to the monomorphism

$$\lambda_{\bar{x}, \bar{y}, \delta}\colon gp(X_{\bar{x}, \delta}) \to F(Y^B).$$

Let us denote by ϕ_δ the set of restrictions of monomorphisms of the form $\lambda_{\bar{x}, \bar{y}, \delta}$ on arbitrary finite subsets, and let

$$\phi_{1i}(m) = \bigcup_{\delta \in \Delta_i(m)} \phi_\delta.$$

It is evident that $\phi_{11}(m) \subseteq \cdots \subseteq \phi_{1m}(m)$ is a succession of non-empty sets of finite partial isomorphisms from $F(X^A)$ to $F(Y^B)$. Let ϕ_{3i} be the set of restrictions of the identity map $N \to N$ on finite subsets of cardinality $\leq i$. Finally we set

$$\phi_i(m) = \{(\varphi_\delta, \delta, \mu) | \varphi_\delta \in \phi_\delta, \delta \in \Delta_i(m), \mu \in \phi_{3i}\}$$

By construction, $\phi_1(m) \subseteq \cdots \subseteq \phi_m(m)$ is a succession of non-empty sets of partial isomorphisms $\Gamma(X, A)$ in $\Gamma(Y, B)$. Let us prove that it satisfies the condition ($*$). Let

$$\varphi = (\varphi\delta, \delta, \mu) \in \phi_i(m), \quad i < m,$$

be arbitrary. If $n \in N$ there obviously exists an extension of φ to a covering $\psi \in \phi_{i+1}(m)$ with $n \in \operatorname{dom} \psi$ since the isomorphism φ_δ preserves length therefore it is sufficient to take the covering $\bar{\mu} \in \phi_{3i+1}$ for μ such that $u \in \operatorname{dom} \bar{\mu}$ and assume $\psi = (\varphi_\delta, \delta, \bar{\mu})$. If $a \in A$ there exists a covering $\bar{\delta} \in \Delta_{i+1}(m)$ for δ such that $a \in \operatorname{dom} \bar{\delta}$; in this case

$$\psi = (\varphi_{\bar{\delta}}, \bar{\delta}, \mu) \in \phi_{i+1}(m)$$

is a covering for φ and $a \in \operatorname{dom} \psi$. Now let $u = x_{j_1}^{a_1} \dots x_{j_s}^{a_s} \in F(X^A)$. By property $(**)$ one can select a covering $\bar{\delta} \in \Delta_{i+1}(m)$ for δ such that $a_1, \dots, a_s \in \operatorname{dom} \bar{\delta}$. By construction, φ_δ is a restriction of some monomorphism $\lambda_{\bar{x}, \bar{y}, \delta}$ on the finite subset $U \subseteq gp(X_{\bar{x}, \delta})$. Let us extend the procession $\bar{x}$ by adding (arbitrarily) to the right-hand side new different letters from the set $\{x_{j1}, \dots, x_{js}\}$ not occurring in $\bar{x}$. The procession obtained will be denoted by $\bar{x}_u$. Since the cardinalities of X and Y are either equal or infinite, in a similar way, the procession $\bar{y}$ can be extended on the right-hand side by different letters from Y to some procession $\bar{Z}$. Let $\varphi_{\bar{\delta}}$ be the restriction of the monomorphism $\lambda_{\bar{x}_u, \bar{z}, \delta}$ on the finite subset $U \cup \{u\}$. Then the partial isomorphism

$$\psi = (\varphi_{\bar{\delta}}, \bar{\delta}, \mu) \in \phi_{i+1}(m)$$

is a covering for φ, where $u \in \operatorname{dom} \psi$. Thus, for any m the succession $\phi_1(m) \subseteq \cdots \subseteq \phi_m(m)$ satisfies condition $(*)$, so $\Gamma(X, A) \equiv \Gamma(Y, B)$. The theorem is proved.

To conclude, let us give an example of groups A and B such that $A \equiv B$ but, for any set X, $\Gamma(X, A) \not\equiv \Gamma(X, B)$.

Let $A = \mathbf{Z}$, $B = \mathbf{Z} \oplus \mathbf{Q}$ where $\mathbf{Z}, \mathbf{Q}$ are the additive groups of the ring of integers and the rational numbers. According to Szmielev's theorem [4], $A \equiv B$. Note that for any prime number p in the group A there are no elements of infinite p-height, but there are these elements in the group B. According to Lemma 7, the predicate $h_p(a) = \infty$ is formula definable in $\Gamma(X, A)$ and $\Gamma(X, B)$. Consequently, $\Gamma(X, A) \not\equiv \Gamma(X, B)$.

5.2. Elementary equivalence in groups of operators admitting rational coordinatization.

THEOREM 5. *Let X and Y be sets, A and B groups, A admitting rational coordinatization. Then $\Gamma(X, A) \equiv \Gamma(Y, B)$ iff the cardinalities of X and Y are either equal or infinite and groups A and B are isomorphic.*

Proof. Let $\Gamma(X, A) \equiv \Gamma(Y, B)$. By Lemma 2, the cardinalities $|X|$ and $|Y|$ are either equal or infinite. Let us prove that $A \simeq B$. Let N_A and N_B be formula enrichments of N in $\Gamma(X, A)$ and $\Gamma(Y, B)$ respectively. By hypothesis, the group A allows rational coordinatization; consequently, by Proposition 5, A is absolutely definable in N_A. Let $A \simeq \psi(N_A)$ (see 2.4) where

$$\psi = \{A(\bar{x}), \mathscr{E}(\bar{x}, \bar{y}), \phi(\bar{x}, \bar{y}, \bar{z})\}$$

is the set of formulas defining A in N_A. Isomorphism of the interpretation μ: $A \to \psi(N_A)$ is defined in $\Gamma(X, A)$ by the formula

$$\mathrm{Is}(\bar{a}, \bar{r}) \overset{\mathrm{df}}{\rightleftharpoons} \exists \bar{c} BR(\bar{c}) \,\&\, Cf(a, \bar{r}, \bar{c}),$$

where Cf, BR are given in Lemma 10. Then the very fact that any formula ψ defines in N_A a group $\psi(N_A)$ and the formula $\mathrm{Is}(a, \bar{r})$ defines isomorphism μ is expressed in the language $\Gamma(X, A)$ by means of a certain closed formula which will be denoted by ϕ_ψ. Thus the validity of ϕ_ψ in $\Gamma(Y, B)$ follows from $\Gamma(X, A) \equiv \Gamma(Y, B)$, and from the construction of ϕ_ψ we see that any formula ψ defines in $\Gamma(Y, B)$ on N_B some group $\psi(N_B)$ isomorphic to B. Let us prove that formulas $A(\bar{x})$, $\mathscr{E}(\bar{x}, \bar{y})$ and $\phi(\bar{x}, \bar{y}, \bar{z})$ define the same predicates in N_A and in N_B, i.e., $\psi(N_B) \simeq \psi(N_A)$. In fact, let $\mathscr{R}(x)$ be an arbitrary formula of the model $\Gamma(X, A)$ defining in N_A some set of processions R. Each procession $\bar{m} \in N^n$ is formula definable in $\langle N; +, 0, 1\rangle$; therefore, in the model $\Gamma(X, A)$, hence in $\Gamma(Y, B)$, the system of closed formulas

$$\{\mathscr{R}(\bar{m}), \neg\mathscr{R}(\bar{k}) | \forall \bar{m} \in R \ \forall \bar{k} \notin R\}$$

is true, guaranteeing that the formula $\mathscr{R}(x)$ defines the same predicate both in $\Gamma(X, A)$ and $\Gamma(Y, B)$ on N. Thus, the groups $\psi(N_A)$ and $\psi(N_B)$, and hence the groups A and B, are isomorphic.

The theorem is proved.

6. Elementary equivalence of free products with the length function

By a simple modification in the proofs, the results of Sections 3, 4, 5 for free operator groups may be extended to free products of groups with a length function.

Let $G = A_1 * A_2$, $H = B_1 * B_2$ be free products of groups, $\Pi(G)$ and $\Pi(H)$ respective two-base models from 2.1.

THEOREM 6. *If the groups A_1, B_1 and A_2, B_2 are pairwise ω-isomorphic, then* $\Pi(G) \equiv \Pi(H)$.

THEOREM 7. *Let the groups A_1 and A_2 allow rational coordinatization. Then* $\Pi(G) \equiv \Pi(H)$ *iff, after a suitable enumeration of the factors,* $A_1 \simeq B_1$, $A_2 \simeq B_2$.

Let us also note that the conditions $A_1 \equiv B_1$ and $A_2 \equiv B_2$ are not sufficient for $\Pi(G) \equiv \Pi(H)$. In fact $\mathbf{Z} \equiv \mathbf{Z} \oplus \mathbf{Q}$ (see 5.1.) but $\Pi(\mathbf{Z} * \mathbf{Z}) \not\equiv \Pi(\mathbf{Z} * (\mathbf{Z} \oplus \mathbf{Q}))$.

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