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DEFINABILITY IN THE TURING DEGREES[1]

BY

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In Memoriam W.W. Boone

1. Introduction

Definability has provided the most fruitful approach to understanding the model-theoretic structure of $\mathscr{D}$, the Turing degrees ordered by Turing reducibility.

The origin of this approach is Spector's result [9] that every countable ideal in $\mathscr{D}$ is uniformly definable from parameters in $\mathscr{D}$. Spector's theorem also shows that the first order theory of $\mathscr{D}$ includes an interpretation of quantification over such ideals.

Simpson [8] used this result, the embedding theorems for upper semi-lattices as initial segments of $\mathscr{D}$, and a coding of models of arithmetic as initial segments of $\mathscr{D}$, to show that there is a faithful interpretation of second order arithmetic in the first order theory of $\mathscr{D}$. In this interpretation, second order quantification over the coded model of arithmetic is interpreted by quantification over ideals in $\mathscr{D}$.

Nerode and Shore [4], [5] gave a simplified way to code models by initial segments. With their method of coding, the degree of the code of a set of integers is close to the degree of the set itself. They applied their method of coding to show that every automorphism of $\mathscr{D}$ with a predicate for the arithmetic degrees is the identity on a cone of degrees. Subsequently, see [2, Harrington-Shore] and [3, Jockusch-Shore], the arithmetic degrees were shown to be definable in $\mathscr{D}$. The best result currently known is that every automorphism is the identity on the cone above 0^{ω} and maps every degree to one arithmetic in it. In further work, Shore [7] showed that the relation "$\vec{c}$ is a code for an element of x" is first order definable in $\mathscr{D}$ for those x above 0^{ω}. This last result explains why every automorphism is the identity above 0^{ω}. Namely, both $\vec{c}$ and its isomorphic image must code the same set of integers, so the degree of the set coded by $\vec{c}$ must be fixed by any isomorphism. Shore's result

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can also be used to conclude that a predicate on the degrees above 0^ω is definable in $\mathscr{D}$ if and only if it is definable in second order arithmetic.

Perhaps the most outstanding application of definability to the study of $\mathscr{D}$ is Shore's Non-Homogeneity Theorem [6], which states that there is a degree x such that $\mathscr{D}$ is not isomorphic to $\mathscr{D}(\geq x)$, the degrees above x. The difference between $\mathscr{D}$ and $\mathscr{D}(\geq x)$ is that there may be a set of integers that is not arithmetically definable but is coded by a degree that is arithmetic in x while every set that is coded by an arithmetic degree is arithmetic. The conclusion follows once arithmetically definable codings for representatives of Turing degrees, and corresponding decodings, are found.

In this paper, we extend Spector's theorem to all countable relations: Any countable relation on Turing degrees is uniformly definable from parameters in $\mathscr{D}$. The parameters used to define a given relation $\mathscr{R}$ are constructed by forcing over $\mathscr{R}$ with essentially finite conditions. This provides some advantage when $\mathscr{D}$ is replaced by the more restrictive model $\mathscr{D}(\leq 0')$ in later sections.

In Sections 3 and 4, we give some applications of this result. In Section 3, we show that any elementary function from $\mathscr{D}$ to $\mathscr{D}$ must be an automorphism. We also show that the definability in $\mathscr{D}$ of the relation "$\vec{c}$ is a code for an element of x" for all degrees x is equivalent to a notion of local rigidity for $\mathscr{D}$. In Section 4, by an effective analysis of the forcing construction, we show that recursive enumerability is a definable property in the structure $\mathscr{D}(\leq 0')$.

1.1. NOTATION. Use upper case letters X and Y to denote real numbers. Given X and Y, let x and y be their Turing degrees. $X \oplus Y$ denotes the recursive disjoint union of X and Y and $x \vee y$ denotes its degree. Let (X) denote the set of reals that are recursive in X; let (x) be the induced ideal in $\mathscr{D}$. Also, use $\wedge$ and $\vee$ to denote the operations of meet and join both between sets (X) and (Y) and between ideals (x) and (y). A finite sequence of sets is denoted by $\vec{X}$ and a finite sequence of degrees by $\vec{x}$. The eth Turing functional applied to the real Y is $\{e\}^Y$; write $\{e\}^Y(n)\downarrow$ if $\{e\}^Y(n)$ has a value and $\{e\}^Y(n)\uparrow$, otherwise. If p is a finite initial segment of a real, then $\{e\}^p(n) = m$ means that the eth Turing functional relative to any real extending p converges at argument n with value m, with a computation length less than the length of p.

The symbol $\diamondsuit$ marks the end of a proof.

2. The coding apparatus

This section includes a proof of the main theorem: the uniform definability of countable relations in $\mathscr{D}$. The proof involves several intermediate steps, each of which is a proof that a certain special type of relation can be defined.

2.1. DEFINITION. A set $\mathscr{I}$ of Turing degrees is an *antichain* if, whenever a_0 and a_1 are distinct elements of $\mathscr{I}$, they are incomparable.

The first step is to show that any countable antichain can be uniformly defined from parameters in $\mathscr{D}$. The proof of this result uses a notion of forcing $\mathscr{P}$ in which the conditions are essentially finite. The fact that the conditions are so simple will be important when it is necessary to find generic reals below $\varnothing'$.

2.2. NOTATION. A real can be identified in a recursive way either with a countable sequence of reals or with a set of finite sets of integers. In the first case, let $X^{(n)}$ denote the nth real in the sequence associated with X.

2.3. LEMMA. [1, Dekker-Myhill]. *Suppose that X is a real number. There is a real Y with the same Turing degrees as X which is recursive in any of its infinite subsets.*

Proof. Given X, let Y be the set of initial segments of X. Y is recursive in X, as X knows its own initial segments; X is recursive in any infinite subset of Y, as any atomic question about X can be answered by every sufficiently long initial segment of X. $\diamondsuit$

The previous lemma shows that any set of degrees can be represented by a set of reals, each of which is recursive in any of its infinite subsets.

2.4. DEFINITION. Let $\mathscr{I}$ be a set of reals whose degrees form an antichain. Suppose that for every real X in I and for any real Y, if Y is an infinite subset of X then X is recursive in Y. Define the forcing partial order $\mathscr{P}$ associated with I as follows.

(1) A condition p is a triple $\langle p_0, p_1, F(p)\rangle$. Here p_0 and p_1 are binary sequences of the same length and $F(p)$ is a finite sequence from I. The set of conditions is denoted by P.
(2) Suppose that p and q are elements of P. Then p is stronger than q or below q if p_0 extends q_0, p_1 extends q_1, and $F(p)$ extends $F(q)$. In addition, for every integer n less than the length of $F(q)$, if k is less than the length of $p_0^{(n)}$ but not less than the length of $q_0^{(n)}$ and k belongs to the nth element $F(q)$, then $p_0^{(n)}(k) = p_1^{(n)}(k)$.

Let G be $\mathscr{P}$-generic. G is easily seen to be equivalent to the pair of reals G_0 and G_1 formed by taking the unions of G's first coordinates. For each X in I, $\mathscr{P}$ generically introduces a real into $(G_0 \oplus X) \wedge (G_1 \oplus X)$ that is not in (X). Genericity will imply that whenever Y is a real with the property that $(G_0 \oplus Y) \wedge (G_1 \oplus Y)$ is not equal to (Y), there is an X in I such that X is recursive in Y.

2.5. PROPOSITION. *Suppose that $\mathscr{I}$ is a countable antichain. There are degrees g_0, g_1 and d such that the elements of $\mathscr{I}$ are the degrees below d that are* *minimal solutions for x in the following equation*:

(2.6.) $$(g_0 \vee x) \wedge (g_1 \vee x) \neq (x)$$

Proof. The proof is presented in more detail than is traditional; the fine points will be necessary in the effective analysis of the construction appearing in Section 4.

Let I be a set of representatives of $\mathscr{I}$ such that each element of I is recursive in any of its infinite subsets. Let $\mathscr{P}$ be the notion of forcing associated with I. Let D be a real such that the elements of I are uniformly recursive in D; let d be its Turing degree. There is a partial ordering of the integers that is isomorphic to $\mathscr{P}$ and recursive in D; one example is the partial ordering of D's codes for the elements of $\mathscr{P}$. In what follows, identify $\mathscr{P}$ with its isomorphic copy. Suppose that G is $\mathscr{P}$-generic with respect to all of the dense sets that are arithmetic in D; let g_0 and g_1 be the degrees of the reals G_0 and G_1 associated with G. Let $\Vdash$ denote the forcing relation for $\mathscr{P}$. Since the statement of equation 2.6 is arithmetic in D, for fixed $x <_T d$, 2.6 will hold if the empty condition in $\mathscr{P}$ forces it. The important dense sets, those used to show that forcing 2.6 is the same as its truth for the generic sets, are all arithmetic in D.

Suppose that X is an element of I. First, verify that the following equation holds, namely that the empty condition forces equation 2.6:

(2.7.) $$\Vdash (g_0 \vee x) \wedge (g_1 \vee x) \neq (x).$$

The fact of the matter is that a real that is Cohen generic with respect to all of the dense sets arithmetically definable in D is coded into both $G_0 \oplus X$ and $G_1 \oplus X$. The proof of equation 2.7 uses only a small part of this genericity.

2.8. Definition. Suppose that p is a condition. Say that n is a *coding location* for p if n is equal to $\langle i, x\rangle$ where i is less than the length of $F(p)$ and x is an element of the ith member of $F(p)$ and is greater than the common length of the first two coordinates of p.

The set of coding locations for p of the form $\langle i, m\rangle$ is recursive in the ith element of $F(p)$. This set is called the set of coding locations in the ith column.

2.9. Notation. Let $D(X)$ be the set of conditions p such that X appears in the finite sequence $F(p)$.

Clearly, $D(X)$ is dense in P. Suppose that p is an element of $D(X)$ and that X is the ith element of $F(p)$. Then define a real $Z(X)$ as follows.

$$Z(X) = \left\{ m \,\middle|\, \begin{array}{l}\text{The } m\text{th coding location for } p \text{ of the form} \\ \langle i, n\rangle \text{ is an element of } G_0.\end{array} \right\}$$

Since every extension of p is required to make its first two coordinates agree at all coding locations for p, $Z(X)$ is recursive in both $G_0 \oplus X$ and $G_1 \oplus X$. It remains to show that $Z(X)$ is not recursive in X. Let e be an index for a Turing reduction.

2.10. NOTATION. Let $C(e, X)$ be the set of conditions p such that for some n less than the length of p_0, either $\{e\}^X(n)\uparrow$, or $\{e\}^X(n)\downarrow$ and its value is unequal to the value of p_0 at the nth coding location for p_0 in the ith column.

Suppose that p and X are given, and X equals the ith element of $F(p)$. If there is an n greater than the length of p_0 such that $\{e\}^X(n)\downarrow$, it is possible to compute $\{e\}^X(n)$ and then define q extending p to diagonalize; thus, $C(e, X)$ is dense.

It remains to show that if y is a degree below d then

$$\Vdash (\exists x \in I)[(y \geq x) \text{ or } ((g_0 \vee y) \wedge (g_1 \vee y) = y)].$$

Let y be a degree below d and let Y be an element of y. Suppose that e_0 and e_1 are indices for Turing reductions.

2.11. NOTATION. Let $M(Y, e_0, e_1)$ be defined by the following set of conditions.

$$M_0(Y, e_0, e_1) = \{p \mid (\exists n)(\{e_0\}^{p_0 \oplus Y}(n) = k_0 \,\&\, \{e_1\}^{p_1 \oplus Y}(n) = k_1 \,\&\, k_0 \neq k_1)$$
$$M_1(Y, e_0, e_1) = \{p \mid \text{There is no extension of } p \text{ in } M_0(Y, e_0, e_1)\},$$
$$M(Y, e_0, e_1) = M_0(Y, e_0, e_1) \cup M_1(Y, e_0, e_1).$$

$M(Y, e_0, e_1)$ is a dense subset of $\mathcal{P}$. If p is an element of $M_0(Y, e_0, e_1)$ then p forces that the pair $\langle e_0, e_1\rangle$ does not provide a set in both $(G_0 \oplus Y)$ and $(G_1 \oplus Y)$. Suppose that p is a condition in $M_1(Y, e_0, e_1)$ and show that either p forces the possible common values of $\{e_0\}^{G_0 \oplus Y}$ and $\{e_1\}^{G_1 \oplus Y}$ to be non-total or recursive in Y or else there is an element of $F(p)$ that is recursive in Y.

There are two cases to consider.

2.12. *Case* 1. *For every n, there is a y such that the following equation holds*:

$$p \Vdash (\forall x)(\{e_0\}^{G_0 \oplus Y}(n) = x \Rightarrow x = y).$$

If p is a condition as in Case 1, p forces that if $\{e_0\}^{G_0 \oplus Y}$ is total then it is recursive in Y. Below p, Y can compute $\{e_0\}^{G_0 \oplus Y}$ at n by finding any extension r_0 of p_0 such that $\{e_0\}^{r_0 \oplus Y}(n)\downarrow$ with use no greater than the length of r_0. For any such sequence r_0, there is a condition r extending p and having r_0 as its first coordinate. The value of $\{e_0\}^{G_0 \oplus Y}$ at n must be the same as

$\{e_0\}^{r_0 \oplus Y}(n)$ since any two extensions of p assigning a value must assign the same one.

2.13. *Case* 2. *For any q extending p in $\mathscr{P}$, there are r and r' extending q and an integer n such that r and r' force different values for $\{e_0\}^{G_0 \oplus Y}(n)$.*

By sequentially changing r_0 at points beyond those mentioned in p to agree with r_0', it is possible to find two conditions forcing different values for $\{e_0\}^{G_0 \oplus Y}(n)$ such that their first coordinates disagree at exactly one point k. Assume that r and r' have this property. Moreover, it is safe to assume that the computations involved have use no greater than the length of the first coordinate of their associated conditions as this property is dense. Regarding a real as a countable cartesian product, let k equal $\langle x, m \rangle$.

Suppose x does not belong to the mth element of $F(p)$. If q is a stronger condition than p then $q_0(k)$ does not have to agree with $q_1(k)$. Then, it is possible to find a condition extending p and forcing $\{e_0\}^{G_0 \oplus Y}(n) \neq \{e_1\}^{G_1 \oplus Y}(n)$ as follows. First, find an extension $\hat{r}$ of r deciding the value of $\{e_1\}^{G_1 \oplus Y}(n)$ and forcing the use of the computation to be less than the length of its second coordinate. Either $\hat{r}$ forces the inequality or the condition resulting from $\hat{r}$ by changing its value at k forces the inequality. Either case contradicts the original assumption that p forces the two functions to be equal.

Thus, any two conditions extending p, forcing different values for $\{e_0\}^{G_0 \oplus Y}$ at some argument and having first coordinates which disagree at exactly one point must have the disagreement at a coding location. Namely, if they disagree at $\langle x, m \rangle$, then x must be an element of the mth element of $F(p)$. By the condition of Case 2, no single condition above p can decide all the values of $\{e_0\}^{G_0 \oplus Y}$ so Y can compute an infinite set of points $\langle x, m \rangle$ each being the point of difference in a pair of conditions forcing different values at some argument. As $F(p)$ is finite, infinitely many of these points must involve the same m. The elements of I were chosen to have the property of being recursive in any infinite subset; thus, Y can compute some element of $F(p)$.

The two cases exhaust all of the possibilities. $\diamond$

The next step is to reduce defining an arbitrary countable set of Turing degrees to defining an antichain. The following lemma is standard. A proof is indicated in Section 4, where the effectiveness of the construction is important.

2.14. LEMMA. *Suppose that D is a real of Turing degree d. Let G be a set of reals that are pairwise mutually Cohen generic with regard to meeting all the dense sets in the Cohen partial order that are arithmetic in D. For any A and B that are recursive in D and any sequence $G_0, G_1, \ldots, G_n$ belonging to G, $A \oplus G_0$ is recursive in $B \oplus G_1 \oplus G_2 \oplus \cdots \oplus G_n$ if and only if A is recursive in B and for some i, G_0 is equal to G_i.*

2.15. PROPOSITION. *Suppose that $\mathscr{C}$ is a countable set of Turing degrees and d is an upper bound on the elements of $\mathscr{C}$. Let D be an element of d and suppose that G is a countable set of reals that are mutually Cohen generic with regard to meeting every dense set that is arithmetic in D. If ψ is a bijection between $\mathscr{C}$ and the set of degrees of elements of G, then ψ is definable with parameters in $\mathscr{D}$.*

Proof. Let $\mathscr{G}$ be the set of degrees with representatives in G; let $\mathscr{I}$ be the set of degrees of the form $x \vee \psi(x)$ where x is a degree in $\mathscr{C}$. Lemma 2.14 implies that both $\mathscr{G}$ and $\mathscr{I}$ are antichains and that the operation $\vee$ is injective on $\mathscr{D}(\leq d) \times \mathscr{G}$. Proposition 2.5 states that each of these sets can be defined in $\mathscr{D}$ uniformly using finitely many parameters.

This implies the definability of $\mathscr{C}$ and ψ by the following equations:

$$x \in \mathscr{C} \Leftrightarrow (x < d \,\&\, (\exists g \in \mathscr{G})(\exists z \in \mathscr{I})(x \vee g = z)),$$
$$\psi(x) = g \Leftrightarrow (x < d \,\&\, g \in \mathscr{G} \,\&\, (x \vee g) \in \mathscr{I}) \qquad \diamond$$

It remains to prove the main technical theorem.

2.16. THEOREM. *Suppose that $\mathscr{R}$ is a countable relation on $\mathscr{D}$. Then $\mathscr{R}$ is definable from parameters in $\mathscr{D}$. Furthermore, for each k there is a formula $\varphi(x_1, \ldots, x_k, y_1, \ldots, y_m)$ such that for each countable k-place relation R on $\mathscr{D}$ there is a sequence of parameters $a_1, \ldots, a_m$ such that R is defined by $\varphi(x_1, \ldots, x_k, a_1, \ldots, a_m)$ in $\mathscr{D}$.*

Proof. Suppose that $\mathscr{R}$ is a countable subset of $\mathscr{D}^n$. For each m smaller than n, let $R(m)$ be defined by

$$R(m) = \{a \mid (\exists v \in \mathscr{R})(v(m) = a)\}.$$

Let d be a uniform upper bound on all of the $R(m)$; let D be an element of d. Let G be a countable set of reals that are mutually Cohen generic with regard to meeting all of the dense sets in the Cohen partial order arithmetically definable in D. Write G as a disjoint union of sets $G(m)$, each of which has the same cardinality as $R(m)$. Fix bijections ψ_m between $R(m)$ and $G(m)$.

By the preceding propositions 2.5 and 2.15, each ψ_m, $R(m)$, $G(m)$ is definable from parameters in $\mathscr{D}$. Define $\mathscr{C}$ by

$$\mathscr{C} = \left\{ g_0 \vee g_1 \cdots \vee g_{n-1} \middle| \left\langle \psi_0^{-1}(g_0), \psi_1^{-1}(g_1), \ldots, \psi_{n-1}^{-1}(g_{n-1}) \right\rangle \in \mathscr{R} \right\}.$$

By Lemma 2.14, $\mathscr{C}$ is an antichain; it is definable from parameters in $\mathscr{D}$ by proposition 2.5. Now $\mathscr{R}$ can be defined by

$$\mathscr{R} = \{\langle a_0, a_1, \ldots, a_{n-1}\rangle \mid (\forall m \leq n)(a_m \in R(m)) \,\&\, \psi_0(a_0) \vee \psi_1(a_1) \vee \cdots \vee \psi_{n-1}(a_{n-1}) \in \mathscr{C}\}. \qquad \diamond$$

3. Global properties of $\mathscr{D}$

The most fruitful approach to understanding the global nature of $\mathscr{D}$ has been the use of definability. For example, in a delightful piece of work, Simpson [8] showed that there is a faithful interpretation of the second order theory of arithmetic in the first order theory of $\mathscr{D}$. He showed that it is possible to code models of arithmetic as initial segments of $\mathscr{D}$, and then to use Spector's theorem that every countable ideal in $\mathscr{D}$ is uniformly definable as the intersection of two principal ideals to code second order quantifications over the model.

Nerode and Shore [4] simplified the proof of Simpson's theorem by showing that it is possible to uniformly represent every countable, symmetric, irreflexive binary relation as a definable relation on an initial segment of $\mathscr{D}$. Then models of arithmetic and second order quantification over these models can be obtained from Spector's theorem. They also used their simplified definability methods to show that every automorphism is the identity on a cone of degree in $\mathscr{D}$. In later solo work, Shore [7] showed that there is a coding of arithmetic such that the relation "$\vec{c}$ is a code for an element of x" is a definable relation between degrees on the cone above 0^{ω} in $\mathscr{D}$. This last result gives a good explanation for the fact that any automorphism on $\mathscr{D}$ is the identity on a cone; the automorphism cannot change the set of integers coded by $\vec{c}$ so it cannot move x. Notice that the same reasoning shows that any elementary function from $\mathscr{D}$ to $\mathscr{D}$ is also the identity on a cone. Since this result is used later, it is isolated below.

3.1. THEOREM [7, Shore] *Suppose that ψ is an elementary function from $\mathscr{D}$ to $\mathscr{D}$. There is a degree a such that for all degrees x, if x is greater than a then $\psi(x) = x$.*

These results can now be given somewhat unified proofs using theorem 2.16. For example, to prove Simpson's theorem, notice that the usual second order characterization of a standard model of arithmetic involves specifying a countable set N, a distinguished element "0", and a unary function s, such that $\mathscr{N} = \langle N, 0, s\rangle$ satisfies second order induction. All of the objects can be represented by countable relations on the Turing degrees; the second order variables over N are interpreted by first order variables over the degrees using the translation provided by theorem 2.16. Of course, the new contribution provided by the definability theorem is a unified way to code the relations on $\mathscr{D}$ by elements of $\mathscr{D}$; the rest of the work must be done as before.

Elementary functions from $\mathscr{D}$ to $\mathscr{D}$. The next result couples the ability to code and recognize standard models of arithmetic together with the extended power to make definitions provided by Theorem 2.16.

3.2. THEOREM. *Suppose that ψ is an elementary mapping from $\mathscr{D}$ to $\mathscr{D}$. Then ψ is an automorphism.*

Proof. Since ψ is an elementary function, it is an injective homomorphism. To show that ψ is an automorphism, it remains to verify that it is surjective.

Let x be a degree. By Shore's theorem, there is a degree a such that ψ is equal to the identity on the cone above a. By Theorem 2.16, let $\vec{c}$ be a finite sequence of degrees which codes a standard model of arithmetic $\mathscr{N} = \langle N, 0, s \rangle$; let f be a function from N onto the degrees below $x \vee a$ which is coded by $\vec{d}$. The statement that $\vec{c}$ codes a standard model and that $\vec{d}$ codes a counting of the degrees below $x \vee a$ is a first order statement in $\mathscr{D}$ about $x \vee a$, $\vec{c}$ and $\vec{d}$.

Now consider this statement after the application of ψ. Since ψ is an elementary function, the same statement must hold of $\psi(x \vee a)$, $\psi(\vec{c})$ and $\psi(\vec{d})$. Thus, the finite sequence $\psi(\vec{c})$ codes a standard model of arithmetic $\psi(\mathscr{N})$ and $\psi(\vec{d})$ codes a function $\psi(f)$ from the universe of that model onto the degrees below $\psi(x \vee a)$. By the choice of a, $\psi(x \vee a)$ is equal to $x \vee a$; so there is an integer n such that x is the value of $\psi(f)$ at the nth element of $\psi(\mathscr{N})$. But this must mean that x is equal to ψ applied to f of the nth element of N as all of the relevant definitions are preserved by ψ. Thus, x is in the range of ψ as was desired. $\diamondsuit$

The key ingredients in the proof of Theorem 3.2 were the facts that the range of ψ formed a elementary substructure of $\mathscr{D}$ and that it was cofinal in $\mathscr{D}$. The same argument can then be used to show the following result.

3.3. THEOREM. *Suppose that $\mathscr{D}^*$ is an elementary substructure of $\mathscr{D}$ and that $\mathscr{D}^*$ is cofinal in $\mathscr{D}$. Then, $\mathscr{D}^*$ is equal to $\mathscr{D}$.*

The reader may wish to check that Theorem 3.2 holds when $\mathscr{D}$ is replaced by the Turing degrees of the arithmetic sets and the hypothesis ψ is elementary is replaced with ψ is elementary and maps $0''$ to $0''$. The proof is the same using Shore's result that every such ψ is the identity on a cone of degrees of arithmetic sets and the fact that if an arithmetic sequence $\vec{c}$ codes a nonstandard model of arithmetic then there is a code for its standard part in the arithmetic degrees.

3.4. *Question.* Let $\mathscr{A}$ be the partially ordered structure with universe the Turing degrees of the arithmetic sets of integers. Is every elementary function from $\mathscr{A}$ to $\mathscr{A}$ an automorphism? Is every elementary substructure of $\mathscr{A}$ equal to $\mathscr{A}$?

Local rigidity. It follows from the definability of the relation "x is arithmetic in y" and the existence of codes, obtained arithmetically from x, for elements of x, that if ψ is an automorphism of $\mathscr{D}$, then for any degree x, $\psi(x)$ is arithmetic in x.

3.5. DEFINITION. Suppose that $\mathscr{I}$ is an ideal in $\mathscr{D}$. Then $\mathscr{I}$ is a *jump ideal* if $\mathscr{I}$ is also closed under application of the Turing jump.

Note that whenever x belongs to a jump ideal and y is arithmetic in x, y also belongs to the jump ideal. By the preceding remarks, if ψ is an automorphism of $\mathscr{D}$ and $\mathscr{J}$ is a jump ideal in $\mathscr{D}$, then ψ induces an automorphism of $\mathscr{J}$ by restriction.

3.6. DEFINITION Suppose that $\mathscr{P}$ is an upper semi-lattice with an external increasing order preserving operation called the $\mathscr{P}$-jump.

(1) $\mathscr{P}$ is *rigid* if it has no automorphisms other than the identity.

(2) $\mathscr{P}$ is *absolutely rigid* if $\mathscr{P}$ is rigid in every model of ZFC that includes $\mathscr{P}$.

(3) $\mathscr{P}$ is *locally rigid* if each element of $\mathscr{P}$ belongs to a countable, rigid $\mathscr{P}$-jump ideal.

The next theorem establishes the connections between these notions for $\mathscr{D}$.

3.7. THEOREM. *The following conditions are equivalent.*

(1) *$\mathscr{D}$ is locally rigid.*

(2) *The relation "$\vec{c}$ is a code for an element of x" is definable in $\mathscr{D}$.*

Moreover, the defining formula in (2) *can be taken to be equivalent to a* Σ^1_2 *formula in analysis.*

Proof. First show that (1) implies (2). Assume that $\mathscr{D}$ is locally rigid.

As has already been established, the set of finite sequences $\vec{c}$ that are codes for standard models of arithmetic is $\mathscr{D}$-definable. Similarly, the set of codes for countable sets of reals whose degrees form a jump ideal I is $\mathscr{D}$-definable. As the rigidity of I can be expressed in terms of quantifiers over countable functions, the set of codes for sets of reals whose degrees form a countable rigid jump ideal is $\mathscr{D}$-definable. By Jockusch-Shore [3], it is possible to express "$\vec{c}$ codes a set of degrees closed under application of the Turing jump" by a first order formula in $\mathscr{D}$. Thus, "$\vec{c}$ codes a rigid jump ideal" is a first order property of $\vec{c}$ in $\mathscr{D}$.

3.8. CLAIM. *Assume $\mathscr{D}$ is locally rigid. Let x be a Turing degree and suppose that $\vec{c}$ is a code for a set of integers X. Then, X has degree x if and only if there is a set of reals I including X whose degrees form a rigid countable jump ideal, a rigid jump ideal $\mathscr{I}$ and an isomorphism ψ mapping the degrees of I to $\mathscr{I}$ such that ψ maps the degree of X to x.*

The claim is immediate once it is shown that any two isomorphic jump ideals are identical. This follows from the fact that a jump ideal consists of exactly the set of degrees of reals which are coded by sequences in the ideal. This in turn follows from the fact that the code for a set of integers X is produced by a finitary forcing relative to X, by meeting those dense sets which are recursive in a few jumps of X.

The claim is enough to prove that (1) implies (2), as it provides a definition of "$\vec{c}$ codes an element of x" expressible in the language of $\mathscr{D}$. Moreover, the definition is Σ_2^1; it demands the existence of a rigid jump ideal including the degree of x.

Next, show that (2) implies (1). Suppose that the relation "$\vec{c}$ codes an element of x" is defined in $\mathscr{D}$ by the formula ψ. A standard Skolem hull construction shows that for each degree y, there is a countable jump ideal $\mathscr{I}$ including y that is an elementary substructure of $\mathscr{D}$. It must be the case that $\mathscr{I}$ is rigid as any isomorphism must map standard models to standard models, thereby map codes for a real X to codes for X, and thereby map the degree of X to the degree of X.

Finally, notice that no assumption was made as to the form of the definition in the proof of (2) implies (1). If there is any definition then there is one of the correct form. $\diamondsuit$

3.9. COROLLARY. *Suppose that $\mathscr{D}$ is locally rigid.*

(1) *$\mathscr{D}$ is absolutely rigid.*

(2) *The Turing degrees are locally rigid in L.*

Proof. (1) follows immediately from the preceding theorem. To check (2), note that the definition of "$\vec{c}$ is a code for an element of x" is simple enough for the Shoenfield absoluteness theorem to apply. Thus, if "$\vec{c}$ is a code for an element of x" is definable in V, then it is also definable in L. $\diamondsuit$

3.10. *Question.* *Is $\mathscr{D}$ locally rigid?*

4. Definability in the Δ_2^0 degrees

This section is for the dedicated recursion theorists. Let $\varnothing'$ denote the canonical representative for $0'$. Let $D(\leq 0')$ be the set of degrees below $0'$ and let $\mathscr{D}(\leq 0')$ be the structure $\langle D(\leq 0'), \leq_T \rangle$. The main result of the section is to show that the predicate of recursive enumerability is definable from parameters in $\mathscr{D}(\leq 0')$. The proof is divided into two parts. First, prove an effective version of theorem 2.16, where the generic codes are constructed recursively in $\varnothing'$. Second, apply a result of Welch [10] which states that the recursively enumerable degrees are generated by sets which can be defined using the effective coding theorem.

4.1. DEFINITION. (1) Say that a degree is *low* if its Turing jump is $0'$.

(2) Say that a set of degrees $\mathscr{R}$ contained in $D(\leq 0')$ is uniformly recursive in $0'$ if there is a sequence of representatives for the degrees in $\mathscr{R}$ which is uniformly recursive in $\varnothing'$.

(3) With the same notation as in (2), say that $\mathscr{R}$ is uniformly low if it is uniformly recursive in $\varnothing'$ by means of the sequence $\langle X(n) | n \in w \rangle$ and there is a $\varnothing'$-recursive function f such that $\{f(n)\}^{\varnothing'}$ is the Turing jump of $\langle X(m) | m \leq n \rangle$.

The technical result needed to establish the definability of the recursively enumerable degrees is the following effective analog of theorem 2.16. Suppose that $\mathscr{R}$ is a subset of $D(\leq 0')$ which is uniformly low and bounded by a low element of $D(\leq 0')$. Then $\mathscr{R}$ is definable from parameters in $\mathscr{D}(\leq 0')$. The proof follows the same general outline as the one in Section 2. The new feature is that the definition of the set $\mathscr{R}$ is slightly more complicated in order to make the construction of a generic object simpler.

4.2. Proposition. *Suppose that $\mathscr{I}$ is an antichain which is uniformly low and bounded by a low element a of $D(\leq 0')$. There are degrees g_0 and g_1, which are recursive in $0'$, such that I is defined in $\mathscr{D}(\leq 0')$ as the set of minimal solutions below a for x in the following equation*:

$$((g_0) \vee (x)) \wedge ((g_1) \vee (x)) \neq (x). \tag{4.3}$$

Proof. The proof of this result parallels to that of proposition 2.5.

Let X be equal to $\langle X(n) | n \in \omega \rangle$ and be a sequence recursive in $\varnothing'$, representing $\mathscr{R}$ and uniformly low. By Lemma 2.3 and the uniformity of its proof, it is safe to assume that each element of X is recursive in any of its infinite subsets.

Let $\mathscr{P}$ be the notion of forcing introduced in definition 2.3 associated with X. Recall that a condition p is a triple $\langle p_0, p_1, F(p)\rangle$ where p_0 and p_1 are finite conditions and $F(p)$ is a finite sequence from I. The condition q extends p, if q extends p pointwise and the first two coordinates agree on the coding locations specified by the elements of $F(p)$. It was shown in Proposition 2.5 that if G is sufficiently $\mathscr{P}$-generic, then the degrees of G_0 and G_1, the two reals associated with the first and second coordinates of G, satisfy the statement of the theorem. The only problem is to build a set that is sufficiently generic and whose associated reals G_0 and G_1 are recursive in $\varnothing'$. The dense sets used in the proof of 2.5 were of three forms: $D(X(n)), C(e, X(n)), M(Y, e_0, e_1)$, where $X(n)$ lies in X, and Y is recursive in any set of degree a. It is easy to check from the definitions of these dense sets that for any condition p, either p belongs to the set or a $\Sigma_1(p)$ property holds and there is a condition q below p in $\mathscr{P}$ which is uniformly recursive in p, and which is in the dense set.

With the above remarks in mind, let G be a subset of $\mathscr{P}$ such that for every $\Sigma_1(A)$ subset S of $\mathscr{P}$ either G extends an element of S or G contains a condition which has no extension in S. It is easy to find such a set recursively in $\varnothing'$, because a is low. G is sufficiently generic for the proof of Section 2 to apply. Therefore the degrees of G's associated reals G_0 and G_1 satisfy the proposition.◇

The next step, as in Section 2, is to use the coding of antichains to code further sets. For this application, an effective version of Lemma 2.14 must be formulated.

4.4. LEMMA. *Suppose that a is a low element of $D(\leq 0')$ and A is a representative of a. Suppose that R is a uniformly low sequence of reals of degree below a. There is a sequence of reals G such that if B and C are recursive in A, and $G_0, G_1, \ldots, G_n$ is a sequence of reals from G, then $B \oplus G_0$ is recursive in $C \oplus G_1 \oplus G_2 \oplus \cdots \oplus G_n$ if and only if B is recursive in C and, for some i, G_0 is equal to G_i. In addition $A \oplus G$ is low and the sequence $R + G$, defined to be $\langle R(i) \oplus G(i) | i \in \omega \rangle$, is uniformly low.*

Proof. Construct a set of reals which are mutually Cohen generic with regard to certain dense sets. Namely, build G to be an ultrafilter on $\mathscr{P}_C$, the Cohen partial order for adding countably many reals by sequentially meeting the $\Sigma_1(A)$ subsets of $\mathscr{P}_C$ as possible. The question of whether there is an extension of the current condition in the next $\Sigma_1(A)$ set can be answered recursively in $\varnothing'$ by the assumption that A' is recursive in $\varnothing'$. The desired lowness is insured as every Σ_1 and Π_1 statement is forced along the way.

Suppose that B and C are recursive in A. Let e be an index for a recursive functional and let $i_0, \ldots, i_n$ be a sequence of integers. Let $G(\vec{i})$ denote $G_{i_0}, \ldots, G_{i_n}$, and let $q(\vec{i})$ denote the condition on $G(\vec{i})$ imposed by a condition q on G. Suppose that p is a Cohen condition which appears during the construction of G. If i_0 is not equal to any i_j $(i_j \geq 1)$, it is easy to come up with a $\Sigma_1(A)$ set of conditions q extending p making $\{e\}^{B \oplus G(\vec{i})}$ unequal to G_{i_0}. Let k be any number greater than the length of p. Let S be the set of product Cohen conditions q of the form $\langle q_0, q_1, \ldots, q_n \rangle$ extending p for which $\{e\}^{B \oplus q(\vec{i})}(k)\downarrow$ but has value different than $q_{i_0}(k)$. If there is any extension of $p(\vec{i})$ which makes the computation converge, the set S is not empty. Similarly, if there is no extension q of p and no k such that $\{e\}^{B \oplus q(\vec{i})}(k)$ is unequal to $C(k)$ then either there is a k such that $\{e\}^{B \oplus q(\vec{i})}(k)\uparrow$ for every q_i extending p_i, or C is recursive in B as C can be computed from B and the finite Cohen condition p. Either G meets the $\Sigma_1(A)$ set establishing an inequality, there is a k such that $\{e\}^{B \oplus G(\vec{i})}(k)\uparrow$, or C is recursive in B. This is exactly what is desired in Lemma 4.4. $\diamondsuit$

4.5. THEOREM. *Suppose that $\mathscr{R}$ is a uniformly low subset of $D(\leq 0')$ bounded by a low degree a. $\mathscr{R}$ is definable from parameters in $\mathscr{D}(\leq 0')$.*

Proof. Let R be the sequence which represents the elements of $\mathscr{R}$, perhaps with duplications but uniformly low. Let G be a set of reals as guaranteed by the preceding lemma. Let $R + G$ be the sequence defined above. Let $\mathscr{G}$ be the degrees of the reals in G and let $\mathscr{R} + \mathscr{G}$ be the degrees of the reals in $R + G$.

By the lemma, both $\mathscr{G}$ and $\mathscr{R} + \mathscr{G}$ are antichains and uniformly low. Proposition 4.2 implies that they are definable from parameters in $\mathscr{D}(\leq 0')$. Now lemma 4.4 implies that $\mathscr{R}$ can be defined from the following equation:

$$r \in \mathscr{R} \Leftrightarrow (\exists g \in \mathscr{G})((r \vee g) \in \mathscr{R} + \mathscr{G}). \qquad \diamondsuit$$

The final ingredient for the definition of the recursively enumerable degrees below $0'$ is a result of Welch.

4.6. THEOREM [10]. *There are two uniformly low sets $\mathscr{R}$ and $\mathscr{S}$ of recursively enumerable degrees such that each is bounded by a low degree and the recursively enumerable degrees are exactly the set of joins of one element from each set.*

Proof. Let K be the recursively enumerable set defined by

$$K = \{\langle e, n\rangle | n \in W_e\}.$$

The Sacks Splitting Theorem implies that K is the disjoint union of two low recursively enumerable sets A and B. The two uniformly low sequences that establish the theorem are R equal to $\langle A^{(n)} | n \in \omega\rangle$ and S equal to $\langle B^{(n)} | n \in \omega\rangle$. Recall the notation used here is that $A^{(n)}$ is the nth column of A. Every recursively enumerable set is the union of two elements, one from each sequence; any join of recursively enumerable degrees is recursively enumerable. $\diamondsuit$

4.7. THEOREM. *Let $\mathscr{RE}$ be the set of recursively enumerable degrees. The $\mathscr{RE}$ is definable from parameters in $\mathscr{D}(\leq 0')$.*

Proof. Let $\mathscr{R}$ and $\mathscr{S}$ be the sets stated to exist in Welch's theorem. By Theorem 4.5, both of these are definable from parameters in $\mathscr{D}(\leq 0')$. Welch's theorem states that the following equation defines $\mathscr{RE}$ in $\mathscr{D}(\leq 0')$:

$$\omega \in \mathscr{RE} \Leftrightarrow (\exists r \in \mathscr{R})(\exists s \in \mathscr{S})(w = (r \vee s)). \qquad \diamondsuit$$

The methods of this section can be used to obtain sharper definability results than the one mentioned in theorem 4.5. For example, the assumption that a is low can be replaced with the condition that a be incomplete. However, the known definability results are very limited.

4.8. *Question.* Suppose that R is an arithmetically definable set of degrees in $D(\leq 0')$. Is R definable from parameters in $\mathscr{D}(\leq 0')$?

4.9. *Question.* Is the set of recursively enumerable degrees definable without parameters in $\mathscr{D}(\leq 0')$?

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