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SOME QUESTIONS OF EDJVET AND PRIDE ABOUT INFINITE GROUPS

BY

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Dedicated to the memory of Bill Boone

1. The height of Pride

In his paper [9] Stephen Pride describes a pre-order $\preccurlyeq$ on the class of groups. In effect, as modified slightly in [2] the definition is that $H \preccurlyeq G$ if there exist:

$$(*) \quad \begin{cases} \text{(i)} \quad \text{a subgroup } G_0 \text{ of finite index in } G \\ \text{and a normal subgroup } G_1 \text{ of } G_0; \\ \text{(ii)} \quad \text{a subgroup } H_0 \text{ of finite index in } H \\ \text{and a finite normal subgroup } H_1 \text{ of } H_0; \\ \text{(iii)} \quad \text{an isomorphism } G_0/G_1 \to H_0/H_1. \end{cases}$$

If $H \preccurlyeq G$ and $G \preccurlyeq H$ then we write $G \sim H$, and we use $[G]$ to denote the equivalence class consisting of all such groups H. The relation $\preccurlyeq$ induces a partial order, also denoted $\preccurlyeq$, on the collection of all equivalence classes, with the class $[\{1\}]$ of all finite groups as its unique least member. The ideal $\mathrm{Id}[G]$ is defined to be the partially ordered set consisting of all equivalence classes $[H] \preccurlyeq [G]$. A group G is said to be *atomic* if $\mathrm{Id}[G]$ consists of $[\{1\}]$ and $[G]$; it is said to be of *height* h, and we write $\mathrm{ht}[G] = h$, if $\mathrm{Id}[G]$ is of height h as partially ordered set. In the papers [2], [9] a number of questions about these concepts are raised. These, and one or two others, are stated in §2 below. Answers are given in §§3–8. In a final section (§9) I prove some small results relating the pre-order $\preccurlyeq$ and the property max-N. The fact that a finitely generated atomic group satisfies max-N is typical of these.

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2. The questions

Question A. Does there exist a countable group that is SQ-universal but of finite height?

This is part of Problem 8 on page 333 of [9]. Hurley [5, pp. 207, 212] announces an affirmative answer, indeed, the existence of a countable atomic group that is SQ-universal, but his construction does not appear to have been published. I shall give such an example in §3 below.

Question B. Does there exist a finitely generated group that is SQ-universal but of finite height?

This is another part of Problem 8 of [9], and it occurs also as Problem 4 in [5]. In §4 I shall produce a finitely generated group of height 3 that is SQ-universal. On the other hand, finitely generated atomic groups satisfy max-N (see §9 below) and therefore cannot be SQ-universal. I do not know whether or not there exist finitely generated SQ-universal groups of height 2. Probably not —but I have no real evidence.

Question C_1. Do there exist finitely generated just infinite groups not satisfying max-SN?

A group is said to be just infinite if all its non-trivial normal subgroups are of finite index. The question is Problem 5 on page 323 of [9] and Problem 4′ of [2]. It arises in connection with:

Question C_2. Is every finitely generated atomic group finite-by-$\mathfrak{D}_2$-by-finite?

Here $\mathfrak{D}_2$ is the class of groups in which every non-trivial subnormal subgroup has finite index. The question is put in the form of a conjecture on page 12 of [2]. In §5 I construct a group that supplies a positive answer to Question C_1 and a negative answer to Question C_2.

Question D. If G has finite height, are all maximal chains in $\mathrm{Id}[G]$ of the same length?

This is Problem 1 of [2]. A counterexample is given in §6 below.

Question E. Let G be a countable group with normal subgroups K_1, K_2 such that $K_1 \cap K_2 = \{1\}$. Is it true that if G/K_1 and G/K_2 both have finite height then G has finite height (bounded by a function of $\mathrm{ht}[G/K_1]$ and $\mathrm{ht}[G/K_2]$)?

The question was suggested by Theorem 4.5 of [9]. It has a negative answer that will be given in §7 below.

Question F. If G is a finitely generated group of finite height, must $\mathrm{Id}[G]$ be finite?

This is Problem 5 of [2]. A counterexample is produced in §8 below.

3. Answer A

Example A. A countable group A that is SQ-universal but atomic.

Construction. Let P be a countable perfect group that is SQ-universal. For definiteness let us take P to be the triangle group with presentation

$$\left\langle a, b \mid a^2 = b^3 = (ab)^7 = 1\right\rangle$$

or, with an eye to future developments, the free product $\mathrm{Alt}(5) * \mathrm{Alt}(5)$ (see [8] for a proof of SQ-universality of these groups). Let $A := \mathrm{wr}^{\omega} P$ as defined by P. Hall [3] (except that, as in [7], I use wr to denote the restricted wreath product and reserve Wr for unrestricted wreath products). We think of A as a direct limit as follows. Define $A_0 := P$ and thereafter $A_{i+1} := A_i \,\mathrm{wr}\, P$, with A_i embedded in A_{i+1} as the first factor of the base group; then $A = \cup A_i$. Let P_i be the top group in A_i (with $P_0 := A_0$). Then

$$A_i = \langle P_0, P_1, P_2, \ldots, P_i \rangle,$$

and if

$$B_i := \langle P_{i+1}, P_{i+2}, P_{i+3}, \ldots \rangle = \mathrm{wr}^{(\omega - \{0,1,\ldots,i\})} P \cong A,$$

then $A = A_i \,\mathrm{wr}\, B_i$ where here the wreath product is a permutational one. Let K_i denote the base group in this wreath product. It is the normal closure of A_i in A and is isomorphic to a (restricted) direct power of A_i.

It should be clear that A is countable. Also, A is SQ-universal because if X is any countable group then there is a normal subgroup Q of P such that X is embeddable into P/Q, and so X is embeddable into $P/Q \,\mathrm{wr}\, B_0$, which is a homomorphic image of $P_0 \,\mathrm{wr}\, B_0$, that is, of A. It only remains to show that A is atomic.

If $x \in A_{i+1} - K_i$ then, since A_i is perfect and $A_{i+1} = A_i \,\mathrm{wr}\, P$, the normal closure of x in A_{i+1} contains the whole of the base group of this wreath product (see, for example, [7, Lemma 8.2]), and so the normal closure of x in A contains the whole of K_i. Consequently, if N is a proper normal subgroup of

A then $K_i \leq N \leq K_{i+1}$ for some value of i. Now if $H \preccurlyeq A$ then there exist subgroups H_0, H_1 of H as in $(*)$ such that H_0/H_1 is isomorphic to a quotient group of a subgroup of finite index in A. It follows, since A has no proper subgroups of finite index, that either $H_0/H_1 = \{1\}$, in which case $[H] = [\{1\}]$, or $H_0/H_1 \cong A/N$ for some proper normal subgroup N of A, in which case H_0/H_1 has a quotient group isomorphic to A/K_{i+1} for some i and, since $A/K_{i+1} \cong A$, we then have $[H] = [A]$. Thus A is atomic.

4. Answer B

Example B. A finitely generated group that is SQ-universal and of height 3.

Construction. The first ingredient is the free product $P := \mathrm{Alt}(5) * \mathrm{Alt}(5)$ which we use to manufacture the group $A := \mathrm{wr}^{\omega} P$ as in §3. The remaining ingredients are:
a finitely generated infinite simple group S;
an infinite subset Σ of S such that $|\Sigma \cap \Sigma x| \leq 1$ for all $x \in S - \{1\}$;
an enumeration $\Sigma = \{\sigma_0, \sigma_1, \sigma_2, \dots\}$ of Σ;
an automorphism α of S such that $\sigma_i \alpha = \sigma_{i+1}$ for all $i \geq 0$.
This is not too much to ask: if we embed the group with presentation

$$\langle s, t \mid t^{-1} s t = s^2 \rangle$$

into a finitely generated simple group S (using, for example, the methods of P. Hall [4]) then we can take Σ to be $\{s, s^2, s^4, s^8, \dots\}$ and α to be the inner automorphism consisting of conjugation by t.

Now let $W := A \,\mathrm{Wr}\, S$, the unrestricted standard wreath product. Elements of the base group in W, the cartesian power A^S, will be written as sequences $(x_\sigma)_{\sigma \in S}$. Let P_i be the subgroup of A given that name in §3 and let Q_i, R_i be its two free factors Alt(5). Choose generators a_i, b_i of Q_i and c_i, d_i of R_i such that

$$a_i^2 = b_i^3 = (a_i b_i)^5 = 1 \quad \text{and} \quad c_i^2 = d_i^3 = (c_i d_i)^5 = 1.$$

Let $u, v \in A^S$ be the elements $(u_\sigma)_{\sigma \in S}, (v_\sigma)_{\sigma \in S}$ defined by

$$u_\sigma := \begin{cases} 1 & \text{if } \sigma \notin \Sigma \\ a_i & \text{if } \sigma = \sigma_{2i} \\ c_i & \text{if } \sigma = \sigma_{2i+1}, \end{cases}$$

$$v_\sigma := \begin{cases} 1 & \text{if } \sigma \notin \Sigma \\ b_i & \text{if } \sigma = \sigma_{2i} \\ d_i & \text{if } \sigma = \sigma_{2i+1}. \end{cases}$$

The group B that we want is the subgroup $\langle u, v, S\rangle$ of W. Obviously B is finitely generated. What has to be proved is that B is SQ-universal and of height 3.

Let $M := B \cap A^S$, so that $B/M \cong S$, and let $L := A^{(S)}$, the restricted direct power of A consisting of all sequences of finite support in A^S. The crux of the matter is the fact that $L \leq B$ and $B/L \cong \mathrm{Alt}(5) \operatorname{wr} S$ with base group M/L.

The idea of the proof that $L \leq B$ is exactly that of [6, pp. 469, 470]. First we observe that M is a subcartesian power, that is, its projection to each factor in A^S is surjective. Therefore if

$$A^* := \{(w_\sigma)_{\sigma \in S} \mid w_\sigma = 1 \text{ if } \sigma \neq 1\},$$

the "first coordinate subgroup" in A^S, then $M \cap A^* \trianglelefteq A^*$. Consider the commutator

$$[s_1 u s_1^{-1}, s_2 u s_2^{-1}],$$

where $s_1, s_2 \in S$ and $s_1 \neq s_2$. It is the sequence $(w_\sigma)_{\sigma \in S}$ where $w_\sigma = [u_{\sigma s_1}, u_{\sigma s_2}]$. If $w_\sigma \neq 1$ then $u_{\sigma s_1} \neq 1$ and $u_{\sigma s_2} \neq 1$, and so $\sigma s_1 \in \Sigma$ and $\sigma s_2 \in \Sigma$, that is, $\sigma \in \Sigma s_1^{-1} \cap \Sigma s_2^{-1}$. But $\Sigma s_1^{-1} \cap \Sigma s_2^{-1} = (\Sigma \cap \Sigma s_2^{-1} s_1) s_1^{-1}$, and this is either empty or a singleton. Therefore $(w_\sigma)_{\sigma \in S}$ has at most one non-identity component and (by definition) is a member of one of the "coordinate subgroups" of A^S. If we take s_1 to be σ_{2i} and s_2 to be σ_{2i+1} we find that $[\sigma_{2i} u \sigma_{2i}^{-1}, \sigma_{2i+1} u \sigma_{2i+1}^{-1}]$ is the sequence $(w_\sigma^{(i)})_{\sigma \in S}$ such that

$$w_\sigma^{(i)} = \begin{cases} [a_i, c_i] & \text{if } \sigma = 1 \\ 1 & \text{if } \sigma \neq 1, \end{cases}$$

and so $[\sigma_{2i} u \sigma_{2i}^{-1}, \sigma_{2i+1} u \sigma_{2i+1}^{-1}] \in A^* \cap M$. Obviously $A^* \cong A$, and we observed in §3 that any proper normal subgroup of A is contained in the subgroup K_r for some r. Since $w_1^{(r+1)} \notin K_r$ we must have $A^* \cap M = A^*$. Thus $A^* \leq M$ and, as L is generated by the conjugates $s^{-1} A^* s$ for $s \in S$, also $L \leq M$. The fact that $B/L \cong \mathrm{Alt}(5) \operatorname{wr} S$ now follows easily. For, since $u^2 = v^3 = (uv)^5 = 1$ we have $\langle u, v\rangle \cong \mathrm{Alt}(5)$; moreover, if $s \in S - \{1\}$ then $s^{-1}us$ and $s^{-1}vs$ commute with both u and v modulo L; and of course the conjugates $s^{-1}\langle u, v\rangle s$ for $s \in S$ are independent modulo L.

In §3 we defined normal subgroups K_i of A and we saw that every proper normal subgroup of A lies between K_i and K_{i+1} for some i. It follows easily that if N is a proper normal subgroup of B then $N = M$ or $N = L$ or $K_i^{(S)} \leq N \leq K_{i+1}^{(S)}$ for some i. We need to prove that $B/K_{i+1}^{(S)} \cong B$. Now $B/K_{i+1}^{(S)} \cong \langle u', v', S\rangle$, where u' and v' are obtained from u, v respectively by replacing the $\sigma_0, \sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_{2i+2}, \sigma_{2i+3}$ coordinates by 1. And the map

$$u \mapsto u', \quad v \mapsto v', \quad s \mapsto s\alpha^{2i+4} \quad \text{if } s \in S$$

gives an isomorphism $B \to \langle u', v', S\rangle$. Thus if $K_i^{(S)} \le N \le K_{i+1}^{(S)}$ then B/N has a quotient group isomorphic to B. It follows easily that Id[B] consists of [{1}], [S], [Alt(5) wr S] and [B], and hence that ht[B] = 3.

Now let X be any countable group. Since A is SQ-universal there is a normal subgroup N of A such that X is embeddable into A/N. The homomorphism of A onto A/N induces a homomorphism of W onto $(A/N)\,\mathrm{Wr}\,S$ that maps L to the direct power $(A/N)^{(S)}$ and so X is embeddable into the image of B. Thus B is SQ-universal, as required.

5. Answers C_1 and C_2

Before describing the relevant construction here we prove a result that sets the scene. Throughout this section S will be a non-abelian finite simple group and Σ a faithful transitive S-space. In due course we take Σ to be $\{1,2,3,4,5,6\}$ and S to be Alt(Σ).

THEOREM 5.1. *Let G be a group such that*:
(i) *G is perfect*;
(ii) *G is residually finite*;
(iii) $G \cong G\,\mathrm{wr}_\Sigma\, S$.
Then also:
(iv) *every non-trivial normal subgroup has finite index in G (that is, G is just infinite)*;
(v) *every subnormal subgroup is isomorphic to a finite direct power of G*;
(vi) *nevertheless, G does not satisfy* max-SN;
(vii) *G is atomic.*

Proof. We have $G = G_1\,\mathrm{wr}_\Sigma\, S$, where $G_1 \cong G$. Consequently $G = G_n\,\mathrm{wr}_{\Delta_n} W_n$, where $G_n \cong G$,

$$W_n := S\,\mathrm{wr}_\Sigma\, S\,\mathrm{wr}_\Sigma\, \cdots\, \mathrm{wr}_\Sigma\, S \ (n \text{ factors})$$

and

$$\Delta_n := \Sigma \times \Sigma \times \cdots \times \Sigma \ (n \text{ factors}).$$

Let K_n be the base group in this wreath product, so that $K_n \cong G^{\Delta_n}$, and $G/K_n \cong W_n$. Put $K_0 := G$.

LEMMA 5.2. *If* $N \trianglelefteq G$ *and* $N \ne \{1\}$ *then* $N = K_n$ *for some n.*

Proof. First we prove that $K_0, K_1, K_2, \ldots$ are the only normal subgroups of finite index. Let X be a finite group and $f\colon G \to X$ a homomorphism. If m is large enough there must be two distinct coordinate subgroups G_{mi}, G_{mj} (direct factors isomorphic to G) of K_m that have the same image under f.

Since G_{mi}, G_{mj} centralise each other it follows that their common image is abelian and since G is perfect that image must be $\{1\}$; then, since K_m is the normal closure of G_{mi}, also $K_m \leq \mathrm{Ker}(f)$. Therefore $\mathrm{Im}(f)$ is a homomorphic image of G/K_m, that is, of W_m. In W_m, however, the base group $S^{\Delta_{m-1}}$ is a minimal (non-trivial) normal subgroup (because it is a direct power of the non-abelian simple group S and its simple direct factors are permuted transitively under conjugation in W_m) and its centraliser is trivial. Therefore it is the unique minimal normal subgroup. That is, K_{m-1}/K_m is the unique minimal normal subgroup of W_m, and it follows by induction that $K_0, K_1, \ldots, K_{m-1}, K_m$ are the only normal subgroups of G that contain K_m. Thus $\mathrm{Ker}(f) = K_n$ for some n.

Now let N be any non-trivial normal subgroup of G. Since G is residually finite we must have $\bigcap K_m = \{1\}$ and so there exists n such that $N \leq K_n$ and $N \nleq K_{n+1}$. If $x \in N - K_{n+1}$ then, as one sees by a very small modification of the argument used to prove Lemma 8.2 of [7], the normal closure of x in G contains K_{n+1}. Thus $K_{n+1} < N \leq K_n$ and, as we have already seen, it follows that $N = K_n$, as required.

This deals with assertion (iv) of Theorem 5.1. Now every non-trivial normal subgroup of G is isomorphic to a finite direct power of G and so to prove (v) we need to show that a normal subgroup of a finite direct power of G is itself isomorphic to a finite direct power of G. But if $X_1, X_2, \ldots, X_k$ are groups all of whose normal subgroups are perfect, and if $N \trianglelefteq X_1 \times X_2 \times \cdots \times X_k$ then, as is very easy to prove, $N = Y_1 \times Y_2 \times \cdots \times Y_k$ where $Y_i \trianglelefteq X_i$ for $1 \leq i \leq k$.

To prove (vi) we proceed as follows. Suppose, as inductive hypothesis, that G has a subnormal subgroup $X_n \times Y_n$ with $Y_n \cong G$. This is certainly true for $n = 0$ with $X_0 := \{1\}$, $Y_0 := G$. Now $Y_n \cong G \operatorname{wr}_\Sigma S$ and we can take $X_{n+1} := X_n \times Z_1$, $Y_{n+1} := Z_2$, where Z_1, Z_2 are two of the direct factors in the base group of the wreath product. Then $X_{n+1} \times Y_{n+1}$ is subnormal in G, so induction supplies a properly increasing sequence $X_0 < X_1 < X_2 < \cdots$ of subnormal subgroups of G.

Suppose now that $H \preccurlyeq G$. There exist subgroups H_0, H_1 as in $(*)$, such that H_0/H_1 is a homomorphic image of a subgroup G^* of finite index in G. Moreover, we can take G^* to be normal in G. Then G^* is a finite direct power of G and, by what has already been shown, it follows that H_0/H_1 is a direct product of finitely many groups, each of which is finite or isomorphic to G. Therefore either H is finite or $G \preccurlyeq H$, and so $\mathrm{Id}[G]$ consists of $[\{1\}]$ and $[G]$, that is, G is atomic. This completes the proof of Theorem 5.1.

Assertions (iv) and (vi) applied to the following example give a positive answer to Question C_1, and (vii) gives a negative answer to Question C_2.

Example C. A finitely generated group C that is perfect, residually finite and isomorphic to $C \operatorname{wr}_\Sigma S$.

Construction. We take $\Sigma := \{1,2,3,4,5,6\}$ and $S := \mathrm{Alt}(\Sigma)$. Define $\Delta_n := \Sigma^n$ (n-fold cartesian power) and $W_n := \mathrm{wr}^n S$ with its natural action as a permutation group on Δ_n. Embed W_{n-1} into W_n as the top group in the representation $W_n = S \,\mathrm{wr}_{\Delta_{n-1}} W_{n-1}$, and take S_n to be one of the direct factors (coordinate subgroups) of the base group. Now define W to be the direct limit $\bigcup W_n$—so that W is, in fact, P. Hall's wreath power $\mathrm{wr}^{-\mathbf{N}} S$. If $V_n := \langle S_{n+1}, S_{n+2}, \ldots \rangle$ then $V_n \cong W$ and $W = V_n \,\mathrm{wr}_{\Delta_n} W_n$. And if L_n is the normal closure of V_n in W, that is, the base group in this wreath product, then $L_n \cong V_n^{\Delta_n} \cong W^{\Delta_n}$. It is not hard to see directly that $W, L_1, L_2, L_3, \ldots$ are the only non-trivial normal subgroups of W—although this also follows from Lemma 5.2.

There is a natural surjective homomorphism $W_n \to W_{n-1}$ for each n, and we define $\overline{W}$ to be the inverse limit $\varprojlim W_n$. Elements of W_n can be expressed uniquely in the form $t_n t_{n-1} \cdots t_2 t_1$, where t_i is in the base group of W_i (we take the 'base group' of W_1 to be W_1 itself); then elements of $\overline{W}$ may be uniquely described by left-infinite sequences $\cdots t_n t_{n-1} \cdots t_2 t_1$, where t_i is in the base group of W_i. Each factor t_i may in turn be written as a product $\prod_{\delta \in \Delta_{i-1}} s_i(\delta)$, where $s_i(\delta) \in S$, this expression being unique up to the order of its factors (we take Δ_0 to be a singleton set so that t_1 is simply a member of S). The rule for multiplication in $\overline{W}$ is determined by that in the finite wreath products. Since it is quite complicated, and since we shall need only very special cases, I do not write it down explicitly. The group W may be seen as that subgroup of $\overline{W}$ that consists of sequences in which $t_i = 1$ for all except finitely many values of i. In fact, $\overline{W}$ is the completion of W with respect to the topology that has the groups L_n as a base for the neighbourhoods of 1. If $\overline{V}_1$ is the closure of V_1 in $\overline{W}$ then $\overline{W} = \overline{V}_1 \,\mathrm{wr}_\Sigma S$ and $\overline{V}_1 \cong \overline{W}$. We can describe $\overline{V}_1$ explicitly as the set of all sequences

$$\cdots t_n t_{n-1} \cdots t_2 t_1$$

in which $t_1 = 1$ and for all $i > 1$, $s_i(\sigma_{i-1}, \ldots, \sigma_2, \sigma_1) = 1$ if $\sigma_1 \neq 1$. And we can define an isomorphism $\overline{W} \to \overline{V}_1$ explicitly:

$$\cdots t_n t_{n-1} \cdots t_2 t_1 \mapsto \cdots v_n v_{n-1} \cdots v_2 v_1$$

where

$$v_1 = 1, \quad v_i = \prod_{\Delta_{i-1}} u_i(\sigma_{i-1}, \ldots, \sigma_2, \sigma_1),$$

and

$$u_i(\sigma_{i-1}, \ldots, \sigma_2, \sigma_1) = \begin{cases} 1 & \text{if } \sigma_1 \neq 1 \\ s_{i-1}(\sigma_{i-1}, \ldots, \sigma_2) & \text{if } \sigma_1 = 1. \end{cases}$$

Similarly, if $\overline{V}_n$ is the closure of V_n then $\overline{V}_n \cong \overline{W}$ and $\overline{W} = \overline{V}_n \operatorname{wr}_{\Delta_n} W_n$. If $\overline{L}_n$ is the closure of L_n then $\overline{L}_n$ is the base group in this wreath product and $\overline{L}_n$ consists of those sequences $\cdots t_i t_{i-1} \cdots t_2 t_1$ such that $t_i = 1$ if $i \leq n$. Since $\overline{L}_n$ has finite index in $\overline{W}$ and $\bigcap_n \overline{L}_n = \{1\}$, $\overline{W}$ is residually finite.

Calculation in $\overline{W}$ may be simplified if we represent it as a permutation group. There is a natural surjective map $\Delta_n \to \Delta_{n-1}$ that is compatible with the actions of W_n on Δ_n and W_{n-1} on Δ_{n-1} and with our surjective homomorphism $W_n \to W_{n-1}$. It follows that there is a natural action of $\varprojlim W_n$ on $\varprojlim \Delta_n$; that is, if we define $\overline{\Delta} := \Sigma^{-\mathbf{N}} = \varprojlim \Delta_n$, there is a natural action of $\overline{W}$ on $\overline{\Delta}$. Elements of $\overline{\Delta}$ may be thought of as left-infinite sequences

$$(\ldots, \sigma_n, \sigma_{n-1}, \ldots, \sigma_2, \sigma_1),$$

where $\sigma_i \in \Sigma$ for all i. The action of $\overline{W}$ on $\overline{\Delta}$ is the following:

$$(\ldots, \sigma_n, \sigma_{n-1}, \ldots, \sigma_2, \sigma_1) \cdots t_n t_{n-1} \cdots t_2 t_1 = (\ldots, \rho_n, \rho_{n-1}, \ldots, \rho_2, \rho_1),$$

where, if $t_i = \prod_{\delta \in \Delta_{i-1}} s_i(\delta)$ as before, then

$$\rho_i = \sigma_i s_i(\sigma_{i-1}, \ldots, \sigma_2, \sigma_1).$$

The set $\overline{\Delta}_1$ of all sequences $(\ldots, \sigma_n, \sigma_{n-1}, \ldots, \sigma_2, 1)$ is a block of imprimitivity for $\overline{W}$ in $\overline{\Delta}$. Its stabiliser is $\overline{V}_1$. The map $\overline{\Delta} \to \overline{\Delta}_1$ given by

$$(\ldots, \sigma_n, \sigma_{n-1}, \ldots, \sigma_2, \sigma_1) \mapsto (\ldots, \sigma_{n-1}, \sigma_{n-2}, \ldots, \sigma_1, 1)$$

induces the isomorphism $\overline{W} \to \overline{V}_1$ described above; and the natural bijection $\overline{\Delta} \to \overline{\Delta}_1 \times \Sigma$ induces our isomorphism $\overline{W} \to \overline{V}_1 \operatorname{wr}_\Sigma S$.

We are now ready to define the group C. For each permutation $t \in \operatorname{Alt}(\Sigma)$ and each element $\tau \in \Sigma$ define

$$w(t, \tau) := \ldots t_{\tau\tau\tau} t_{\tau\tau} t_\tau t \in \overline{W}.$$

By this I mean that the i-th component of $w(t, \tau)$ is the element t in that coordinate subgroup of the base group of W_i that is indexed by $(\tau, \ldots, \tau, \tau)$ in Δ_{i-1}: that is,

$$t_{\tau \cdots \tau\tau} = \prod_{\Delta_{i-1}} s_i(\sigma_{i-1}, \ldots, \sigma_2, \sigma_1),$$

where

$$s_i(\sigma_{i-1}, \ldots, \sigma_2, \sigma_1) := \begin{cases} 1 & \text{if } (\sigma_{i-1}, \ldots, \sigma_2, \sigma_1) \neq (\tau, \ldots, \tau, \tau) \\ t & \text{if } (\sigma_{i-1}, \ldots, \sigma_2, \sigma_1) = (\tau, \ldots, \tau, \tau). \end{cases}$$

Now define

$$C := \langle w(t,\tau) | t \in \mathrm{Alt}(\Sigma), \tau \in \mathrm{fix}(t) \rangle.$$

Certainly C is finitely generated: the given generating set has 360 members but very much smaller ones will suffice. Consider for the moment a fixed element σ of Σ and all generators $w(t,\sigma)$ with $\sigma \in \mathrm{fix}(t)$. It is easy to see that

$$w(t_1,\sigma)w(t_2,\sigma) = w(t_1t_2,\sigma)$$

and so these $w(t,\sigma)$ form a subgroup of $\overline{W}$ that is isomorphic to Alt(5). Thus C is generated by six subgroups isomorphic to Alt(5) and therefore C is perfect. Since $\overline{W}$ is residually finite, also C is residually finite, and all we still have to prove is that $C \cong C \mathrm{wr}_\Sigma S$.

Define $w^*(t,\tau) := \ldots t_{\tau\tau\tau}t_{\tau\tau}t_\tau$, so that $w(t,\tau) = w^*(t,\tau)t$. If s, t are permutations in $\mathrm{Alt}(\Sigma)$ that fix both 5 and 6 then, as is easy to see, $w(s,5)^*$ and $w(t,6)^*$ commute with each other and with both s and t. Computing commutators we therefore have that

$$[w(s,5), w(t,6)] = [s,t] \in W_1 \leq \overline{W}.$$

If we take s, t to be the 3-cycles (123), (134) respectively then we find that $(12)(34) \in C$. Similarly all other double transpositions lie in C and so $W_1 \leq C$. Consequently $C = (C \cap I_1).W_1$. Now let τ be any element of Σ and t any permutation in $\mathrm{Alt}(\Sigma)$ fixing τ. Choose $s \in \mathrm{Alt}(\Sigma)$ that maps τ to 1. Then

$$s^{-1}w^*(t,\tau)s = \cdots t_{\tau\tau 1}t_{\tau 1}t_1$$

and this is the element corresponding to $w(t,\tau)$ in our isomorphism of $\overline{W}$ to $\overline{V}_1$. Since C is generated by the members of W_1 together with the elements $w^*(t,\tau)$ (with $\tau \in \mathrm{fix}(t)$) it is generated by W_1 together with these conjugates $s^{-1}w^*(t,\tau)s$. Clearly the latter generate a copy C_1 of C inside $\overline{V}_1$. Therefore $C = C_1 \mathrm{wr}_\Sigma S$ with $C_1 \cong C$: thus C is a finitely generated group satisfying conditions (i), (ii), (iii) of Theorem 5.1, as required.

Comment 5.3. Let Δ be the C-orbit in $\overline{\Delta}$ that contains the sequence $(\ldots,1,1,\ldots,1,1)$ and let $\Delta_1 \times \{1\}$ be the C_1-orbit of this sequence. If $t \in W_1$ maps 1 to τ then

$$(\ldots,1,1,\ldots,1,1)t = (\ldots,1,1,\ldots,1,\tau)$$

and the $t^{-1}C_1t$-orbit of this sequence is $\Delta_1 \times \{\tau\}$. Consequently the obvious bijection $\Delta \to \Delta_1 \times \Sigma$ induces our isomorphism

$$C \to C_1 \mathrm{wr}_\Sigma \mathrm{Alt}(\Sigma),$$

and the obvious bijection $\Delta \to \Delta_1$ induces our isomorphism $C \to C_1$. We shall need the permutation representation of C on Δ as an ingredient in our next construction.

Comment 5.4. The construction of C can be varied in many ways. Here is one. Let $S_1, S_2, S_3, \ldots$ be non-abelian finite simple groups acting faithfully and transitively on sets $\Sigma_1, \Sigma_2, \Sigma_3, \ldots$. Suppose that for some integer d every group S_i can be generated by d subgroups, each of which is isomorphic to Alt(5) and each of which fixes at least two members of Σ_i (if (S_i, Σ_i) is Alt(n_i) in its natural action then this can be achieved with $d = 3$ provided that $n_i \geq 7$ for all i). Define $\Delta_1 := \Sigma_1$, $W_1 := S_1$, and thereafter

$$\Delta_n := \Sigma_n \times \Delta_{n-1}, \quad W_n := S_n \operatorname{wr}_{\Delta_{n-1}} W_{n-1}.$$

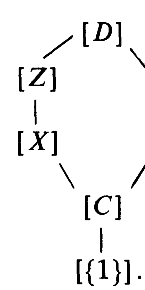
As before we can find a finitely generated subgroup C of the inverse limit $\overline{W} := \varprojlim W_n$ which has the property that all non-trivial normal subgroups have finite index in C. But if the sequence $(S_1, \Sigma_1), (S_2, \Sigma_2), (S_3, \Sigma_3), \ldots$ is not periodic then the proper subnormal subgroups are direct products of groups of the same kind of structure as C, but none of which is isomorphic to C. Under these circumstances C is a finitely generated just infinite group of infinite height.

6. Answer D

Example D. A group D of height 4 such that Id$[D]$ has a maximal chain of length 3.

Construction. We begin with the group C of §5 and with two faithful transitive C-spaces. The first is the C-space Δ described in Comment 5.3, the second Γ is the coset space $(C : W)$. A subgroup of finite index in C contains the normal subgroup K_n, the kernel of the homomorphism of C onto W_n, for some n, and $WK_n = C$. Therefore every subgroup of finite index in C is transitive on Γ.

Let S be a non-abelian simple group and define $X := S \operatorname{wr}_\Gamma C$, $Y := S \operatorname{wr}_\Delta C$, $Z := X \times C$ and $D := X \times Y$. I shall prove that the partially ordered set Id$[D]$ is



A subgroup of finite index in D contains a subgroup $X_0 \times Y_0$ where X_0 is normal and of finite index in X, and Y_0 is normal and of finite index in Y. Since the base groups $S^{(\Gamma)}$, $S^{(\Delta)}$ (restricted direct powers of S) are the unique minimal normal subgroups of X and Y respectively, we have $S^{(\Gamma)} \leq X_0$, $S^{(\Delta)} \leq Y_0$ and

$$X_0 = S \operatorname{wr}_\Gamma K_m, \quad Y_0 = S \operatorname{wr}_\Delta K_n$$

for some m, n. Since K_m is transitive on Γ, $S^{(\Gamma)}$ is the unique minimal normal subgroup of X_0. Since K_n has 6^n (as it happens) orbits on Δ, on each of which it acts like C on Δ, $Y_0 \cong Y^{6^n}$. Every normal subgroup of X_0 and of Y_0 is perfect, so a normal subgroup of $X_0 \times Y_0$ is of the form $X_1 \times Y_1$ where $X_1 \leq X_0$ and $Y_1 \leq Y_0$. If $X_1 \neq \{1\}$ then X_0/X_1 is isomorphic to a quotient group of $X_0/S^{(\Gamma)}$, that is of K_m, and therefore $X_0/X_1 \cong C^k \times Q$ for some k and some finite group Q; similarly, $Y_0/Y_1 \cong Y^k \times C^l \times R$ for some integers k, l and some finite group R. Clearly therefore $Y \preccurlyeq Y^k \times C^l \preccurlyeq Y$ and so $Y \sim Y \times C \sim Y^k$ for any positive integer k. It follows immediately that $\mathrm{Id}[D]$ consists of $[\{1\}], [C], [X], [Y], [X \times C], [D]$, and that it is ordered as shown in the diagram.

7. Answer E

Example E. A group E of infinite height that is a subdirect product of two atomic groups.

Construction. Let S be a non-abelian finite simple group, let R be the countable restricted direct power $S^{(\aleph_0)}$, let $R_1 := R$ and $R_2 := R \operatorname{wr} R$. We define Q to be the wreath power $\operatorname{wr}^\omega R$ and Ω to be the set on which it naturally acts, that is, since R is to be thought of as acting on itself regularly (by right multiplication), Ω is the countable restricted direct power $R^{(\omega)}$ (see [3]). Define $E := (R_1 \times R_2) \operatorname{wr}_\Omega Q$.

If $K_1 := R_1^{(\Omega)}$ and $K_2 := R_2^{(\Omega)}$, so that $K_1 \times K_2$ is the base group in the wreath product, then

$$E/K_1 \cong R_2 \operatorname{wr}_\Omega Q \cong Q \quad \text{and} \quad E/K_2 \cong R_1 \operatorname{wr}_\Omega Q \cong Q,$$

and so E is a subdirect product in $Q \times Q$. But Q is atomic (compare §3). So E is a subdirect product of two atomic groups.

On the other hand E has quotient groups of the form

$$E_{m,n} \cong (S^m \times (S^n \operatorname{wr} R)) \operatorname{wr}_\Omega Q.$$

If $m = 0$ or $n = 0$ then $E_{m,n} \sim Q$, but if $m \neq 0$ and $n \neq 0$ then it is easy to see that $E_{m,n} \preccurlyeq E_{m',n'}$ if and only if $m \leq m'$ and $n \leq n'$. Thus E has infinite height.

8. Answer F

Recall that a soluble minimax group is a group G with a subnormal series $\{1\} = G_0 \leq G_1 \leq \cdots \leq G_n = G$ in which every factor G_i/G_{i-1} either is cyclic or is a quasi-cyclic group Z_{p^∞} for some prime number p. The number of infinite factors in such a series is an invariant $m(G)$, the minimax length. As preparation for our final example we require:

LEMMA 8.1. *If G is a soluble minimax group then*
(i) $\mathrm{ht}[G] \leq m(G)$,
and
(ii) $[G]$ *consists of only countably many isomorphism classes of groups.*

Proof. We use induction on $m(G)$ to prove (i). Let H be a group that is strictly smaller than G in Pride's sense. There exist G_0, G_1, H_0 and H_1 as in $(*)$ and an isomorphism $G_0/G_1 \to H_0/H_1$, and G_1 must be infinite. Therefore $m(G_0/G_1) < m(G)$ and, by inductive hypothesis, $\mathrm{ht}[G_0/G_1] \leq m(G_0/G_1)$. Consequently

$$\mathrm{ht}[H] = \mathrm{ht}[G_0/G_1] \leq m(G) - 1,$$

and so

$$\mathrm{ht}[G] = 1 + \sup\{\mathrm{ht}[H] | H \prec G\} \leq m(G),$$

as required.

To prove (ii) we first show that if $G_1 \trianglelefteq G_0 \leq G$, $|G : G_0|$ is finite and G_1 is infinite, then $G_0/G_1 \prec G$. Let $Y := G_0/G_1$ and suppose, if possible, that $Y \sim G$. Then there exist subgroups Y_0, Y_1 of Y with $|Y : Y_0|$ finite and $Y_1 \trianglelefteq Y$, and there exist subgroups X_0, X_1 of G with $|G : X_0|$ finite, $X_1 \trianglelefteq X_0$ and X_1 finite, such that $Y_0/Y_1 \cong X_0/X_1$. But

$$m(X_0/X_1) = m(G) = m(G_0) = m(Y) + m(G_1) > m(Y) \geq m(Y_0/Y_1).$$

This contradiction shows that $G_0/G_1 \prec G$.

Now if $H \sim G$ then there exist $G_0, G_1 \leq G$ and H_0, $H_1 \leq H$ as in $(*)$. We may suppose moreover that $H_0 \trianglelefteq H$. Since

$$G_0/G_1 \sim H_0/H_1 \sim H \sim G$$

we must have that G_1 is finite. Therefore there are only countably many possibilities for the group G_0/G_1, that is, for H_0/H_1 up to isomorphism. Using the Lyndon-Hochschild-Serre spectral sequence one may show that if X is a soluble minimax group and Y is a finite $\mathbf{Z}X$-module then all cohomology groups $H^n(X, Y)$ are finite (see [10]). From the finiteness of the second cohomology groups it follows easily that there are only countably many extensions of a finite group by a given soluble minimax group: thus there are only countably many possibilities for H_0. And it is easy to see that there are only countably many extensions of a given countable group by a finite group. Thus there are (up to isomorphism) only countably many possibilities for H.

Example F. A finitely generated group F such that $\mathrm{ht}[F] = 9$ and $\mathrm{Id}[F]$ has $2^{\aleph_0}$ members.

Construction. Let N be the group that is generated by elements u_n, v_n, w_n, y_n, z_n $(n \geq 0)$ subject to the relations (for all relevant m, n):

y_n, z_n are central;
$u_{n+1}^2 = u_n;\ v_{n+1}^2 = v_n;\ w_{n+1}^2 = w_n;\ y_{n+1}^2 = y_n;\ z_{n+1}^2 = z_n;$
$y_0 = z_0 = 1;$
$[v_m, w_n] = 1;\ [u_m, v_n] = y_{m+n};\ [u_m, w_n] = z_{m+n}.$

This group is nilpotent of class 2, its centre $Z(N)$ is isomorphic to $Z_{2^\infty} \times Z_{2^\infty}$ generated by the elements y_n, z_n, and $N/Z(N)$ is a direct product of three copies of $2^{-\infty}\mathbf{Z}$. There is an automorphism that fixes all y_n and z_n, and maps u_n to u_n^2, v_n to v_{n+1}, w_n to w_{n+1} for all n. We take F to be the semi-direct product of N with an infinite cyclic group inducing this automorphism: thus

$$F := \left\langle N, x \mid x u_n x^{-1} = u_{n+1}, x^{-1} v_n x = v_{n+1}, x^{-1} w_n x = w_{n+1} \right\rangle.$$

Clearly F is generated by $\{x, u_1, v_1, w_1\}$, so F is a finitely generated group. Also, F is a soluble minimax group built from four infinite cyclic groups and five copies of Z_{2^∞}, so $m(F) = 9$ and therefore, by Lemma 8.1(i), $\mathrm{ht}[F] \leq 9$. In fact it is quite easy to see that $\mathrm{ht}[F] = 9$. Now

$$Z(F) = \left\langle y_n, z_n\ (n \geq 0) \right\rangle \cong Z_{2^\infty} \times Z_{2^\infty}.$$

Thus $Z(F)$ has $2^{\aleph_0}$ subgroups (see [1]), that is, F has $2^{\aleph_0}$ normal subgroups. Since F is finitely generated there are only countably many homomorphisms of F to a given countable group and so F must have $2^{\aleph_0}$ non-isomorphic quotient groups. From Lemma 8.1(ii) it follows that these must fall into $2^{\aleph_0}$ equivalence classes, and so $\mathrm{Id}[F]$ has $2^{\aleph_0}$ members, as claimed.

9. Finitely generated atomic groups

The constructions that I have described in this paper mostly seem to have slightly negative consequences for Pride's theory. Therefore it is a pleasure to report some small positive results.

LEMMA 9.1. *If G satisfies* max-N *and $H \preccurlyeq G$ then H satisfies* max-N.

Proof. Given that $H \preccurlyeq G$ there exist subgroups G_0, G_1 of G and H_0, H_1 of H as in $(*)$. By a theorem of John S. Wilson [11], G_0 satisfies max-N. Then G_0/G_1 and therefore also H_0/H_1 satisfies max-N. Since H_1 is finite H_0 satisfies max-N and now by Wilson's theorem again H satisfies max-N.

THEOREM 9.2. *A finitely generated atomic group satisfies* max-N.

Proof. Let G be a finitely generated atomic group. Since G is finitely generated and infinite it has a just-infinite quotient group H. Then $H \preccurlyeq G$ and since G is atomic $H \sim G$, whence $G \preccurlyeq H$. It follows from Lemma 9.1 with the roles of G and H reversed that G satisfies max-N, as required.

There is a slightly more general version of this theorem.

THEOREM 9.3. *Let G be a finitely generated group of height n. If there are n inequivalent atomic groups $H_1, \ldots, H_n$ such that $H_i \preccurlyeq G$ for all i then G satisfies* max-N.

Proof. By Theorem 2 of [2], $H_1 \times \cdots \times H_n \preccurlyeq G$, by Theorem 1(ii) of [2], $\mathrm{ht}[H_1 \times \cdots \times H_n] = n$, and so $G \sim H_1 \times \cdots \times H_n$. It follows that this direct product is finitely generated, so each group H_i is finitely generated and, by Theorem 9.2, satisfies max-N. Then $H_1 \times \cdots \times H_n$ satisfies max-N and so G satisfies max-N.

REFERENCES

1. GERHARD BEHRENDT and PETER M. NEUMANN, *On the number of normal subgroups of an infinite group*, J. London Math. Soc. (2), vol. 23 (1981), pp. 429–432.
2. M. EDJVET and STEPHEN J. PRIDE, 'The concept of "largeness" in group theory II' in *Groups–Korea* 1983 (Proceedings edited by A.C. Kim and B.H. Neumann), Lecture Notes in Mathematics, Vol. 1098, Springer-Verlag 1985, pp. 29–54.
3. P. HALL, *Wreath powers and characteristically simple groups*, Proc. Cambridge Philos. Soc., vol. 58 (1962), pp. 170–184.
4. ______, *On the embedding of a group in a join of given groups*, J. Australian Math. Soc., vol. 17 (1974), pp. 434–495.
5. BERNARD M. HURLEY, 'Small cancellation theory over groups equipped with an integer-valued length function' in *Word problems, II: the Oxford book* (Proceedings edited by S.I. Adjan, W.W. Boone and G. Higman), North Holland 1980, pp. 157–214.
6. B.H. NEUMANN and HANNA NEUMANN, *Embedding theorems for groups*, J. London Math. Soc., vol. 34 (1959), pp. 465–479.
7. PETER M. NEUMANN, *On the structure of standard wreath products of groups*, Math. Zeitschrift, vol. 84 (1964), 343–373.

8. ______, *The* SQ-*universality of some finitely presented groups*, J. Australian Math. Soc., vol. 16 (1973), pp. 1–6.
9. STEPHEN J. PRIDE, 'The concept of "largeness" in group theory' in *Word problems, II: the Oxford book* (Proceedings edited by S.I. Adjan, W.W. Boone and G. Higman), North Holland 1980, pp. 299–335.
10. DEREK J.S. ROBINSON, *On the cohomology of soluble groups of finite rank*, J. Pure Appl. Algebra, vol. 6 (1975), pp. 155–164.
11. JOHN S. WILSON, *Some properties of groups inherited by normal subgroups of finite index*, Math. Zeitschrift, vol. 114 (1970), pp. 19–21.

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