ILLINOIS JOURNAL OF MATHEMATICS
Volume 30, Number 2, Summer 1986

RECURSIVE FUNCTIONS IN GROUP THEORY

BY

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In Memoriam W.W. Boone

1. Introduction

In this paper we develop representations of partial recursive functions and recursively enumerable (r.e.) sets by equations in finitely presented groups. These representations are an attempt to provide an algebraic method for studying concepts from recursive function theory. Using our representations we are able to obtain a new characterization of finitely generated (f.g.) groups with solvable word problems.

This work is an outgrowth of a result by Boone and Higman [1].

THEOREM 1. (Boone-Higman) *Let G be any finite presentation of a group and S be any set of words of G that is closed under equality in G. Then S is recursively enumerable if and only if there exists a finitely presented group G' in which G is embedded and having an expression $\Phi(X)$ such that for each word Σ of G, $\Phi(\Sigma)$ holds if and only if $\Sigma \in S$.*

In [1] a combinatorial proof of Theorem 1 was outlined by giving a proof for the important case where the set S was a semigroup. Boone and Higman also developed an algebraic proof based on the Higman embedding theorem. This proof is a subargument of our argument and will be pointed out along the way. It should be noted that the proof in [1] does not use the Higman embedding theorem, so there is a possibility of getting a new proof of the Higman embedding theorem from [1].

2. Terminology and statements of the main results

We envision ourselves as doing recursive function theory on elements of groups, and we identify elements with the equivalence classes of words that represent them. When it is necessary to distinguish between elements and

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Received April 15, 1985.

[1] The results of this paper were part of the author's doctoral thesis [4] under the direction of Professor W.W. Boone at the University of Illinois. The work could not have been done without his advice and encouragement.

words, we use $[w]$ to denote the element that is represented by the word w. A set S of elements of a group is *recursively enumerable* (*r.e.*) if the underlying set $\{w: [w] \in S\}$ is r.e.

Our representations are in terms of *parametric equations*, where a parametric equation of n variables in a group G is an element of the free product of G with a free group F of rank n. The generators of the free group F are regarded as variables ranging over G. Our Theorem 2 below differs from the Boone-Higman theorem in that the restriction on the domain of the variable in the parametric equation is removed.

THEOREM 2. *Let G be a finitely generated, recursively presented group, and let S be a set of elements of the group. Then S is recursively enumerable if and only if there exist a finitely presented group H into which G is embedded and a parametric equation $f(x) = 1$ in H such that for each element h of H, $f(h) = 1$ if and only if h is in S.*

DEFINITION. Let G and H be groups with finite, fixed sets of generators. A partial function φ from G to H is *partial recursive* if there is a partial recursive function φ' from the words of G to the words of H such that for each word w of G,

(i) whenever $\varphi'(w)$ is defined, $\varphi([w])$ is defined and equals $[\varphi'(w)]$, and
(ii) whenever $\varphi([w])$ is defined, there exists a word u for $[w]$ such that $\varphi'(u)$ is defined.

The partial function φ' is called a *skeleton for* φ.

In the above definition, (i) assures that the skeleton is consistent with φ and is not defined for any extra arguments. Requirement (ii) assures that the skeleton is large enough to support φ. Note also that this paper deals only with finitely generated groups, so the definition always makes sense.

THEOREM 3. *Let G be a finitely generated, recursively presented group and φ a partial function from G to G. Then φ is partial recursive if and only if there exist a finitely presented group H into which G is embedded and a parametric expression $f(x)$ in H such that for each element h in H,*

(i) *$f(h) = \varphi(h)$, if $\varphi(h)$ is defined, and*
(ii) *$f(h) \notin G$, if $\varphi(h)$ is undefined.*

DEFINITION. A parametric expression $f(x)$ satisfying (i) and (ii) of the above theorem is said to *represent* φ.

The simultaneous representation of all partial recursive functions from a group to itself would be the analog in our group theoretic setting of the enumeration theorem of recursive function theory [5]. The enumeration theorem is the theorem that asserts the existence of a universal partial function.

The last theorem in this paper asserts that the enumeration theorem holds for a finitely generated group if and only if the group has a solvable word problem.

THEOREM 4. *Let G be a finitely generated group. Then G has a solvable word problem if and only if there exist a finitely presented group H into which G is embedded and a parametric expression $p(z, y)$ of H such that for each partial recursive function α from G to G there is an element of h of H for which $p(h, y)$ represents $\alpha(y)$ in H. Furthermore, if G is infinite and has a solvable word problem, the element h can be taken to lie in G itself.*

3. The proof of Theorem 2 and the basic construction

We prove the theorems in an order that avoids introducing all the details in one proof. The proof of Theorem 2, which follows, illustrates the basic algebraic construction for all three theorems in its most simple form.

Proof of Theorem 2. In one direction the proof of Theorem 2 is trivial. Given a finite presentation for H, an effective enumeration of the words equal to the identity exists and can be used to give an effective enumeration of the solution set of $f(x) = 1$.

To demonstrate Theorem 2 in the other direction, we give an explicit construction of the group H and the parametric expression $f(x)$. Let G and S be as in the statement of the theorem. Let G_1 be $G * \langle s \rangle$, the free product of G with the infinite cyclic group $\langle s \rangle$. Let G_2 be the group given by the presentation

$$\langle G_1, r\colon r^{-1}g^{-1}sgr = g^{-1}sg \text{ for } g \in S \rangle.$$

Here we are using the notation

$$G = \langle q_1, \dots, q_j\colon r_1, \dots, r_k \rangle$$

to show that G is given by a presentation with generators q_1 through q_j and relators r_1 through r_k. We use the shorthand notation $\langle G, h\colon s \rangle$ to abbreviate the addition of an additional generator h and relator s to the presentation of G.

Let F denote the set of elements $\{ g^{-1}sg\colon g \in S \}$ in G_1. Note that the set F freely generates a free subgroup of G_1. The group G_2 is the HNN extension of G_1 with respect to the identity isomorphism on this free subgroup. Since S is recursively enumerable and G_1 is finitely generated and recursively presented, the group G_2 is finitely generated and recursively presented. By the Higman embedding theorem [2], G_2 can be embedded in a finitely presented group G_3.

The group G_3 suffices for the role of the group G' in the original Boone-Higman result, Theorem 1. The construction so far, together with the next lemma, complete the algebraic proof by Boone and Higman of their result. The equation

$$r^{-1}x^{-1}sxrx^{-1}s^{-1}x = 1$$

is the required expression $\Phi(x)$ for this theorem.

LEMMA 3.1. *For each x in G, $r^{-1}x^{-1}sxr = x^{-1}sx$ in G_3 if and only if x is in S.*

Proof. Suppose that x is a word of G satisfying

$$r^{-1}x^{-1}sxr = x^{-1}sx \quad \text{in } G_3.$$

Since x, r, and s are all in G_2, the equation

$$r^{-1}x^{-1}sxrx^{-1}s^{-1}x = 1$$

holds in G_2. Britton's lemma then shows that there must be an r-pinch between r^{-1} and r. Thus $x^{-1}sx$ must be in the free subgroup of G_1 generated by F. Since this subgroup is free and x is s-free, we conclude that $x^{-1}sx$ must be a generator of the free subgroup. Thus x is a word for an element of S.

On the other hand, if x is in S, that $r^{-1}x^{-1}sxr$ equals $x^{-1}sx$ is an immediate consequence of the defining relations.

To continue with the proof of Theorem 2, let G_4 be given by

$$\langle G_3, p\colon\ p^{-1}yp = y \textit{ for each generator } y \textit{ of } G\rangle.$$

Let H be the free product $G_4 * \langle t\rangle$. The group G_4 is an HNN extension of G_3, and it is finitely presented since the distinguished subgroup G is finitely generated. The group H is then also finitely presented.

We diagram the groups defined so far, together with some groups yet to be defined that are to play a role later in the proof. The reader's attention is drawn to the fact that the same diagram is valid for future proofs and will be cited again.

$$\begin{tikzcd}
& G \arrow[dl] \arrow[dr] & & & \\
I \arrow[dr] & & G_1 = G * \langle s\rangle \arrow[dl] \arrow[dr] & & \\
& J \arrow[dr] & & G_2 = \langle G_1, r \mid r^{-1}x^{-1}sxr = x^{-1}sx, \text{ for } x \in S\rangle \arrow[dl] \arrow[dr] & \\
& & K \arrow[dr] & & G_3 \arrow[dl] \\
& & & G_4 = \langle G_3, p \mid p^{-1}yp = y, \text{ for each generator } y \text{ of } G\rangle \arrow[d] & \\
& & & H = G_4 * \langle t\rangle &
\end{tikzcd}$$

The group H is our desired finitely presented group. The desired parametric equation is

$$r^{-1}x^{-1}sxrtp^{-1}x^{-1}pxt^{-1}x^{-1}s^{-1}x = 1. \tag{1}$$

LEMMA 3.2. *Let w be a word of H that satisfies*

$$r^{-1}w^{-1}swrtp^{-1}w^{-1}pwt^{-1}w^{-1}s^{-1}w = 1$$

in H. Then there exists a word v of G_4 such that

(*i*) $w = v$ *holds in* H,

(*ii*) $p^{-1}vp = v$ *holds in* G_4, *and*

(*iii*) $r^{-1}v^{-1}svr = v^{-1}sv$ *holds in* G_4.

Proof. It is more convenient to deal with the equivalent form

$$r^{-1}w^{-1}swrtp^{-1}w^{-1}p = w^{-1}swtw^{-1} \tag{2}$$

of the equation.

First it must be shown that (2) cannot hold for w unless w satisfies (i) for some t-free word v. Suppose that w is not equal to any t-free word. Then w is equal in H to a t-reduced word guh, where g and h are words of G_4 and u begins and ends with (possibly the same) occurrences of t-symbols. We substitute guh for w in (2) to obtain

$$r^{-1}h^{-1}u^{-1}\underbrace{g^{-1}sg}_{A}u\underbrace{hr}_{B}t\underbrace{p^{-1}h^{-1}}_{C}u^{-1}g^{-1}p = h^{-1}u^{-1}\underbrace{g^{-1}sg}_{D}u\underbrace{h}_{E}t\underbrace{h^{-1}}_{F}u^{-1}g^{-1}. \tag{3}$$

Now we analyze (3) in light of the normal form theorem for free products. Observe that if no t-cancellations occur in (3), then (3) cannot hold in H. This is because if no t-cancellations occur, the left and right sides of (3) are in free product normal form, but they begin with the unequal factors $r^{-1}h^{-1}$ and h^{-1}.

Next observe that because u is in t-reduced form and begins and ends with t-symbols, no t-cancellations occur inside u and, indeed, the possible t-cancellations are across the subwords identified in (3). Since $g^{-1}sg \neq 1$, there can be no t-cancellation across the subwords labeled A and D. This leaves the possibilities B, C, E, and F.

Now observe that in (3) as written, the right and left sides are parallel in t, i.e., they have the same sequence of occurrences of t and t^{-1}. If (3) holds, then, by the normal form theorem for free products, a t-cancellation on one side of the equation must be balanced by one on the other side. Suppose now that there is a t-cancellation at either of the remaining right-hand side locations E

or F. Then $h = 1$. However, $h = 1$ implies $hr = r \neq 1$ and $p^{-1}h^{-1} = p^{-1} \neq 1$. Thus there can be no t-cancellation on the left-hand side (locations B and C) that occurs simultaneously with one on the right-hand side.

The possibilities for t-cancellations in (3) are exhausted, and the assumption that w does not have a t-free form and yet satisfies (2) is contradicted. Thus there is a t-free word v that equals w in H and satisfies (2).

We substitute v for w in (2) to obtain

$$r^{-1}v^{-1}svrtp^{-1}v^{-1}p = v^{-1}svtv^{-1},$$

which has only the displayed occurrences of t-symbols. Thus by the normal form theorem for free products, the corresponding pairs of factors from G_4 are equal, that is,

$$p^{-1}v^{-1}p = v^{-1} \quad \text{and} \quad r^{-1}v^{-1}svr = v^{-1}sv,$$

and the proof of the lemma is complete.

Continuing with the proof of Theorem 2, we assume that w is a word satisfying (2). From the last lemma we know w is equal to a t-free word v. From (ii) of that lemma, $pv = vp$. Thus we know that v represents an element of the subgroup of G_4 generated by G together with p. Call this subgroup I, and assume now that v is written on the generators of I. Let J be the subgroup of G_4 generated by G together with p and s. Let K be the subgroup of G_4 generated by G together with p, r, and s. Considered as a subgroup of the HNN extension G_4 of G_3, we see that K has the presentation

$$\langle G, r, s, p\colon r^{-1}x^{-1}sxr = x^{-1}sx, \text{if } x \in S, \qquad p^{-1}yp = y, \text{for each generator } y \text{ of } G\rangle.$$

Thus K is an HNN extension of J with stable letter r. To conclude finally that (2) implies w is in S, we require only the following result.

LEMMA 3.3. *For w in I, the equation $r^{-1}w^{-1}swr = w^{-1}sw$ holds in G_4 if and only if w is in S.*

The statement and the proof of Lemma 3.3 are those of Lemma 3.1 with w, I, J, K, and G_4 replacing x, G, G_1, G_2, and G_3, respectively.

To finish the proof of Theorem 2, observe that, if w is in S, then (2) holds for w as a direct consequence of the relations of the groups constructed.

4. How to represent partial recursive functions

In this section we give the proof of Theorem 3 and show how to represent partial recursive functions in a group.

Proof of Theorem 3. Suppose first that the group H and the parametric expression $f(x)$ satisfying (i) and (ii) exist. A partial recursive skeleton for φ is defined as follows. There exist effective procedures for enumerating all the words in G and for enumerating all pairs of words equal in H. The words on the generators of G alone can be effectively recognized. In one list the words $f(w_i)$, where $\{w_i\colon i = 0, 1, \dots\}$ is an effective listing of words of G, are enumerated. In a second list the pairs of words equal in H are enumerated. After each $f(w_i)$ in the first list is enumerated, the portion of the second list that already exists is checked for pairs consisting of $f(w_i)$ and a word u on the generators of G alone. If such a pair is found, $f(w_i)$ is crossed off and the pair (w_i, u) is added to φ'. Similarly when a pair is added to the second list, the portion of the first list that already exists is checked, and words are crossed off and pairs added to φ' as appropriate. Thus the graph of a skeleton φ' for φ is recursively enumerated, and φ is partial recursive.

To go the other way, we assume that φ is partial recursive and that φ' is a partial recursive skeleton for φ. The construction of H is similar to the construction used in Theorem 2. Let $G_1 = G * \langle s \rangle$ as before. Let G_2 be defined by

$$\langle G_1, r\colon r^{-1}w^{-1}s^2wr = w^{-1}s\varphi'(w)sw \quad \text{for each word } w \text{ for which } \varphi'(w) \text{ is defined}\rangle.$$

Let A and B be the sets

$$\{w^{-1}s^2w\colon \varphi'(w) \text{ is defined}\}$$

and

$$\{w^{-1}s\varphi'(w)sw\colon \varphi'(w) \text{ is defined}\}.$$

These are sets of words. Let $[A]$ and $[B]$ be the sets of elements represented by words in A and B, respectively. Each of the sets $[A]$ and $[B]$ freely generates a free subgroup of G_1. The map sending $g^{-1}s^2g$ to $g^{-1}s\varphi(g)sg$ is a bijection between $[A]$ and $[B]$ and induces an isomorphism between the free subgroups. Thus G_2 is an HNN extension of G_1. Since φ is partial recursive, we can effectively enumerate the set of relations of G_2 that equates words in A to words in B. Thus, since G_1 is finitely generated and recursively presented, so is G_2. Therefore G_2 can be embedded in a finitely presented group G_3 by the Higman embedding theorem. The groups G_4 and H are constructed from G_3 as in the proof of Theorem 2.

The parametric expression $f(x)$ required for the theorem is

$$s^{-1}xr^{-1}x^{-1}s^2xrtpx^{-1}p^{-1}xt^{-1}x^{-1}s^{-1}.$$

We need a lemma similar to Lemma 3.2 in the proof of Theorem 2.

LEMMA 4.1. *Let q be a word of G, and let w be a word of H satisfying $q = f(w)$. Then there is a word v of G_4 such that*

(*i*) $v = w$ *in* H,

(*ii*) $p^{-1}vp = v$ *in* G_4, *and*

(*iii*) $r^{-1}v^{-1}s^2vr = v^{-1}sqsv$ *in* G_4.

Proof. Assume without loss of generality that w is t-reduced. First of all it must be shown that w has no occurrences of t. Suppose that w has an occurrence of t. Then w is guh, where g and h are words of G_4 and u begins and ends with (possibly the same) occurrences of t-symbols. Writing out $f(guh) = q$ and conjugating, we get

$$r^{-1}h^{-1}\underbrace{u^{-1}g^{-1}s^2gu}_{A}\underbrace{hr}_{B}\underbrace{tph^{-1}u^{-1}}_{C}g^{-1} = h^{-1}\underbrace{u^{-1}g^{-1}sqsgu}_{D}\underbrace{ht}_{E}\underbrace{h^{-1}u^{-1}}_{F}g^{-1}p.$$

This equation is now analyzed via the normal form theorem for free products, as in lemma 3.2, to show that the assumption that w has a t-reduced form containing t must be false. Thus there is a v in G_4 with $v = w$ in H.

Substituting v for w in $f(w) = q$ and conjugating, we get

$$r^{-1}v^{-1}s^2vrtpv^{-1} = v^{-1}sqsvtv^{-1}p,$$

which has only the displayed occurrences of t. Equations (ii) and (iii) of the lemma statement then follow from the normal form theorem for free products.

To continue with the proof of Theorem 3, assume that $f(h)$ is in G for some element h of H, that is, $f(w) = q$ for some word w representing h in H and for some word q of G. By arguments parallel to those in the proof of Theorem 2 we find that w is equal to a word v of G for which $v^{-1}s^2v$ is in A. Thus $\varphi'(v)$ is defined and $\varphi(h) = \varphi([w]) = \varphi([v]) = \varphi'(v)$. So (ii) of our theorem holds for f.

That (i) of the theorem holds is a direct consequence of the defining relations.

5. Characterizing f.g. groups with solvable word problem

Recall our last theorem, Theorem 4, asserts that the enumeration theorem holds for a finitely generated group if and only if the group has a solvable word problem. Essentially all of the algebra has been done in the proofs of the previous theorems. Thus the proof that follows involves arguments that have the flavor of recursive function theory.

Proof of Theorem 4. Assume that G has solvable word problem, and let φ be a universal partial function, which for simplicity of notation we consider as going from pairs consisting of a natural number and a word of G to the words of G. Thus $\varphi(i, w) = \varphi_i(w)$, where $\{\varphi_i\colon i = 0, 1, \dots\}$ is an enumeration of the partial recursive functions from the words of G to the words of G. The existence of such a function is a consequence of the enumeration theorem of ordinary recursive function theory.

We define a partial function from the ordered pairs consisting of a natural number and a word of G to the words of G as follows. The partial function φ considered as the set

$$\{(i, v, w)\colon \varphi(i, v) = w\}$$

is r.e. The new partial function ψ is to be a subset of φ such that for each natural number j, $\psi(j, v)$ is a partial function of v that is compatible with equality in G. We construct this new partial function ψ by sorting out extra triples from φ. Enumerate φ. At each stage of the enumeration, add the newly enumerated triple (i, v, w) to ψ, unless there is a triple (i, v', w') already in ψ, for which $v = v'$ but $w \neq w'$. This procedure is effective, since we have an algorithm for the word problem of G. Thus ψ is partial recursive.

Let G' be $G * \langle n \rangle$. Since the solvability of the word problem implies that G has a recursive presentation, G' is recursively presented. If G is infinite, let $S = \{g_i\colon i \in N\}$ be a set of words each of which represents a distinct element of G. Since G has solvable word problem, such a set can be enumerated effectively. It can also be assured in the enumeration of S that when i is less than j, the Godel number of g_i is less than the Godel number of g_j. Thus S may be taken to be recursive. If G is finite, let g_i be n^i. Again $S = \{g_i\colon i \in N\}$ is recursive.

The word $g_i n v$ encodes the pair $\{g_i, v\}$. Define $\psi'(g_i n v)$ to be $\psi(i, v)$, and let ψ' be undefined for other arguments. Apply the construction of Theorem 3, with minor modifications, to ψ' considered as the skeleton of a partial recursive function from G' to G'. The construction is the same up to the construction of G_4, which must be defined by the presentation

$$\langle G_3, p\colon p^{-1} y p = y \text{ if } y \text{ is a generator of } G \rangle,$$

rather than by the presentation

$$\langle G_3, p\colon p^{-1} y p = y \text{ if } y \text{ is a generator of } G' \rangle,$$

which would be the case if the construction were applied exactly as in the proof of Theorem 3. The group H is then $G_4 * \langle t \rangle$ as before.

The parametric expression $p(z, y)$ is then

$$s^{-1}(zny) r^{-1}(zny)^{-1} s^2 (zny) r t p (y^{-1}) p^{-1}(y) t^{-1}(zny)^{-1} s^{-1}.$$

Here zny has been substituted for x in the part of the expression $f(x)$ in the proof of Theorem 3 actually used to define values, and y has been substituted for x in the part where the "domain" is limited to G. The latter substitution assures that y is limited to G, the original group, and not the group G'.

Given some particular partial recursive function α from G to G, let α' be a partial recursive skeleton for α. Then for some i in N, we have $\alpha'(y) = \varphi(i, y)$ as partial functions of y. Since α' is a skeleton, $\varphi(i, y) = \psi(i, y)$. Thus from the relations set forth in the construction, it is clear that $p(g_i, y) = \alpha(y)$ when $\alpha(y)$ is defined.

Consider the parametric equation $q = p(g_i, y)$, which is

$$q = s^{-1}g_i nyr^{-1}y^{-1}n^{-1}g^1 s^2 g_i nyrtpy^{-1}p^{-1}yt^{-1}y^{-1}n^{-1}g_i^{-1}s^{-1}.$$

It is of the same form as the parametric equation in Lemma 4.1 except that $sg_i n$ or $n^{-1}g_i^{-1}s$ appears instead of s. However, the s-weight remains 2 in the subwords

$$n^{-1}g_i^{-1}s^2 g_i n \quad \text{and} \quad n^{-1}g_i^{-1}sqsg_i n$$

(which appears after conjugation) and this is all that is required in the proof of Lemma 4.1. Thus Lemma 4.1 can be used to show that if w is a word of H, then $q = p(g_i, w)$ for some word q of G implies that there is a word v of G_4 satisfying

(i) $v = w$ in H,
(ii) $p^{-1}vp = v$ in G_4, and
(iii) $r^{-1}v^{-1}n^{-1}g_i^{-1}s^2 g_i nvr = v^{-1}n^{-1}g_i^{-1}sqsg_i nv$ in G_4.

Proceeding as in the proof of Theorem 3 and using the fact that G_4 has relations $p^{-1}yp = y$ for each generator y of G (not G'), we conclude that $p(g_i, y)$ is in G only if $\alpha(y)$ is defined. Thus $p(g_i, y)$ represents $\alpha(y)$.

Now we prove the theorem going in the other direction. Assume the existence of a finitely presented group H in which G is embedded and a parametric expression $p(z, y)$ that is universal for the one variable partial recursive functions from G to G in the manner set forth in the statement of the theorem. Without loss of generality assume that G is non-trivial. Let z be a fixed word of G that is not equal to the identity.

Since H is finitely presented, the set of words of H that are equal to the identity and the set of words of G that are equal to the identity are each r.e. The parametric expression $p(x, y)$ will be used to show that the set of words that are not equal to the identity is also r.e. We call this last set NE.

Let $\#w$ be the Godel number of a word w of H. Let $\langle i, j, k\rangle$ be the image of (i, j, k) under a recursive bijection from N^3 onto N. We enumerate the set NE in stages.

At stage $\langle \#w, \#v, k\rangle$ we do k steps in the enumeration of the sets

$$A_v = \{u\colon u \text{ is a word of } G \text{ and } u = p(v, 1)\}$$

and

$$B_{v,w} = \{u\colon u \text{ is a word of } G \text{ and } u = p(v, w)\}$$

If, at this stage, 1 appears in A_v and z appears in $B_{v,w}$, we put w in our listing of NE.

It remains to show that all words in NE are enumerated. Suppose that $w \neq 1$ for a word w of G. Then

$$\beta'(x) = \begin{cases} 1 & \text{if } x \text{ is } 1, \\ z & \text{if } x \text{ is } w, \\ \text{undefined} & \text{otherwise,} \end{cases}$$

is a skeleton of a partial recursive function β from G to G. This partial recursive function must be represented by $p(h, y)$ for some h in H. Thus z eventually appears in $B_{h,z}$ and 1 eventually appears in A_h. Therefore w is listed as a member of NE by our listing.

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