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DIAGONAL EMBEDDINGS OF NILPOTENT GROUPS

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Dedicated to the memory of William Boone

1. Introduction

Among various embeddings of a group G into $G \times G \times G$ are the embeddings

$$\phi_1: g \to (g, g, 1) \quad \text{and} \quad \phi_2: g \to (1, g, g)$$

which yield a weak form of permutability between the isomorphic groups G^{ϕ_1} and G^{ϕ_2}, namely, $g^{\phi_1}g^{\phi_2} = g^{\phi_2}g^{\phi_1}$ for all $g \in G$. This natural situation leads to the study of the double group

$$\mathbf{D}(G) = \langle G^{\phi_1}, G^{\phi_2};\ g^{\phi_1}g^{\phi_2} = g^{\phi_2}g^{\phi_1} \quad \text{for all } g \in G \rangle$$

as the quotient group of the free product $G^{\phi_1} * G^{\phi_2}$ by the commutator relations $[g^{\phi_1}, g^{\phi_2}] = 1$ for all $g \in G$. When G is finite, $\mathbf{D}(G)$ is finite (Sidki [4]), and when G is a finite p-group of order p^k, p odd, $\mathbf{D}(G)$ is of order dividing $p^{2k}p^{k(k-1)/2}$ (Rocco [3]). In this paper we develop commutator calculus for the double group $\mathbf{D}(G)$ and obtain a detailed description of its lower central series $\gamma_i(\mathbf{D}(G))$, $i \geq 1$, in terms of the lower central series of G. We prove that *if G is an m-generator nilpotent group of class at most c with $m \geq 2$, $c \geq 1$, then $\mathbf{D}(G)$ is nilpotent of class at most* $\max\{m, c+2\}$. *Furthermore, if $m \geq c+3$ then $\gamma_{c+3}(\mathbf{D}(G))$ is an elementary abelian 2-group of rank at most*

$$\sum_{k=c+3}^{m} \binom{m}{k}$$

(Theorems 3.2 and 3.3).

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2. Preliminaries

We use standard commutator notation (see, for instance, [2]). For elements x, y, x_i, y_i in a group G,

$$[x, y] = x^{-1}y^{-1}xy = x^{-1}x^{y};$$
$$[x_1, \dots, x_{n+1}] = [[x_1, \dots, x_n], x_{n+1}];$$
$$[x, ny] = [x, y_1, \dots, y_n] \quad \text{with } y_1 = \cdots = y_n = y;$$
$$[x_1, \dots, x_m;\ y_1, \dots, y_n] = [[x_1, \dots, x_m], [y_1, \dots, y_n]]$$

and so on. If $G_1, \dots, G_n$ are subgroups of G, then $[G_1, \dots, G_n]$ is the subgroup of G generated by all commutators $[g_1, \dots, g_n]$, $g_i \in G_i$. In particular, $\gamma_n(G) = [G_1, \dots, G_n]$ with $G_1 = \cdots = G_n = G$, is the n-th term of the lower central series of G.

For elements x, y, z in G, the following commutator identities are standard and will be used without reference:

$$[x, y = [x, y^{-1}]^{-y} = [x^{-1}, y]^{-x};$$
$$[x, yz] = [x, z][x, y]^{z} = [x, z][x, y][x, y, z];$$
$$[xy, z] = [x, z]^{y}[y, z] = [x, z][x, z, y][y, z];$$
$$[x, y^{-1}, z]^{y}[y, z^{-1}, x]^{z}[z, x^{-1}, y]^{x} = 1;$$

or equivalently

$$[z, [x, y]] = [z, y^{-1}, x^{z}]^{y}[z, x^{-1}, y^{-1}]^{xy} \quad \text{(Witt identity)}$$
$$[x, y, z][y, z, x][z, x, y] \equiv 1 \bmod \gamma_2(\gamma_2\langle x, y, z\rangle) \quad \text{(Jacobi Congruence)}$$

We simplify our notation by redefining the double group $\mathbf{D}(G)$ of G as

$$\mathbf{D} = \mathbf{D}(G) = \langle G, G^{\phi};\ [g, g^{\phi}] = 1 \quad \text{for all } g \in G\rangle,$$

where $\phi\colon G \to G^{\phi}$ is an isomorphism (note that in Sidki [4] and Rocco [3] the notation for $\mathbf{D}(G)$ is $\chi(G)$). In the following lemmas we derive some fundamental relations which hold in the group $\mathbf{D}(G)$.

LEMMA 2.1. *For all $x, y, z, y_i, z_i \in G$ we have*:
(i) $[x^{\phi}, y] = [x, y^{\phi}]$;
(ii) $[x^{\phi}, y]^{z^{\phi}} = [x^{\phi}, y]^{z}$; *and more generally*,
(iii) $[x^{\phi}, y]^{\omega(z_1^{\varepsilon_1}, \dots, z_n^{\varepsilon_n})} = [x^{\phi}, y]^{\omega(z_1, \dots, z_n)}$ *for* $\varepsilon_i \in \{1, \phi\}$ *and* $\omega = \omega(z_1, \dots, z_n) \in G$;
(iv) $[x^{\phi}, y, x] = [x, y, x^{\phi}]$; *and more generally*,
(v) $[x^{\phi}, y_1, \dots, y_n, x] = [x, y_1, \dots, y_n, x^{\phi}]$.

Proof of (i). We use the commuting relations $(xy^{-1})(xy^{-1})^{\phi} = (xy^{-1})^{\phi}(xy^{-1})$, $xx^{\phi} = x^{\phi}x$, $yy^{\phi} = y^{\phi}y$ to obtain, in turn

$$xy^{-1}x^{\phi}y^{-\phi} = x^{\phi}y^{-\phi}xy^{-1}; \qquad x^{-\phi}xy^{-1}x^{\phi} = y^{-\phi}xy^{-1}y^{\phi};$$
$$xx^{-\phi}y^{-1}x^{\phi} = y^{-\phi}xy^{\phi}y^{-1}; \qquad x^{-\phi}y^{-1}x^{\phi}y = x^{-1}y^{-\phi}xy^{\phi};$$
$$[x^{\phi}, y] = [x, y^{\phi}].$$

Proof of (ii). We use (i) to write $[x^{\phi}, yz] = [x, y^{\phi}z^{\phi}]$ which, when expanded, yields, in turn

$$[x^{\phi}, z][x^{\phi}, y]^{z} = [x, z^{\phi}][x, y^{\phi}]^{z^{\phi}};$$
$$[x^{\phi}, y]^{z} = [x, y^{\phi}]^{z^{\phi}}; \qquad [x^{\phi}, y]^{z} = [x^{\phi}, y]^{z^{\phi}}.$$

Proof of (iii). Let $\omega(z_1^{\varepsilon_1}, z_2^{\varepsilon_2}, \ldots, z_n^{\varepsilon_n}) = g_1h_1^{\phi}g_2h_2^{\phi} \cdots g_mh_m^{\phi}$ so that

$$\omega(z_1, z_2, \ldots, z_n) = g_1h_1g_2h_2 \cdots g_mh_m.$$

We prove by induction on $m \geq 1$ that

$$[x^{\phi}, y]^{g_1h_1^{\phi} \cdots g_mh_m^{\phi}} = [x^{\phi}, y]^{g_1h_1 \cdots g_mh_m}.$$

For $m = 1$,

$$[x^{\phi}, y]^{g_1h_1^{\phi}} = [x^{\phi}, y]^{g_1^{\phi}h_1^{\phi}} \text{ (by (ii))} = [x^{\phi}, y]^{(g_1h_1)^{\phi}} = [x^{\phi}, y]^{g_1h_1} \text{ (by (ii))}.$$

For the inductive step, we assume $[x^{\phi}, y]^{g_1h_1^{\phi} \cdots g_mh_m^{\phi}} = [x^{\phi}, y]^{g_1h_1 \cdots g_mh_m}$. Then,

$$\begin{aligned}[x^{\phi}, y]^{g_1h_1^{\phi} \cdots g_mh_m^{\phi}g_{m+1}h_{m+1}^{\phi}} &= [x^{\phi}, y]^{g_1h_1 \cdots g_mh_mg_{m+1}h_{m+1}^{\phi}} \\ &= [x^{\phi}, y]^{(g_1h_1 \cdots g_mh_mg_{m+1})^{\phi}h_{m+1}^{\phi}} \\ &= [x^{\phi}, y]^{g_1h_1 \cdots g_{m+1}h_{m+1}} \text{ (by (ii))}.\end{aligned}$$

Proof of (iv). We use (iii) to write $1 = [y, x^{\phi}; y, x^{\phi}] = [y, x^{\phi}; y, x]$. Then, expansion of $[y, x^{\phi}x] = [y, xx^{\phi}]$ yields, in turn,

$$[y, x][y, x^{\phi}][y, x^{\phi}, x] = [y, x^{\phi}][y, x][y, x, x^{\phi}];$$
$$[y, x^{\phi}, x] = [y, x, x^{\phi}]; \qquad [x^{\phi}, y, x]^{-[y, x^{\phi}]} = [x, y, x^{\phi}]^{-[y, x]};$$
$$[x^{\phi}, y, x] = [x, y, x^{\phi}] \text{ (by (iii))}.$$

Proof of (v). By induction on $n \geq 1$. For $n = 1$ the result is given by (iv). We assume that $n \geq 2$ and that the result holds for $n - 1$. Thus,

$$[x^\phi, y_1, \dots, y_{n-2}, (y_{n-1}y_n), x] = [x, y_1, \dots, y_{n-2}, (y_{n-1}y_n), x^\phi]$$

which upon expansion yields,

$$\begin{aligned}&[[x^\phi, y_1, \dots, y_{n-2}, y_n][x^\phi, y_1, \dots, y_{n-2}, y_{n-1}][x^\phi, y_1, \dots, y_{n-2}, y_{n-1}, y_n], x]\\ &\quad = [[x, y_1, \dots, y_{n-2}, y_n][x, y_1, \dots, y_{n-2}, y_{n-1}]\\ &\quad \times [x, y_1, \dots, y_{n-2}, y_{n-1}, y_n], x^\phi].\end{aligned}$$

Therefore,

$$\begin{aligned}&[x^\phi, y_1, \dots, y_{n-2}, y_n, x]^{[x^\phi, y_1, \dots, y_{n-2}, y_{n-1}][x^\phi, y_1, \dots, y_{n-2}, y_{n-1}, y_n]}\\ &\quad \times [x^\phi, y_1, \dots, y_{n-2}, y_{n-1}, x]^{[x^\phi, y_1, \dots, y_{n-2}, y_{n-1}, y_n]}\\ &\quad \times [x^\phi, y_1, \dots, y_{n-2}, y_{n-1}, y_n, x]\\ &\quad = [x, y_1, \dots, y_{n-2}, y_n, x^\phi]^{[x, y_1, \dots, y_{n-2}, y_{n-1}][x, y_1, \dots, y_{n-2}, y_{n-1}, y_n]}\\ &\qquad \times [x, y_1, \dots, y_{n-2}, y_{n-1}, x^\phi]^{[x, y_1, \dots, y_{n-2}, y_{n-1}, y_n]}\\ &\qquad \times [x, y_1, \dots, y_{n-2}, y_{n-1}, y_n, x^\phi],\end{aligned}$$

which by the induction hypothesis, together with (iii) yields

$$[x^\phi, y_1, \dots, y_n, x] = [x, y_1, \dots, y_n, x^\phi]$$

as desired. This completes the proof of Lemma 2.1.

For subgroups H, K of a group G, we set $[H, 0k] = H$ and denote by $[H, nK]$ the subgroup

$$[H, K_1, \dots, K_n] \quad \text{with } K_i = K \ (1 \leq i \leq n).$$

In particular, $\gamma_{n+1}(G) = [G, nG]$. We now prove:

LEMMA 2.2. (i) $[G^\phi, G, G^{\varepsilon_1}, \dots, G^{\varepsilon_n}] = [G^\phi, (n+1)G]$ *for all* $n \geq 1$ *and all* $\varepsilon_i \in \{1, \phi\}$;

(ii) $[G^\phi, mG; \gamma_n(G)] \leq [G^\phi, (m+n)G]$ *for all* $m \geq 0$, $n \geq 1$;

(iii) $[G^{\varepsilon_1}, \dots, G^{\varepsilon_n}] \leq [G^\phi, (n-1)G]\gamma_n(G)^{\mathbf{D}}\gamma_n(G^\phi)^{\mathbf{D}}$, *where* $H^{\mathbf{D}}$ *denotes the normal closure of* H *in* $\mathbf{D} = \mathbf{D}(G)$.

Proof. The proof of (i) is an immediate consequence of Lemma 2.1 (iii).

The proof of (ii) is by induction on $n \geq 1$. For $n = 1$, there is nothing to prove. For the inductive step we assume $n \geq 2$ and that the result holds for $n-1$. With $x \in \gamma_{n-1}(G)$, $y \in G$, $z \in [G^{\phi}, mG]$, the equivalent form of the Witt identity $[z,[x, y]] = [z, y^{-1}, x^{z}]^{y}[z, x^{-1}, y^{-1}]^{xy}$, together with Lemma 2.1 (iii), yields

$$\begin{aligned}[G^{\phi}, mG, \gamma_n(G)] &\leq [G^{\phi}, (m+1)G, \gamma_{n-1}(G)]^{G}[G^{\phi}, mG, \gamma_{n-1}(G), G]^{G}\\ &\leq [G^{\phi}, (m+1)G, \gamma_{n-1}(G)][G^{\phi}, mG, \gamma_{n-1}(G), G]\\ &\leq [G^{\phi}, (m+n)G].\end{aligned}$$

For the proof of (iii) we may assume $n \geq 2$ and

$$(\varepsilon_1, \ldots, \varepsilon_n) \neq (1, \ldots, 1), (\phi, \ldots, \phi).$$

Then, without loss of generality,

$$(\varepsilon_1, \ldots, \varepsilon_n) = (\phi, \ldots, \phi, 1, \varepsilon_{i+2}, \ldots, \varepsilon_n) \quad \text{for some } 1 \leq i < n.$$

Thus

$$\begin{aligned}[G^{\varepsilon_1}, \ldots, G^{\varepsilon_n}] &= [\gamma_i(G)^{\phi}, G, G^{\varepsilon_{i+2}}, \ldots, G^{\varepsilon_n}]\\ &= [\gamma_i(G)^{\phi}, G, (n-i-1)G] \text{ (by Lemma 2.1 (iii))}\\ &= [\gamma_i(G), G^{\phi}, (n-i-1)G]\\ &= [G^{\phi}, \gamma_i(G), (n-i-1)G]\\ &\leq [G^{\phi}, (n-1)G] \text{ (by (ii))}.\end{aligned}$$

As a corollary of Lemma 2.2 we obtain:

LEMMA 2.3. *Let* $\gamma_{c+1}(G) = \{1\}$ *and* $\mathbf{D} = \mathbf{D}(G)$. *Then*
(i) $[\gamma_{c+1}(\mathbf{D}), \gamma_2(\mathbf{D})] = \{1\}$,
(ii) $[\gamma_i(\mathbf{D}), \gamma_2(\mathbf{D}), (2c-1-i)\mathbf{D}] = \{1\}$ *for all* $i \geq 2$.

Proof. For the proof of (i) we have, by Lemma 2.2 (iii), $\gamma_{c+1}(\mathbf{D}) = [G^{\phi}, cG]$. Thus,

$$\begin{aligned}[\gamma_{c+1}(\mathbf{D}), \gamma_2(\mathbf{D})] &= [G^{\phi}, cG; G^{\varepsilon_1}, G^{\varepsilon_2}] \quad (\varepsilon_1, \varepsilon_2 \in \{1, \phi\})\\ &= [[G^{\phi}, cG], [G, G]] \quad \text{(by Lemma 2.1 (iii))}\\ &= [[G^{\phi}, cG], [G^{\phi}, G]]\\ &= [[G^{\phi}, G], [G, cG] \quad \text{(by Lemma 2.1 (iii))}\\ &= \{1\}.\end{aligned}$$

For the proof of (ii) we make repeated application of the inclusion

$$[A, B, C] \leq [A, C, B][B, C, A]$$

for normal subgroups A, B, C of $\mathbf{D}$ to obtain, for $i \geq 2$,

$$[\gamma_i(\mathbf{D}), \gamma_2(\mathbf{D}), (2c-1-i)\mathbf{D}] \leq \prod_{\substack{m+n=2c+1\\ m,n\geq 2}} [\gamma_m(\mathbf{D}), \gamma_n(\mathbf{D})].$$

Since $m \geq c+1$ or $n \geq c+1$, the result follows by (i).

As in Levin [1], an immediate consequence of Lemma 2.3 yields:

LEMMA 2.4. *If* $\gamma_{c+1}(G) = \{1\}$, *then for all* $g_i \in G$ *and* $\varepsilon_i \in \{1, \phi\}$,

$$\left[g_1^{\varepsilon_1}, g_2^{\varepsilon_2}, g_3^{\varepsilon_3}, \dots, g_{2c+1}^{\varepsilon_{2c+1}}\right] = \left[g_1^{\varepsilon_1}, g_2^{\varepsilon_2}, g_{3\sigma}^{\varepsilon_{3\sigma}}, \dots, g_{(2c+1)\sigma}^{\varepsilon_{(2c+1)\sigma}}\right]$$

for all permutations σ *of* $\{3, \dots, 2c+1\}$.

An important consequence of Lemma 2.4 is the following Lemma on local nilpotency of $\mathbf{D}(G)$.

LEMMA 2.5. *If* G *is a locally nilpotent group then* $\mathbf{D}(G)$ *is also locally nilpotent.*

Proof. Let $\{h_1, \dots, h_n\}$ be a set of elements of $\mathbf{D} = \mathbf{D}(G)$ and let $\{g_1, \dots, g_m\}$ be its support in G. We wish to prove that $\langle h_1, \dots, h_n\rangle$ is a nilpotent subgroup of $\mathbf{D}$. Clearly, we may assume $m \geq 2$. Since $\langle g_1, \dots, g_m\rangle$ is a nilpotent subgroup of G, say of class c, by Lemma 2.2 (iii), it suffices to prove that

$$[x^\phi, y, z_1, \dots, z_{c^*}] = 1$$

for some large $c^* > c$ and all $x, y, z_1 \in \langle g_1, \dots, g_m\rangle$. With $c^* \geq 2cm$, by Lemma 2.3 and 2.4, $[x^\phi, y, z_1, \dots, z_{c^*}]$ can be written as a product of commutators of the form

$$[x^\phi, y, k_1 g_1', \dots, k_m g_m']$$

where $\{g_1', \dots, g_m'\} = \{g_1, \dots, g_m\}$, $k_1 \geq \cdots \geq k_m \geq 0$ and $\sum_{i=1}^m k_i \geq c^* \geq 2cm$. It follows that $k_1 \geq 2c$ and, therefore, it suffices to prove that $[x^\phi, y, kz] = 1$ for all $k \geq 2c$ and $x, y, z \in \langle g_1, \dots, g_m\rangle$.

Let $\overline{G} = \langle x, y, z\rangle$. Then, by hypothesis, $\gamma_{c+1}(\overline{G}) = \{1\}$. By Lemma 2.3, we may use the Jacobi congruence to write

$$[x^\phi, y, z, (k-1)z] = [x^\phi, z, y, (k-1)z][x^\phi, [y, z], (k-1)z]$$

and

$$\begin{aligned}[x^{\phi},[y,z],(k-1)z] &= [z,y,x^{\phi},(k-1)z]\\ &= [z,y,(k-1)z,x^{\phi}]\\ &= 1.\end{aligned}$$

Thus,

$$\begin{aligned}[x^{\phi},y,z,(k-1)z] &= [x^{\phi},z,y,(k-1)z]\\ &= [z^{\phi},x,y,(k-2)z,z]^{-1}\\ &= [z,x,y,(k-2)z,z^{\phi}]^{-1} \text{ (by Lemma 2.1 (iv))}\\ &= 1.\end{aligned}$$

This completes the proof of Lemma 2.5.

3. The main results

Let G be a nilpotent group of class at most c, $c \geq 1$. Then, by Lemma 2.1. (iv), $\mathbf{D} = \mathbf{D}(G)$ satisfies the identity

$$[x^{\phi}, y_2, \ldots, y_{c+1}, x] = 1 \qquad (3.1)$$

for all x, $y_i \in G$.

If $G = \langle x, y\rangle$ then, modulo $\gamma_{c+3}(\mathbf{D})$, $\gamma_{c+2}(\mathbf{D})$ is generated by elements of the form $[x^{\phi}, z_2, \ldots, z_{c+1}x]$ and $[y^{\phi}, z_2, \ldots, z_{c+1}, y]$, with $z_1 \in \{x, y\}$, each of which is trivial by (3.1). It follows that $\gamma_{c+2}(\mathbf{D}) = \gamma_{c+3}(\mathbf{D})$. Since $\mathbf{D}$ is nilpotent (Lemma 2.5), we have $\gamma_{c+2}(\mathbf{D}) = \{1\}$. We record this as follows:

THEOREM 3.1. *If G is a 2-generator nilpotent group of class at most c, then $\mathbf{D}(G)$ is nilpotent of class at most $c + 1$.*

We now investigate the general case with $\gamma_{c+1}(G) = \{1\}$. Working modulo $\gamma_{c+3}(\mathbf{D})$, the identity (3.1) yields

$$1 \equiv [x^{\phi}y_1^{\phi}, y_2, \ldots, y_{c+1}, xy_1] \equiv [x^{\phi}, y_2, \ldots, y_{c+1}, y_1][y_1^{\phi}, y_2, \ldots, y_{c+1}, x]$$

which on commuting with x and using (3.1) gives

$$[y_1^{\phi}, y_2, \ldots, y_{c+1}, x, x] \equiv 1 \pmod{\gamma_{c+4}(\mathbf{D})} \qquad (3.2)$$

for all $x, y_i \in G$. Furthermore, modulo $\gamma_{c+4}(\mathbf{D})$, for $2 \le k \le c$, we have

$$\begin{aligned}
&[[y_1^\phi, y_2, \dots, y_k], [x, y_{k+1}], y_{k+2}, \dots, y_{c+1}, x] \\
&\quad \equiv [[y_1^\phi, y_2, \dots, y_k], [x^\phi, y_{k+1}], y_{k+2}, \dots, y_{c+1}, x] \quad \text{(by Lemma 2.1 (iii))} \\
&\quad \equiv [[x^\phi, y_{k+1}], [y_1, \dots, y_k], y_{k+2}, \dots, y_{c+1}, x]^{-1} \\
&\quad \equiv [[x, y_{k+1}], [y_1, \dots, y_k], y_{k+2}, \dots, y_{c+1}, x^\phi]^{-1} \quad \text{(by (3.1))} \\
&\quad \equiv 1 \quad (\text{since } \gamma_{c+1}(G) = \{1\}).
\end{aligned}$$

We record this as

$$[[y_1^\phi, y_2, \dots, y_k], [x, y_{k+1}], y_{k+2}, \dots, y_{c+1}, x] \equiv 1 \pmod{\gamma_{c+4}(\mathbf{D})} \tag{3.3}$$

for all $x, y_i \in G$ and all $2 \le k \le c$. By (3.3), for $2 \le k \le c$, we have

$$\begin{aligned}
&[y_1^\phi, y_2, \dots, y_k, x, y_{k+1}, y_{k+2}, \dots, y_{c+1}, x] \\
&\quad \equiv [y_1^\phi, y_2, \dots, y_{k+1}, x, y_{k+2}, \dots, y_{c+1}, x] \\
&\quad\ \vdots \\
&\quad \equiv [y_1^\phi, y_2, \dots, y_{c+1}, x, x] \\
&\quad \equiv 1 \quad \text{(by (3.2))}.
\end{aligned}$$

Also, $[y_1^\phi, x, y_2, \dots, y_{c+1}, x] \equiv [x^\phi, y_1, \dots, y_{c+1}, x]^{-1} \equiv 1$ by (3.1). Thus we have

$$[y_1^\phi, y_2, \dots, y_k, x, y_{k+1}, \dots, y_{c+1}, x] \equiv 1 \pmod{\gamma_{c+4}(\mathbf{D})} \tag{3.4}$$

for all $1 \le k \le c+1$. Replacing x by xz in (3.4) and expanding modulo $\gamma_{c+4}(\mathbf{D})$ yields the congruence

$$[y_1^\phi, \dots, y_k, x, y_{k+1}, \dots, y_{c+1}, z] \equiv [y_1^\phi, \dots, y_k, z, y_{k+1}, \dots, y_{c+1}, x]^{-1}. \tag{3.5}$$

Using (3.5) it follows that every commutator of weight $c+3$ in $\mathbf{D}$ with a repeated entry x can be expressed, modulo $\gamma_{c+4}(\mathbf{D})$, as a product of commutators of the form

$$[y_1^\phi, \dots, y_k, x, y_{k+1}, \dots, y_{c+1}, x], \quad 1 \le k \le c+1,$$

which is trivial by (3.4). In particular, if G is an m-generator group with

$\gamma_{c+1}(G) = \{1\}$ and $m \le c+2$, then $\gamma_{m+3}(\mathbf{D}) = \gamma_{m+4}(\mathbf{D}) = \cdots = \{1\}$, by Lemma 2.5. We have thus proved:

THEOREM 3.2. *Let G be an m-generator nilpotent group of class at most c with $m \ge 2$, $c \ge 1$. Then for $m \le c+2$, $\gamma_{c+3}(\mathbf{D}(G)) = \{1\}$.*

Let G be nilpotent of class at most c. The congruence (3.5) also yields

$$\begin{aligned}[y_1^{\phi}, \ldots, y_{c+1}, x, z] &\equiv [y_1^{\phi}, \ldots, y_{c+1}, z, x]^{-1} \\ &\equiv [y_1^{\phi}, \ldots, y_{c+1}, x, z]^{-1} \quad \text{(by Lemma 2.3(i))},\end{aligned}$$

so that

$$[y_1^{\phi}, \ldots, y_{c+1}, x, z]^2 \equiv 1 \pmod{\gamma_{c+4}(\mathbf{D})}$$

By Theorem 3.2 every commutator of weight $c+4$ in $\mathbf{D}$ with entries from the set

$$\{y_1, \ldots, y_{c+1}, x, z\}$$

is trivial. Thus we have

$$[y_1^{\phi}, \ldots, y_{c+1}, x, z]^2 = 1. \tag{3.6}$$

Repeated application of (3.5) yields

$$[y_1^{\phi}, y_2, \ldots, y_{c+3}] = [y_1^{\phi}, y_{2\sigma}, \ldots, y_{(c+3)\sigma}]^{|\sigma|}$$

where σ is a permutation of $\{2, \ldots, c+3\}$ and $|\sigma| = 1$ or -1 according as σ is even or odd. Thus, if G is an m-generator group with $m \ge c+3$, then for $c+3 \le k \le m$, there are $\binom{m}{k}$ choices for distinct k-element sets from the generators of G. This fact together with (3.6) gives us the following theorem.

THEOREM 3.3. *Let G be an m-generator nilpotent group of class at most c with $m \ge 2$, $c \ge 1$. Then, for $m \ge c+3$, $\gamma_{c+3}(\mathbf{D}(G))$ is an elementary abelian 2-group of rank at most*

$$\sum_{k=c+3}^{m} \binom{m}{k}.$$

COROLLARY 3.4. (c.f. Rocco [3]) *Let G be a p-group of class c with p odd. Then $\mathbf{D}(G)$ is a p-group of class at most $c+2$.*

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