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DIMENSION SUBGROUPS OF FREE CENTER-BY-METABELIAN GROUPS

BY

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Dedicated to the memory of W.W. Boone

Introduction

Let $\mathbf{Z}G$ be the integral group ring of a group G and $\Delta(G)$ be its augmentation ideal. For each $n \geq 1$, the subgroup $D_n(G) = G \cap (1 + \Delta^n(G))$ is the n-th dimension subgroup of G. It is easily verified that $D_n(G) \supseteq \gamma_n(G)$, the n-th term of the lower central series of G. The validity of the reverse inequality, namely, $D_n(G) \subseteq \gamma_n(G)$, is known as the dimension subgroup problem for G. Rips [8] has constructed an example of a finite 2-group such that $D_4(G) \neq \gamma_4(G)$. On the other hand, a well-known result due to P. Hall and S.A. Jennings states that if the lower central factors $\gamma_k(G)/\gamma_{k+1}(G)$ are torsion free for all $k \geq 1$, then $D_n(G) = \gamma_n(G)$ for all $n \geq 1$ (cf. [6, Corollary 3.1]). In particular, it follows that if $G = F/F''$ is a free metabelian group, then $D_n(G) = \gamma_n(G)$ for all n. For a free center-by-metabelian group $G = F/[F'', F]$, the lower central factors have elementary abelian 2-subgroups (Ridley [7], Hurley [5]), and hence these groups are not covered by the Hall-Jennings result. The purpose of this paper is to prove that if G is a free center-by-metabelian group, then $D_n(G) = \gamma_n(G)$ for all n. This answers a question of I.B.S. Passi (verbal communication).

In terms of the free group ring $\mathbf{Z}F$, with $G = F/R$, the dimension subgroup problem reduces to identifying the subgroup $F \cap (1 + \mathbf{r} + \mathbf{f}^n)$ as $R \cdot \gamma_n(F)$, where $\mathbf{f} = \Delta(F) = \mathbf{Z}F(F - 1)$ and $\mathbf{r} = \mathbf{Z}F(R - 1)$. If $R = [F'', F]$, then $\mathbf{r} \subseteq \mathbf{faf}$, where $\mathbf{a} = \mathbf{Z}F(F' - 1)$, so as a first approximation to the identification of $F \cap (1 + \mathbf{r} + \mathbf{f}^n)$, in Section 3 we identify $F \cap (1 + \mathbf{faf} + \mathbf{f}^n)$ for all n. (The identification $F \cap (1 + \mathbf{fa} + \mathbf{f}^n) = F'' \cdot \gamma_n(F)$ for all n has been shown by N.D. Gupta [2]).

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2. Notation and preliminaries

Our commutator notation is as follows:

$$[g_1, g_2] = g_1^{-1}g_2^{-1}g_1g_2,$$
$$[g_1, g_2, g_3] = [[g_1, g_2], g_3],$$
$$[g_1, g_2; g_3, g_4] = [[g_1, g_2], [g_3, g_4]]$$

for group elements g_i, and

$$(r_1, r_2) = r_1r_2 - r_2r_1, \quad (r_1, r_2, r_3) = ((r_1, r_2), r_3),$$
$$((r_1, r_2), (r_3, r_4)) = (r_1, r_2; r_3, r_4)$$

for ring elements r_i. Also, $\gamma_n(G)$ is the n-th term of the lower central series of $G, G' = \gamma_2(G)$, $G'' = \gamma_2(G')$. Other unexplained notation follows Gupta-Hurley-Levin [3].

As in [3], [4], [5], our basic tool is the following power series representation of $F/[F'', F]$: Let $\mathbf{Z}[[y_1, y_2, \cdots]]$ be the free associative power series ring over $\mathbf{Z}$ generated by y_i, $i \geq 1$, and denote by C the ideal generated by all elements $y_i(y_j, y_k)y_1$. Set $\mathbf{P} = \mathbf{Z}[[y_1, y_2, \cdots]]/C$ and denote the generators of $\mathbf{P}$ by $x_i = y_i + C$, $i \geq 1$. The group $U(\mathbf{P})$ of units of $\mathbf{P}$ is a center-by-metabelian group (Hurley [5]). If F is a free group freely generated by $f_1, f_2, \ldots,$ then

$$\theta : f_i \to 1 + x_i$$

defines a homomorphism of F into $U(\mathbf{P})$. For any word $w \in F$, $w\theta$ is a power series of the form

$$w\theta = 1 + \sum_{i \geq 1} \langle w\theta \rangle_i,$$

where $\langle w\theta \rangle_i$ denotes the component of $w\theta$ of terms of total degree i. In particular, if $w \in \gamma_k(F)$, then $\langle w\theta \rangle_i = 0$ for all $i \leq k - 1$. The linear extension of θ yields the ring homomorphism

$$\theta : \mathbf{Z}F \to \mathbf{P}$$

with $\operatorname{Ker} \theta = \mathbf{faf}$. Thus, we obtain a power series representation of $\mathbf{Z}F/\mathbf{r}$, where $\mathbf{r} = \mathbf{Z}F(R - 1)$, $R = [F'', F]$.

Apart from the subgroups $[F'', F]$ and $\gamma_{c+1}(F)$ of F, in the sequel we shall refer to the fully invariant subgroups $K_6(F)$, $U_c(F)$, c even, $c \geq 6$, and $T_c(F)$, c odd, $c \geq 5$, defined as follows.

(i) $K_6(F)$ is the fully invariant closure of

$$\begin{aligned} u_6 = \prod_{\tau} & [f_{1\tau}, f_{2\tau}; f_{1\tau}, f_{2\tau}, f_{3\tau}, f_{4\tau}] \\ \times & [f_2, f_4, f_3; f_2, f_4, f_1, f_3][f_4, f_3, f_2; f_2, f_3, f_1, f_4] \\ \times & [f_3, f_1, f_4; f_3, f_1, f_2, f_4][f_4, f_3, f_1; f_4, f_1, f_2, f_3] \\ \times & [f_4, f_1, f_2; f_4, f_1, f_2, f_3][f_2, f_4, f_1; f_2, f_1, f_3, f_4] \\ \times & [f_2, f_1, f_3; f_2, f_1, f_3, f_4][f_3, f_2, f_1; f_3, f_1, f_2, f_4], \end{aligned}$$

where τ runs over those permutations of $\{1, 2, 3, 4\}$ with $1\tau < 2\tau, 3\tau < 4\tau$.

(ii) $U_c(F)$ is the fully invariant closure of $[f_1, f_2; f_1, f_2, \ldots, f_{c-2}]$.

(iii) $T_c(F)$ is the fully invariant closure of $v_c^* = w_c^* g_{c+1}^{-1} h_{c+1}$, where

$$\begin{aligned} w_c^* &= [f_1, f_2; f_3, f_4, f_5, \ldots, f_c] \\ &\quad \times [f_1, f_3; f_4, f_2, f_5, \ldots, f_c] \\ &\quad \times [f_1, f_4; f_2, f_3, f_5, \ldots, f_c]; \\ &= \prod_{\sigma} [f_1, f_{2\sigma}; f_{3\sigma}, f_{4\sigma}, f_5, \ldots, f_c], \end{aligned}$$

where σ ranges over the powers of the permutation $(2, 3, 4)$;

$$\begin{aligned} g_{c+1} = \prod_{\sigma} & [f_1, f_{2\sigma}; f_1, f_{3\sigma}, f_{4\sigma}, f_5, \ldots, f_c] \\ \times & [f_{2\sigma}, f_{3\sigma}; f_{2\sigma}, f_1, f_{4\sigma}, f_5, \ldots, f_c] \\ \times & [f_{2\sigma}, f_1; f_{2\sigma}, f_{4\sigma}, f_{3\sigma}, f_5, \ldots, f_c] \\ \times & [f_{2\sigma}, f_{4\sigma}; f_{2\sigma}, f_{3\sigma}, f_1, f_5, \ldots, f_c], \end{aligned}$$

σ as above;

$$h_{c+1} = \prod_{k=5}^{c} \prod_{\sigma} [f_k, f_{2\sigma}; f_k, f_{4\sigma}, f_{3\sigma}, f_1, f_5, \ldots, \hat{f}_k, \ldots, f_c]$$

σ as above, where $\hat{f}_k$ indicates f_k missing from the sequence $f_5, \ldots, f_c$.

The following Lemma follows from the definitions and the identity

$$[r, s] = 1 + r^{-1}s^{-1}(r, s),$$

valid for any ring units.

LEMMA 2.1 (i) [4, LEMMA 3.4]. *Let*

$$w\theta = [1 + z_1, 1 + z_2; 1 + z_3, \dots, 1 + z_n],$$

where $1 + z_i \in F\theta$. *Then*

$$w\theta = 1 + (-1)^{n-4}(z_1, z_2)z_5 \dots z_n(1 + z_3)^{-1} \dots (1 + z_n)^{-1}(z_3, z_4)$$
$$-(z_3, z_4)z_5 \dots z_n(1 + z_1)^{-1}(1 + z_2)^{-1}(z_1, z_2), \quad n \geq 4.$$

(ii)

$$\langle [1 + z_1, 1 + z_2, \dots, 1 + z_n] \rangle_n$$
$$= (z_1, z_2)z_3 \dots z_n + (-1)^n z_n \dots z_3(z_1, z_2), \quad n \geq 3.$$

Lemma 2.1 (ii) follows by a straight-forward expansion. Using Lemma 2.1 (i) the various degree components $\langle w\theta \rangle_i$ of the power series $w\theta$ can be determined directly by using the power series expansion

$$(1 + z)^{-1} = 1 - z + z^2 - z^3 + \cdots .$$

The main properties required of the fully invariant subgroups listed earlier are stated in the following Lemmas.

LEMMA 2.2 (C.K. GUPTA [1]). (i) $F \cap (1 + \mathbf{faf}) = K_6(F) \cdot [F'', F]$.
(ii) $K_6(F) \subseteq [F'', F]$ *if and only if rank* $F \leq 3$.
(iii) $u^2 \in [F'', F]$ *for all* $u \in K_6(F)$.

LEMMA 2.3. *Let c be odd,* $c \geq 5$.
(i) $\langle w_c^* \theta \rangle_c = 0$.
(ii) $v_c^{*2} \in [F'', F]\gamma_{c+2}(F)$.
(iii) $\langle v_c^* \theta \rangle_c = \langle v_c^* \theta \rangle_{c+1} = 0$. *Hence, for any* $w \in T_c(F)$, $\langle w\theta \rangle_c = \langle w\theta \rangle_{c+1} = 0$.
(iv) *If* $\langle v_c^* v\theta \rangle_{c+1} = 0$ *for some* $v \in \gamma_{c+1}(F)$, *then* $\langle v_c^* v\theta \rangle_{c+2} \neq 0$. *In particular,* $\langle v_c^* v\theta \rangle_{c+2} \neq 0$ *for any* $v \in \gamma_{c+2}(F)$.
(v) $T_c(F) \subseteq [F'', F] \cdot \gamma_{c+1}(F)$ *if F has rank less than c.*
(vi) $w_c^*(f_1, \dots, f_c) \equiv w_c^*(f_{1\sigma}, \dots, f_{c\sigma})$ *modulo* $[F'', F] \cdot \gamma_{c+1}(F)$, *for any permutation* σ *of* $\{1, 2, \dots, c\}$.
(vii) *If* $\langle w\theta \rangle_c = 0$ *for some* $w \in F'' \cap \gamma_c(F)$, *then*

$$w \in T_c(F) \cdot K_6(F) \cdot [F'', F] \cdot \gamma_{c+1}(F).$$

Proof. (i), (iv) and (vi) are proved in [4, Lemma 3.8]. (ii) follows by direct expansion, using Lemma 2.1. (iii) follows from (ii) and the fact that

$$\langle v^2\theta\rangle_{c+1} = 2\langle v\theta\rangle_{c+1} \quad \text{for any } v \in \gamma_c(F).$$

(v) follows from the fact that $w_c^* \in [F'', F] \cdot \gamma_{c+1}(F)$ if F has rank less than c [4, Lemma 4.1 (ii)]. Finally, (vii) follows from Lemma 4.1 (ii) and (iv) of [4].

LEMMA 2.4. *Let $u \in U_c(F)$, c even, $c \geq 6$.*

(i) $u^2 \in [F'', F] \cdot \gamma_{c+1}(F)$.

(ii) $\langle u\theta\rangle_c = 0$.

(iii) *If*

$$u = [f_{i,1}, f_{i,2}; f_{i,1}, f_{i,2}, f_{i,3}, \dots, f_{i,c-2}],$$

then $u \in [F'', F] \cdot \gamma_{c+1}(F)$ if and only if each $f_{i,j}$ occurs an even number of times.

(iv) *If $\langle uv\theta\rangle_{c+1} = 0$ for some $v \in \gamma_{c+1}(F)$, then*

$$u \in [F'', F] \cdot \gamma_{c+1}(F).$$

(v) *If $\langle w\theta\rangle_c = 0$ for some $w \in F'' \cap \gamma_c(F)$, then*

$$w \in U_c(F) \cdot K_6(F) \cdot [F'', F] \cdot \gamma_{c+1}(F).$$

(vi) *For $c \geq 8$ define $u_c = \prod_\sigma [f_{1\sigma}, f_{2\sigma}; f_{1\sigma}, f_{2\sigma}, f_{3\sigma}, \dots, f_{(c-2)\sigma}]$, where σ ranges over all those permutations of $\{1, \dots, c-2\}$ with $1\sigma < 2\sigma$ and $3\sigma < 4\sigma < \cdots < (c-2)\sigma$. Then $u_c \in [F'', F] \cdot \gamma_{c+1}(F)$.*

Proof. (i) and (iii) are straight forward consequences of Lemma 3.1 (i) of [4]. (ii) follows directly by using Lemma 2.1. (v) follows from Lemma 4.1 (i) and (iii) of [4]. (vi) is proved in Lemma 4.3 of [4]. For the proof of (iv) we proceed as follows: For any $w \in F$ let $\alpha_{ij}\langle w\theta\rangle_n$ denote the component of $\langle w\theta\rangle_n$ of terms beginning with x_i and ending with x_j. By Lemma 4.2 of [4], if u involves $c-2$ generators and $\langle u\theta\rangle_{c+1} \equiv 0$ (2) then $u = u_c$, as defined in (vi). By Lemma 4.4 of [4], if F has rank less than $c-2$ then

$$u \in [F'', F] \cdot \gamma_{c+1}(F) \quad \text{if } \langle u\theta\rangle_{c+1} \equiv 0\ (2).$$

Hence, in each case $u \in [F'', F] \cdot \gamma_{c+1}(F)$ if $\langle u\theta\rangle_{c+1} \equiv 0$ (2). However, in the proofs of these lemmas the weaker hypothesis that $\alpha_{ii}\langle u\theta\rangle_{c+1} \equiv 0$ (2) was all that was used, and it follows that $u \in [F'', F] \cdot \gamma_{c+1}(F)$ if $\alpha_{ii}\langle u\theta\rangle_{c+1} \equiv 0$ (2), for all i. On the other hand, if $v \in \gamma_{c+1}(F)$, then it follows easily from Lemma 2.1 (ii) that $\alpha_{ii}\langle v\theta\rangle_{c+1} \equiv 0$ (2), and, hence, if $\langle u \vee \theta\rangle_{c+1} = 0$, then $\alpha_{ii}\langle u\theta\rangle_{c+1} \equiv 0$ (2), also.

3. $F \cap (1 + \mathbf{faf} + \mathbf{f}^{c+1})$

Let $D(c) = F \cap (1 + \mathbf{faf} + \mathbf{f}^{c+1})$.

THEOREM A. (i) $D(c) = \gamma_{c+1}(F)$ *if* $1 \leq c \leq 4$.
(ii) $D(c) = T_c(F) \cdot K_6(F) \cdot [F'', F] \cdot \gamma_{c+1}(F)$ *for c odd*, $c \geq 5$.
(iii) $D(c) = T_{c-1}(F) \cdot U_c(F) \cdot K_6(F) \cdot [F'', F] \cdot \gamma_{c+1}(F)$ *for c even*, $c \geq 6$.

Proof. It follows from Lemmas 2.2 (i), 2.3 (iii) and 2.4 (ii) that $D(c)$ contains the respective right sides of (i), (ii) and (iii), so it remains to prove the reverse inclusions. Thus, suppose $w \in D(c)$, that is, $\langle w\theta \rangle_i = 0$ for all $i \leq c$, since $\mathbf{faf} = \operatorname{Ker} \theta$. For the proofs of our results it will suffice to assume that all terms of w involve the same set of generators and, since all statements are made modulo some term of the lower central series of F, that all entries in the commutators are generators. Also, by [2], since $\mathbf{faf} \subseteq \mathbf{fa}$, we may further assume that $w \in F''$, and, since $F'' < \gamma_4(F)$, that $w \in F'' \cdot \gamma_5(F)$. In particular, if $w \notin \gamma_5(F)$, then, using these assumptions,

$$\langle w\theta \rangle_4 = a_1(x_1, x_2; x_3, x_4) + a_2(x_1, x_3; x_2, x_4) + a_3(x_1, x_4; x_2, x_3)$$

or

$$\langle w\theta \rangle_4 = a_1(x_1, x_2; x_1, x_3) + a_2(x_1, x_2; x_2, x_3) + a_3(x_1, x_3; x_3, x_2),$$

but in either case it follows by directly expanding that $\langle w\theta \rangle_4 = 0$ only if all $a_i = 0$, that is, $w \in \gamma_5(F)$. This proves (i).

For the proofs of (ii) and (iii) we proceed by induction. Since $K_6(F) \subset \gamma_6(F)$, the case $c = 5$ for (ii) follows immediately from Lemma 2.3 (vii). By induction, suppose (ii) and (iii) hold for $c < k$, $k \geq 6$. If k is even, then (ii) holds for $c = k - 1$ and $w \in T_{k-1}(F) \cdot K_6(F) \cdot [F'', F] \cdot \gamma_k(F)$. Thus, modulo $K_6(F) \cdot [F'', F]$, $w = w_1 w_2$, $w_1 \in T_{k-1}(F)$, $w_2 \in F'' \cap \gamma_k(F)$. By Lemma 2.3 (iii), $\langle w_1\theta \rangle_{k-1} = \langle w_1\theta \rangle_k = 0$ so $\langle w_2\theta \rangle_k = 0$. Thus, by Lemma 2.4 (v),

$$w_2 \in U_k(F) \text{ modulo } K_6(F) \cdot [F'', F] \cdot \gamma_{k+1}(F),$$

whence $D(k)$ has the desired form. Finally, if k is odd, $k \geq 7$, then (iii) holds for $c = k - 1$. Hence,

$$w = w_1 w_2 w_3, w_1 \in T_{k-2}(F), w_2 \in U_{k-1}(F), w_3 \in F'' \cap \gamma_k(F),$$
$$\text{modulo } K_6(F) \cdot [F'', F] \cdot \gamma_{k+1}(F).$$

Since terms in $T_{k-2}(F)$ involve $k - 2$ generators while those in $U_{k-1}(F)$ involve at most $k - 3$ generators, we may write $w = w_1 w_3' w_2 w_3''$, where $w_3 =$

$w_3'w_3''$ and the terms in w_1w_3' involve $k-2$ generators, while those in w_2w_3'' involve at most $k-3$ generators. By Lemma 2.3 (iv), $\langle w_1w_3'\theta\rangle_k \neq 0$, so $w_1w_3' \equiv 1$ since $\langle w\theta\rangle_k = 0$. Similarly, by Lemma 2.4 (iv), we may assume that $w_2 \equiv 1$ and, hence, $\langle w_3''\theta\rangle_k = 0$. Hence, by Lemma 2.4 (vii),

$$w_3'' \in T_k(F) \text{ modulo } K_6(F)\cdot[F'',F]\cdot\gamma_{k+1}(F),$$

which completes the proof.

Remark. By [2],

$$F\cap(1+\mathbf{fa}+\mathbf{f}^{c+1}) = F''\cdot\gamma_{c+1}(F) = (F\cap(1+\mathbf{fa}))\cdot(F\cap(1+\mathbf{f}^{c+1})).$$

A similar result does not, however, hold true for **faf**. In other words,

$$F\cap(1+\mathbf{faf}+\mathbf{f}^{c+1}) \neq (F\cap(1+\mathbf{faf}))\cdot(F\cap(1+\mathbf{f}^{c+1}))$$

for any $c \geq 5$.

4. Free center-by-metabelian groups

In this section we shall complete the proof of our principal result that

$$F\cap(1+\mathbf{r}+\mathbf{f}^{c+1}) = R\cdot\gamma_{c+1}(F),$$

where $R = [F'',F]$ and $\mathbf{r} = \mathbf{Z}F(R-1)$. Since

$$R\cdot\gamma_{c+1}(F) \subseteq F\cap(1+\mathbf{r}+\mathbf{f}^{c+1}),$$

to complete the proof the reverse inclusion must be verified. Since $\mathbf{r}\subset\mathbf{faf}$, Theorem A is directly applicable, and to complete the proof it will be necessary to eliminate the "unwanted" factors $K_6(F)$, $T_c(F)$ and $U_c(F)$. For this purpose we shall need to consider the ideal $\mathbf{f}^2\mathbf{af}$, which contains $\mathbf{fr}$ but not $\mathbf{r}$ itself, and the following power series representation of $\mathbf{Z}F/\mathbf{f}^2\mathbf{af}$.

Let C_1 be the ideal of $\mathbf{Z}[[y_1, y_2, \ldots]]$, the free associative power series ring, generated by all $y_iy_j(y_k, y_l)y_m$. The map $f_i \to 1+z_i$, where $z_i = y_i + C_1$, extends by linearity to a representation φ of

$$\mathbf{Z}F/\mathbf{f}^2\mathbf{af} \quad \text{in } \mathbf{Z}[[y_1, y_2, \ldots]]/C_1.$$

In particular, the elements of $F\cap(1+\mathbf{f}^2\mathbf{af}+\mathbf{f}^{c+1})$ are characterized by $w\in F\cap(1+\mathbf{f}^2\mathbf{af}+\mathbf{f}^{c+1})$ if and only if $\langle w\varphi\rangle_i = 0$, $i\leq c$.

The restriction of φ to $[F'',F]$ can be thought of as being the composition of two maps, the map θ defined in Section 2 and the map ψ of f_i to

$1 + y_i + A$, where A is the ideal generated by all (y_i, y_j), where, with the obvious interpretation,

$$\langle [f, v]\varphi \rangle = f\psi \cdot v\theta \quad \text{for } f \in F \text{ and } v \in F''. \tag{4.1}$$

The fact that the above definition is unambiguous comes again from the identity

$$[r, s] = 1 + (1 + r)^{-1}(1 + s)^{-1}(r, s)$$

for unit elements. With this interpretation the following lemma follows directly from Lemmas 2.3 and 2.4.

LEMMA 4.1. (i) *Let v_c^* be as defined in Section 2. Then for any f_i,*

$$\langle [v_c^*, f_i]\varphi \rangle_{c+1} = \langle [v_c^*, f_i]\varphi \rangle_{c+2} = 0 \quad c \text{ odd}, c \geq 5.$$

(ii) *For any $u \in U_c$, c even, $\langle [u, f_i]\varphi \rangle_{c+1} = 0$.*

The following lemma lists some further extensions of Lemmas 2.3 and 2.4.

LEMMA 4.2. (i) *If $\langle [v_c^*, f_i]v\varphi \rangle_{c+2} = 0$ for any $v \in \gamma_{c+2}(F)$, then $\langle [v_c^*, f_i]v\varphi \rangle_{c+3} \neq 0$ (cf. Lemma 2.3 (iv)).*

(ii) *If $u \in U_c(F)$, c even, but $u \notin [F'', F] \cdot \gamma_{c+1}(F)$, then $\langle [u, f]v\varphi \rangle_{c+2} \neq 0$ for any $v \in \gamma_{c+2}(F)$ (cf. Lemma 2.4 (iv)).*

The following result is essential to eliminate the factors $T_c(F)$.

LEMMA 4.3. *For any c odd, $c \geq 5$, $w_c^* \notin 1 + \mathbf{r} + \mathbf{f}^{c+1}$. In particular, if*

$$w \in (1 + \mathbf{r} + \mathbf{f}^{c+1}) \cap T_c(F),$$

then

$$w \in [F'', F] \cdot \gamma_{c+1}(F).$$

Proof. If $w_c^* - 1 \in \mathbf{r} + \mathbf{f}^{c+1}$, then $w_c^* - 1$ can be expressed as a sum $s_5 + s_6 + \cdots + s_c$, modulo $\mathbf{f}^{c+1}$, where s_i is a sum of terms of the form $g(r_i - 1)$, where $g \in F$ and $r_i \in R \cap \gamma_i(F)$, $r_i \notin \gamma_{i+1}(F)$. However, by Lemma 2.1, for any $i < c$, $\langle r_i\varphi \rangle_c$ will involve repetitions of generators. Since the terms of w_c^* are linear in each generator, we may assume that $\langle w_c^*\varphi \rangle_c = \langle s_c\varphi \rangle_c$. Moreover, since $\mathbf{f}^2\mathbf{af}$ contains $\mathbf{fr}$, $\langle s_c\varphi \rangle_c = \langle r\varphi \rangle_c$, for some $r \in \mathbf{Z}(R - 1)$. In particular, the terms in $\langle w_c^*\varphi \rangle_c$ with left factor z_c, the "z_c-component" of

$\langle w_c^*\varphi\rangle_c$, must be equal to the z_c-component of $\langle r\varphi\rangle_c$. This leads to an equation

$$\begin{aligned} z_c\{&(z_3, z_4)z_5 \cdots z_{c-1}(z_1, z_2) \\ &+(z_4, z_2)z_5 \cdots z_{c-1}(z_1, z_3) + (z_2, z_3)z_5 \cdots z_{c-1}(z_1, z_4)\} \\ &= z_c\sum_i a_i(z_{i,1}, z_{i,2}; z_{i,3}, z_{i,4}, \ldots, z_{i,c-1}), \end{aligned} \tag{4.2}$$

where $a_i \in \mathbf{Z}$, $\{i, 1, \ldots, i, c-1\}$ is a permutation of $\{1, \ldots, c-1\}$. For $c = 5$, (4.2) reduces to

$$\begin{aligned} &z_5\{(z_3, z_4)(z_1, z_2) + (z_4, z_2)(z_1, z_3) + (z_2, z_3)(z_1, z_4)\} \\ &\quad = z_5\{a_1(z_1, z_2; z_3, z_4) + a_2(z_1, z_3; z_2, z_4) + a_3(z_1, z_4; z_2, z_3)\}. \end{aligned}$$

This equation can have no integral solution, which can be observed by an easy comparison of terms. For $c \geq 7$, (4.2) remained valid if we replace each of $z_5, \ldots, z_{c-1}$ by z_4. Using the Jacobi identity, (4.2) reduces to

$$\begin{aligned} z_c\{&(z_3, z_4)z_4^{c-5}(z_1, z_2) + (z_4, z_2)z_4^{c-5}(z_1, z_3) + (z_2, z_3)z_4^{c-5}(z_1, z_4)\} \\ &= z_c\Big\{a_1\Big(z_4, z_1; z_4, z_2, z_3, \underbrace{z_4, \ldots, z_4}_{c-6}\Big) \\ &\quad +a_2\Big(z_4, z_1; z_4, z_3, z_2, \underbrace{z_4, \ldots, z_4}_{c-6}\Big) \\ &\quad +a_3\Big(z_4, z_2; z_4, z_3, z_1, \underbrace{z_4, \ldots, z_4}_{c-6}\Big)\Big\}. \end{aligned}$$

Comparing the z_c-components $z_c z_1 z_3 z_4^{c-4} z_2$ and $z_c z_1 z_2 z_4^{c-4} z_3$ shows that $a = 1 = a_2 = 0$. Now comparing the z_c-components $z_c z_3 z_1 z_4^{c-4} z_2$ and $z_c z_2 z_1 z_4^{c-4} z_3$ gives $1 = a_3$ and $-1 = a_3$ respectively, which is meaningless. This completes the proof of the lemma.

The elimination of $U_6(F)$ and, in particular, of $K_6(F)$, hinges on the following lemma.

LEMMA 4.4. *Let $u \in U_6(F)$ and suppose that u is a nontrivial product of terms*

$$[f_{i,1}, f_{i,2}; f_{i,1}, f_{i,2}, f_{i,3}, f_{i,4}]$$

with $f_{i,3} \neq f_{i,4}$, each term occuring at most once. Then $u \notin 1 + \mathbf{r} + \mathbf{f}^7$. In particular, $K_6(F) \not\subseteq 1 + \mathbf{r} + \mathbf{f}^7$.

Proof. Suppose $u \in 1 + \mathbf{r} + \mathbf{f}^7$. Then, as in Lemma 4.3, we may write $u - 1 = s_5 + s_6$, modulo $\mathbf{f}^7$. Since $u \in \gamma_6(F)$ it follows that $\langle s_5\varphi\rangle_5 = 0$, which, using (4.1), further implies that there is a non-trivial element of $F'' \cap \gamma_4(F)$ with zero 4-weight component under the θ-map. However, this has been shown in the proof of Theorem A (i) to be impossible. Hence, we may assume that $\langle u\varphi\rangle_6 = \langle r\varphi\rangle_6$ for some $r \in R \cap \gamma_6(F)$. Any two commutators of length 6 of the type exhibited in the lemma have either different entries or, if the sets of entries are the same, the number of occurrences of the generators will be different. Hence the existence of a solution for $\langle u\varphi\rangle_6 = \langle r\varphi\rangle_6$ will imply a solution factor-wise, and, in particular, there will be a solution to an equation of the form

$$\begin{aligned}(z_{i,1}, z_{i,2}; z_{i,1}, z_{i,2}, z_{i,3}, z_{i,4}) \\ = \sum a_j(z_{j,1}, z_{j,2}; z_{j,3}, z_{j,4}, z_{j,5}; z_{j,6}).\end{aligned} \tag{4.3}$$

Also, we may assume $i, 3 < i, 4$, so the substitution of z_1 for both $z_{i,1}$ and $z_{i,3}, z_2$ for both $z_{i,2}$ and $z_{i,4}$, will lead to an equation (4.3) with left side

$$(z_1, z_2; z_1, z_2, z_1, z_2)$$

and right side

$$a_1(z_1, z_2; z_1, z_2, z_1; z_2) + a_2(z_1, z_2; z_1, z_2, z_2; z_1)$$

which, modulo C_1, is equal to

$$-a_1z_2(z_1, z_2; z_1, z_2, z_1) - a_2z_1(z_1, z_2; z_1, z_2, z_2),$$

and a straightforward comparison of the z_2-components of each side, that is, $z_2(z_1, z_2)z_1(z_1, z_2)$ with $-a_1z_2(z_1, z_2; z_1, z_2, z_1)$, shows that there is no integral solution for a_1.

Corollary 4.5. (i) *$F \cap (1 + \mathbf{r} + \mathbf{f}^{c+1}) \subseteq U_c(F) \cdot [F'', F] \cdot \gamma_{c+1}(F)$ if c is even, $c \geq 8$.*

(ii) *$F \cap (1 + \mathbf{r} + \mathbf{f}^{c+1}) = [F'', F] \cdot \gamma_{c+1}(F)$ if $c = 6$ or if c is odd, $c \geq 5$.*

Proof. Suppose $w \in F \cap (1 + \mathbf{r} + \mathbf{f}^{c+1})$. By Theorem A,

(a) $w \in T_c(F) \cdot K_6(F) \cdot [F'', F] \cdot \gamma_{c+1}(F)$, c odd, and

(b) $w \in T_{c-1}(F) \cdot U_c(F) \cdot K_6(F) \cdot [F'', F] \cdot \gamma_{c+1}(F)$, c even.

Thus, let $w = w_1w_2$, with $w_2 \in [F'', F] \cdot \gamma_{c+1}(F)$ and w_1 in the remaining factors, accordingly as c is odd or even. Since $w \in F \cap (1 + \mathbf{r} + \mathbf{f}^{c+1})$,

$$w_1 = 1 + \sum a_i g_i(r_i - 1) + s \quad \text{where } a_i \in \mathbf{Z},\ g_i \in F,\ r_i \in [F''F],\ s \in \mathbf{f}^{c+1}.$$

If $c = 5$, then $w_1 \in T_5(F)$ and, $\langle w_1\varphi\rangle_5 = \langle \Sigma a_i(r_i - 1)\varphi\rangle_5$. By Lemma 4.3, it follows that $w_1 \in [F'', F] \cdot \gamma_6(F)$, and, hence $w \in [F'', F] \cdot \gamma_6(F)$. If $c = 6$, then $w_1 \in T_5(F) \cdot U_6(F)$, and by the argument for $c = 5$, we may assume that $w_1 \in U_6(F)$. Thus, by Lemma 4.4, w_1 and, hence, w is in $[F'', F] \cdot \gamma_7(F)$. Finally, if $c \geq 7$, then $w_1 = w_1' w_1''$, where $w_1' \in K_6(F)$ and $w_1'' \in \gamma_7(F)$. Hence,

$$\langle w_1' \varphi \rangle_6 = \langle \Sigma a_i (r_i - 1) \varphi \rangle_6$$

and, by Lemma 4.4, $w_1' \in [F'', F] \cdot \gamma_7(F)$. Since $K_6(F) \cap \gamma_7(F) \subseteq [F'', F]$, and it follows that $w_1 \in T_c(F)$, c odd, or $w_1 \in T_{c-1}(F) \cdot U_c(F)$, c even. The corollary now follows directly from Lemma 4.3.

Thus, by Corollary 4.5, we are left with the factor $U_c(F)$, c even, $c \geq 8$, to resolve. This was relatively easy for $c = 6$ since the number of generators of $U_6(F)$ is small. Modulo $[F'', F] \cdot \gamma_{c+1}(F)$, $U_c(F)$ is an elementary abelian 2-group, and a basis for this group has been determined in N.D. Gupta, Hurley and Levin [3]. Before quoting this basis, in Lemma 4.6, below, we need a definition.

Let $u = [f_i, f_j; f_i, f_j, f_{i,5}, \ldots, f_{i,c}] \in U_c(F)$, and suppose that u involves the generators $f_1, \ldots, f_n$ for $n \leq c$. Then u will be abbreviated by

$$[i, j; p_1, p_2, \ldots, p_n],$$

where p_k is the number of occurrences of f_k in the sequence $f_{i,5}, f_{i,c}$.

LEMMA 4.6 [3]. *Let F be free of rank* $\leq c - 3$. *A basis for* $U_c(F)$ *modulo* $[F'', F] \cdot \gamma_{c+1}(F)$ *is given by the set of elements of the form*

$$[i, j; p_1, \ldots, p_n], \quad i < j,$$

such that $p_k \leq 1$ *for* $k < i$ *and the first non-zero integer reading left to right in the sequence* $p_n, p_{n-1}, \ldots, p_1$ *is odd. If the rank of F is* $c - 2$, *then we must also include the commutators, less any one factor, occuring in* u_c, *as defined in Lemma* 2.4 (vi), *to complete a basis.*

(In the above notation u_c is the product of all commutators

$$[i, j; p_1, \ldots, p_{c-2}]$$

with $1 \leq i < j \leq c - 2$, $p_i = p_j = 0$ and $p_k = 1$ for $k \neq i, j$.)

Before applying Lemma 4.6, for our forthcoming Lemma 4.8, we need a further result from [4].

LEMMA 4.7 [4, LEMMA 3.2 (i)]. *For c even*, $c \geq 8$,

$$\prod_{k=3}^{c-3} [f_1, f_2; f_1, f_2, f_3, \ldots, f_k, f_k, \ldots, f_{c-3}]$$

is in $[F'', F] \cdot \gamma_{c+1}(F)$.

LEMMA 4.8. *If there exists an element* $w \in U_c(F) \cap (1 + \mathbf{r} + \mathbf{f}^{c+1})$, *c even*, $c \geq 8$, *with* $w \notin [F'', F] \cdot \gamma_{c+1}(F)$, *then there exists an element with these properties involving at most* $c - 4$ *distinct generators.*

Proof. Suppose an element w exists as described above. Without loss of generality we may assume that w involves at most $c - 2$ generators $f_1, \ldots, f_{c-2}$. By Lemma 4.6, if w involves precisely $c - 2$ generators, then w is a proper factor of u_c, modulo $[F'', F] \cdot \gamma_{c+1}(F)$. Thus, either for some fixed i, w does not contain all factors $[i, j; p_1, \ldots]$, $j > i$, or for some fixed j, w does not contain all factors $[i, j; p_1, \ldots]$, $i < j$. In either case, after a suitable change of subscripts, we may assume that w has the factor

$$[f_1, f_2; f_1, f_2, f_3, \ldots, f_{c-2}]$$

but not

$$[f_1, f_3; f_1, f_3, f_2, \ldots, f_{c-2}].$$

Let w' be the word obtained from w by identifying f_3 with f_2. Then

$$w' \in U_c(F) \cap (1 + \mathbf{r} + \mathbf{f}^{c+1})$$

and involves $c - 3$ generators. To see that $w' \notin [F'', F] \cdot \gamma_{c+1}(F)$, we use the basis given in Lemma 4.6 as follows. First we observe that the factors of the form $[f_1, f_i; f_1, f_i, \ldots]$ in w' are basis elements since $c \geq 8$. In fact, the only terms that are not basic will have the form

$$[f_i, f_j; f_i, f_j, f_1, f_2, f_2, \ldots] \quad \text{with} \quad i \geq 4.$$

However, by using the Jacobi identity (cf. [1]) modulo $[F'', F] \cdot \gamma_{c+1}(F)$, such a term is equal to the product

$$[f_2, f_i; f_2, f_i, f_1, f_4, \ldots, f_j, f_j, \ldots][f_2, f_j; f_2, f_j, f_1, f_4, \ldots, f_i, f_i, \ldots], \tag{4.4}$$

the left factor in (4.4) is basic unless $j = c - 2$ and the right factor unless $i = c - 3$ and $j = c - 2$. If $j = c - 2$, the left factor of (4.4) has the form

$$[f_2, f_i; f_2, f_i, f_1, f_4, \ldots, f_{c-2}, f_{c-2}],$$

which, by Lemma 4.7, is congruent to

$$[f_2, f_i; f_2, f_i, f_1, f_1, f_4, \ldots, \hat{f}_i, \ldots, f_{c-3}, f_{c-2}]$$
$$\cdot \prod_{k=4}^{c-3} [f_2, f_i; f_2, f_i, f_1, f_4, \ldots, \hat{f}_i, \ldots, f_k, f_k, \ldots, f_{c-3}, f_{c-2}] \tag{4.5}$$

modulo $[F'', F] \cdot \gamma_{c+1}(F)$.

Each term in the product in (4.5) is basic, and again by [1],

$$[f_2, f_i; f_2, f_i, f_1, f_1, f_4, \ldots, \hat{f}_i, \ldots, f_{c-3}, f_{c-2}]$$

is congruent to

$$[f_1, f_2; f_1, f_2, f_4, \ldots, f_i, f_i, \ldots, f_{c-2}][f_1, f_i; f_1, f_i, f_2, f_2, f_4, \ldots, f_{c-2}],$$

which is a product of basis elements. The right factor in (4.4) can be represented analogously as a product of basis elements if $i = c - 3$, $j = c - 2$. However, in all these reductions the basis term

$$[f_1, f_2; f_1, f_2, f_2, f_4, \ldots, f_{c-2}]$$

does not occur. Hence, by Lemma 4.6, $w' \notin [F'', F] \cdot \gamma_{c+1}(F)$.

Finally, suppose w involves precisely the $c - 3$ generators $f_1, \ldots, f_{c-3}$. After a possible change of subscripts, we may assume basis elements of the form $[f_1, f_2; f_1, f_2, \ldots]$ occur as factors of w. There are three possible forms for such elements, based on Lemma 4.6:

(i) $[f_1, f_2; f_1, f_2, f_1, f_3, f_4, \ldots, f_{c-3}]$
(ii) $[f_1, f_2; f_1, f_2, f_2, f_3, f_4, \ldots, f_{c-3}]$
(iii) $[f_1, f_2; f_1, f_2, f_3, \ldots, f_i, f_i, \ldots, f_{c-3}]$ for $3 \leq i \leq c - 4$.

Further, we may assume that basis elements of the form (iii) occur and at least one, say $[f_1, f_2; f_1, f_2, f_3, f_4, f_4, f_5, \ldots, f_{c-3}]$ does not occur in w. Let w' be obtained from w by identifying f_4 with f_3. After identifying f_4 with f_3, the factors of w of the form $[f_1, f_2; f_1, f_2 \ldots]$ will still be in the basis and remain independent modulo $[F'', F] \cdot \gamma_{c+1}(F)$. As in the above case, the non-basic factors of w' will come from those in w which after replacing f_4 by f_3 have the form $[f_i, f_j; f_i, f_j, f_1 f_2, f_3, f_3, \ldots]$ with $i \geq 5$. As in the $(c - 2)$-case, such terms may be expressed as products of basis elements. However, in the present case the resulting basis elements of the form

$$[f_1, f_k; f_1, f_k, \ldots]$$

will appear with $k \in \{i, j, 3\}$ only, and it follows that $w' \in [F'', F] \cdot \gamma_{c+1}(F)$. If either (i) or (ii) is the case, then identifying f_4 with f_3 will, as in the case for $c - 2$ generators, yield an element w' in $c - 4$ generators having the desired properties.

We shall now establish our main result.

THEOREM B. *For any $c \geq 5$,*

$$F \cap (1 + \mathbf{r} + \mathbf{f}^{c+1}) = [F'', F] \cdot \gamma_{c+1}(F),$$

where $\mathbf{r} = \mathbf{Z}F([F'', F] - 1)$.

Proof. Suppose $w \in F \cap (1 + \mathbf{r} + \mathbf{f}^{c+1})$, $w \notin [F'', F] \cdot \gamma_{c+1}(F)$. By corollary 4.5, we may assume that $w \in U_c(F)$, $c \geq 8$, and, by Lemma 4.8, that w involves at most $c - 4$ distinct generators, modulo $[F'', F] \cdot \gamma_{c+1}(F)$. Thus,

$$w = w_1 w_2, \quad w_1 \in U_c(F), \quad w_2 \in [F'', F] \cdot \gamma_{c+1}(F),$$

and it follows as before that

$$w_1\varphi = 1 + \sum a_i(r_i - 1)\varphi + s\varphi, \quad r_i \in [F'', F],\ s \in \mathbf{f}^{c+1},\ a_i \in \mathbf{Z}. \quad (4.6)$$

Since $w_1 \in \gamma_c(F)$,

$$\left\langle \sum a_i(r_i - 1)\varphi \right\rangle_k = 0 \quad \text{for } k < c.$$

Thus, by Lemma 4.2, we may assume that all $r_i \in \gamma_{c-2}(F)$ and, in particular, that the summand of those $r_i - 1$ with $r_i \in \gamma_{c-2}(F) \setminus \gamma_{c-1}(F)$ is a linear combination of $r_i - 1$ with $r_i \in [T_{c-3}(F), F]$. Since w_1 involves less than $c - 3$ distinct generators, it follows, by Lemma 2.3 (v), that these r_i are in

$$[F'', F, F] \cdot \gamma_{c-1}(F).$$

However, $[F'', F, F]$ is in the kernel of φ, so for the purpose of finding a solution to (4.6) we may in fact, assume that all $r_i \in \gamma_{c-1}(F)$. In particular, by Lemma 2.4, we may further assume that the summand with $r_i \in \gamma_{c-1}(F) \setminus \gamma_c(F)$ is a linear combination of terms $r_i - 1$ with $r_i \in [U_{c-2}(F), F]$. Let w_1 be expressed as a product of basis elements as given by Lemma 4.6, and, without loss of generality, suppose that terms of the form

$$[1, 2;\ p_1, \ldots, p_n]$$

occur in this product, where n is the number of generators involved in w_1, $n \leq c - 4$. For any such term $t = [1, 2;\ p_1, \ldots, p_n]$,

$$\langle t\varphi \rangle_c = \sum_{k=1}^{n} p_k z_k (z_1, z_2) z_1^{p_1} \cdots z_k^{p_k - 1} \cdots z_n^{p_n}(z_1, z_2). \quad (4.7)$$

By Lemma 4.6, p_n will be odd for this factor t of w_1. If all r_i in (4.6) are in $\gamma_c(F)$, the terms in the summand which will have left factor z_n will be those terms coming from $[F'', f_n]$, and, in particular, those with left factor $z_n z_1$ and right factor z_1 will come from products of terms of the form

$$[f_1, f_i;\ f_1, f_j, \ldots; f_n].$$

However, $(z_1, z_i;\ z_1, z_j, \ldots, z_n) = -z_n(z_1, z_i;\ z_1, z_j, \ldots) = 2 \cdot z_n z_1 \cdots z_1$

$+ \ldots$, so if there is a solution to (4.6), then not all r_i will be in $\gamma_c(F)$. Hence, in (4.6) we may assume that some of the r_i are in $[U_{c-2}(F), F]$, and, in particular, the terms of the form

$$v = \left[f_1, f_j; f_1, f_j, q_1 f_1, \ldots, q_m f_m; f_k\right] \in \gamma_{c-1}(F)$$

occur. Further, modulo $[F'', F, F]$, we may assume that q_m is odd, $m = n$, using the basis given in Lemma 4.6. However, if $f_k \neq f_n$, one summand of $\langle v\varphi\rangle_c$ is $z_k q_n(z_1, z_j) \ldots z_n^{q_n+1} \ldots (z_1, z_j)$, and since $q_n + 1$ is even, we observe from (4.7) that this term will not compare with one from $\langle w_1\varphi\rangle_c$. Further, since q_m is odd, there will be not term from

$$\langle (r_i - 1)\varphi\rangle_c, \quad r_i \in \gamma_c(F),$$

to compare with this term. Since, as observed above, such terms occur with a coefficient 2 or a multiple of 2. Hence, it follows that $f_k = f_n$. However, c is even and p_n is odd, so in each term of w_1 of the form (4.7) there must be a p_i, $i < n$, with p_i odd. Thus, there will be a term in the expansion (4.7) of this element with an odd coefficient p_i. By the remarks following (4.7) this term with odd p_i cannot be compared with a term from an $r_i - 1$, $r_i \in \gamma_c(F)$. Hence, if equation (4.6) is to be possible, there must be terms in

$$[U_{c-2}, F] \cap \gamma_{c-1}(F)$$

of the form

$$\left[f_1, f_j; f_1, f_j, \ldots; f_i\right].$$

Since $i \neq n$, this is a contradiction, which shows that (4.6) has a solution for $a_i \in \mathbf{Z}$ only if $w_1 \in [F'', F] \cdot \gamma_{c+1}(F)$, which completes the proof of the theorem.

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