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THE SOLUBILITY OF SETS OF EQUATIONS OVER GROUPS

BY

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In memoriam W. W. Boone

1. Introduction

We shall be concerned in this paper with finite sets of equations over a group G. Such a set, S, is

$$S\colon w_1 = w_2 = \cdots = w_n = 1,$$

where w_i, $i = 1, \dots, n$, is an element of the free product $G * F$ of G and the free group F freely generated by the variables $z_1, \dots, z_r$ entering into the equations. If H is a group containing G, the equations S can be solved *in* H if the inclusion map from G to H extends to a homomorphism from $G * F$ to H with w_i in the kernel, $i = 1, \dots, n$. The equations S can be solved *over* G if they can be solved in some group H containing G. It is clear from the theory of free products with a single amalgamated subgroup that any group G is contained in a group H such that every finite set of equations over G which is soluble over G is soluble in H. The case is altered, however, if we seek to find such an H which is in some sense small. The main theorem of this paper is about such a situation.

If N is the normal subgroup of $G * F$ normally generated by the elements w_i, $i = 1, \dots, n$, it is clear that S is soluble over G if and only if the intersection $G \cap N$ is trivial. Whether or not this is the case, we shall write $\langle G; S\rangle$ for the factor group $(G * F)/N$. Thus there is always a natural map from G to $\langle G; S\rangle$, which is an embedding if and only if S is soluble over G. A group H containing G is said to be *finitely presented over* G if it is isomorphic over G to $\langle G; S\rangle$ for some finite set S of equations over G which is soluble over G. Then our main theorem is:

THEOREM 1.1. *If the finitely generated group G is non-trivial there does not exists a group H, finitely presented over G, such that every finite set of equations over G which is soluble over G is soluble in H.*

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We shall use this theorem to put in a more general setting the result of K. Hickin and Angus Macintyre [2], that if finitely generated groups G, H and an existentially closed group M satisfy $G < M < H$, then G and H cannot be isomorphic. If the nontriviality condition is removed, the theorem becomes false, because every set of equations over a trivial group is soluble in the group itself. However, the existence of universal finitely presented groups shows that it fails for $G = 1$ in a more radical way. It could not be saved, for instance, by considering inequalities as well as equations.

The proof of Theorem 1.1 is recursion theoretic; it could, indeed, be said to arise from a consideration of the role of enumeration reducibility in combinatorial group theory. We recall some definitions. If X, Y are subsets of the set $\mathbf{N}$ of natural numbers we say that Y is *enumeration reducible* to X, and write $Y \leq_e X$, if there exists a recursively enumerable set U of pairs (m, A), $m \in \mathbf{N}$, A a finite subset of $\mathbf{N}$, such that $n \in Y$ if and only if $(n, A) \in U$ for some A contained in X. (For this and all other recursion theoretic ideas, see Rogers [4].) Evidently, if ρ, σ are recursive permutations of $\mathbf{N}$, $Y\sigma \leq_e X\rho$ if and only if $Y \leq_e X$. Thus if X, Y are subsets of effectively enumerated infinite sets E, F we can give an unambiguous meaning to the statement $Y \leq_e X$: it means that $Y\phi \leq_e X\theta$, where θ, ϕ are recursive bijections from E, F to $\mathbf{N}$.

Let G be a finitely generated group, generated by $a_1, \dots, a_r$. Let W_r be the set of words in the symbols $x_1, \dots, x_r, x_1^{-1}, \dots, x_r^{-1}$. Any element of W_r has a *value at* $a_1, \dots, a_r$, obtained by replacing each occurence of each letter x_i by a_i and evaluating the result as a product in G. Then $\mathrm{Rel}(a_1, \dots, a_r)$ is the set of words in W_r whose value at $a_1, \dots, a_r$ is 1. If $b_1, \dots, b_s$ also generate G, it is easy to see that each of $\mathrm{Rel}(a_1, \dots, a_r)$ and $\mathrm{Rel}(b_1, \dots, b_s)$ is $(1-1)$-reducible to the other. It follows (e.g., see Rogers [4]) that there is a recursive bijection from W_r to W_s carrying $\mathrm{Rel}(a_1, \dots, a_r)$ to $\mathrm{Rel}(b_1, \dots, b_s)$. We define $\mathrm{Rel}(G)$ to be the image of $\mathrm{Rel}(a_1, \dots, a_r)$ under a recursive bijection from W_r to $\mathbf{N}$. Thus $\mathrm{Rel}(G)$ is a subset of $\mathbf{N}$, determined only up to a recursive permutation of $\mathbf{N}$, but, subject to that ambiguity, independent of the generating set used to define it. Notice that the ambiguity implies that we cannot sensibly say, for instance, $\mathrm{Rel}(G) \subset \mathrm{Rel}(H)$; but that the limitation on the ambiguity implies that we can sensibly say, for instance, $\mathrm{Rel}(G) \leq_e \mathrm{Rel}(H)$.

Recall finally that though the elements of a finitely generated group G are not usually effectively enumerated, the words denoting them, in a fixed set of generators, are. We can therefore, properly apply recursion theoretic expressions, for instance, to sets of finite sets of equations over G. We are now ready to state the technical result on which the proof of Theorem 1.1 depends.

THEOREM 1.2. *Let $\{S_i, i \in \mathbf{N}\}$ be a sequence of finite sets of equations over the finitely generated group G, and suppose that S_i depends recursively on i. Then the following statements are equivalent*:

(i) *There exists a group H, finitely presented over G, such that whenever S_i is soluble over G it is soluble in H.*

(ii) *If $J = \{i \in \mathbf{N} \mid S_i$ is soluble over $G\}$ then $J \leq_e \mathrm{Rel}(G)$.*

It is possible that this theorem has independent interest, if only as a further example of the natural way in which recursion-theoretic ideas arise in group theory.

2. Use of the relative subgroup theorem

The first steps in the exploration of the role of enumeration reducibility in group theory were taken by C. F. Miller III (unpublished) and M. Ziegler [5], who obtained independently the following result.

THEOREM 2.1 (RELATIVE SUBGROUP THEOREM). *If G and H are finitely generated groups then a necessary and sufficient condition for H to be isomorphic to a subgroup of a group finitely presented over G is that* $\mathrm{Rel}(H) \leq_e \mathrm{Rel}(G)$.

Of course, this theorem relativises the subgroup theorem for finitely presented groups, the statement, that is, that a finitely generated group G is a subgroup of a finitely presented group if and only if $\mathrm{Rel}(G)$ is recursively enumerable. Like the theorem it relativises, Theorem 2.1 implies the existence of a certain kind of universal group.

COROLLARY 2.2. *Given any finitely generated group G, there exists a group H, finitely presented over G, such that every group finitely presented over G is isomorphic to a subgroup of H.*

It is worth-while dwelling for a moment on the relationship between Corollary 2.2 and Theorem 1.1. By definition, the group H of the corollary contains G, that is, we have an inclusion map ι: $G \to H$. If S is a finite set of equations over G which is soluble over G then there is a natural embedding of G in $\langle G, S\rangle$, and, by the corollary, an embedding of $\langle G, S\rangle$ in H. The composite is an embedding $\alpha(S)$: $G \to H$. There is no reason whatever to suppose that $\alpha(S) = \iota$; and, indeed, what Theorem 1.1 says is precisely that it cannot be true for every soluble set S that $\alpha(S) = \iota$. For a single set S, the situation is easily rectified. For it is sufficient to make the embeddings ι and $\alpha(S)$ conjugate, and we can do this in an HNN-extension $\langle H, t\rangle$, which will be finitely presented over H, and hence over G, because G is finitely generated. An almost identical argument works for a finite set of sets S, and a rather similar one, using the subgroup theorem for finitely presented groups, works for an infinite set, provided that it is recursively enumerable. What Theorem 1.2 tells us is how far we can relax the condition that the set be recursively enumerable, and still make the argument work. Lemmas 3.1 and 3.2 below tell us that this relaxation is insufficient to include all soluble sets.

We turn next to the proof of Theorem 1.2, beginning with the relatively trivial implication (i) $\to$ (ii).

LEMMA 2.3. *Let* $\{S_i\ i \in \mathbf{N}\}$ *be a sequence of finite sets of equations over the finitely generated group* G, *and suppose that* S_i *depends recursively on* i. *If* $J_0 = \{i \mid S_i \text{ is soluble in } G\}$ *then* $J_0 \leq_e \operatorname{Rel}(G)$.

Suppose that G is generated by $a_1, \dots, a_r$, and consider first a single finite set S of equations over G. We may take S to be the set $w_1 = \cdots = w_n = 1$, where each w_i is a word in the generators $a_1, \dots, a_r$, the variables $z_1, \dots, z_m$, and their inverses. Let $U(S)$ be the set of sets of words in $x_1, \dots, x_r, x_1^{-1}, \dots x_r^{-1}$ obtained from $\{w_1, \dots, w_n\}$ by substituting uniformly $x_1, \dots, x_r$ for $a_1, \dots, a_r$ and $u_1, \dots, u_m$ for $z_1, \dots, z_m$, where $u_1, \dots, u_m$ are arbitrary words in $x_1, \dots, x_r, x_1^{-1}, \dots x_r^{-1}$. Evidently $U(S)$ is a recursively enumerable set, depending recursively on S, and S is soluble in G if and only if $A \subset \operatorname{Rel}(a_1, \dots, a_r)$ for some S in A. Thus if

$$V = \{(i, A) | A \in U(S_i)\},$$

V is recursively enumerable and $i \in J_0$ if and only if $(i, A) \in V$ for some A with $A \subset \operatorname{Rel}(a_1, \dots, a_r)$. Thus $J_0 \leq_e \operatorname{Rel}(a_1, \dots, a_r) \equiv_e \operatorname{Rel}(G)$ as required.

COROLLARY 2.4. *With the hypotheses of Theorem* 1.2, (i) *implies* (ii).

If (i) holds, Lemma 2.3 gives that $J \leq_e \operatorname{Rel}(H)$. But H is finitely presented over G, so that, by Theorem 2.1, $\operatorname{Rel}(H) \equiv_e \operatorname{Rel}(G)$, giving (ii).

Turning now to the reverse implication, we carry out the necessary construction in a slightly more general situation than the theorem requires.

LEMMA 2.5. *Let* G *be a finitely generated group, and let* $\{S_i, i \in \mathbf{N}\}$ *be a recursively enumerable sequence of sets of equations over* G. *Let* J *be a subset of* $\mathbf{N}$ *such that* (a) S_i *is soluble over* G *for all* i *in* J *and* (b) $J \leq_e \operatorname{Rel}(G)$. *Then there exists a group* H *which is finitely presented over* G *such that* S_i *is soluble in* H *for all* i *in* J.

Form first the free product H_0 of G and of all the groups $\langle G, S_i \rangle$, $i \in \mathbf{N}$. There are natural maps λ, μ_i of G into H_0, such that $G\lambda$ is the extraneous factor G and $G\mu_i$ is contained in $\langle G, S_i \rangle$, $i \in \mathbf{N}$. Notice that λ is an embedding, and so is μ_i if S_i is soluble over G, in particular if $i \in J$, and that the map μ_i extends to a map of $\langle G, S_i \rangle$ into H_0. We shall embed H_0 in an increasing sequence of groups, and, by a systematic abuse of notation, we shall use λ, μ_i also for the corresponding maps into the larger groups; the properties of λ and μ_i that we have just noticed will be unaffected by this ambiguity in their denotation. We first embed H_0 in H_1, which is generated by H_0 and generators t_i, $i \in \mathbf{N}$, with relations

$$t_i^{-1} g\lambda t_i = g\mu_i \text{ for all } g \in G, \quad \text{whenever } i \in J.$$

Since λ and μ_i, $i \in J$, are both embeddings, H_1 is an HNN-extension of H_0, and so does indeed embed it. (The generators t_i, $i \in \mathbf{N} \setminus J$ are included solely to make it easier to keep track of the recursive properties of relations).

Next, we embed H_1 in a two-generator group H_2, using the standard method of [3]. Because we need to keep track of relations, it is simplest to follow Ziegler [5], who extracted the essentials of the result in [3], and added a little extra precision, to give it the form:

There exists a recursive sequence $\{w_i, i \in \mathbf{N}\}$ *of words in* $x, y, x^{-1} y^{-1}$ *such that, for any sequence* $\{g_i, i \in \mathbf{N}\}$ *of elements in any group* G, *the equations* $w_i = g_i$, $i \in \mathbf{N}$, *can be solved over* G.

For our purposes, we rewrite the sequence $\{w_i\}$ as a double sequence $\{w_{i,j}, i, j \in \mathbf{N}\}$ with $w_{i,j}$ recursive in (i, j). We also need an explicit listing of a set of generators of H_1. Let $G = \langle a_0, a_1, \dots, a_{r-1} \rangle$, and denote by $x_0^{(i)}, \dots, x_{k(i)}^{(i)}$ both the variables in the equations S_i, and also the corresponding elements of $\langle G, S_i \rangle$. Then H_1 is generated by the elements $a_i\lambda$, $i = 0, \dots, r-1$, $a_i\mu_j$, $i = 0, \dots, r-1$, $j \in \mathbf{N}$, $x_j^{(i)}$, $j = 0, \dots, k(i)$, $i \in \mathbf{N}$, and t_i, $i \in \mathbf{N}$. We embed H_1 in the two-generator group $H_2 = \langle x, y \rangle$, by solving freely over H_1 the equations

$$\begin{aligned} w_{0,j} &= t_j, \quad j \in \mathbf{N}, \\ w_{1,j} &= a_j\lambda, \quad j = 0, \dots, r-1, \\ w_{i+2,j} &= a_j\mu_i, \quad j = 0, \dots, r-1, i \in \mathbf{N}, \\ w_{i+2,r+j} &= x_j^{(i)}, \quad j = 0, \dots, k(i), i \in \mathbf{N}. \end{aligned}$$

There are then three blocks of relations between x and y which can be taken as defining relations for H_2. First, there are the relations which guarantee that the homomorphisms λ, μ_i of G into H_2 really are homomorphisms. These are

R_1 $u(w_{i,0}, \dots, w_{i,r-1}) = 1$ for all words u such that $u(a_0, \dots, a_{r-1}) = 1$ and for $i \in \mathbf{N}$ except $i = 0$.

Evidently $R_1 \leq_e \mathrm{Rel}(G)$, since $a_0, \dots, a_{r-1}$ are generators of G. Second, there are relations which express the fact that the additional generators of the $\langle G, S_i \rangle$ are solutions of the equations S_i. These are

R_2 $u(w_{i+2,0}, \dots, w_{i+2,r+k(r)}) = 1$ whenever the equation

$$u\left(a_0, \dots, a_{r-1}, x_0^{(i)}, \dots, x_{k(r)}^{(i)}\right) = 1$$

belongs to S_i, $i \in \mathbf{N}$.

Then R_2 is recursively enumerable; this is practically what we mean when we say that the sequence $\{S_i, i \in \mathbf{N}\}$ is recursively enumerable. Finally, we need relations to ensure that the elements t_i perform the conjugations they are supposed to. These are

$$R_3 \quad w_{0,r}^{-1} w_{1,j} w_{0,i} = w_{i+2,j}, \; i \in J, \; j = 0, \dots, r-1.$$

Evidently $R_3 \leq_e J$; and since by assumption $J \leq_e \mathrm{Rel}(G)$, we have $R_3 \leq_e \mathrm{Rel}(G)$. Thus the set $R_1 \cup R_2 \cup R_3$, which is a set of defining relations for H_2, satisfies $R_1 \cup R_2 \cup R_3 \leq_e \mathrm{Rel}(G)$. It follows that $\mathrm{Rel}(H_2) \leq_e \mathrm{Rel}(G)$.

By Theorem 2.1, H_2 can be embedded in a group H_3 which is finitely presented over G. Finally, we set $H_4 = \langle H_3, z;\; z^{-1} g \iota z = g\lambda,\; g \in G \rangle$, where ι is the inclusion map $G \to H_3$, so that H_4 is an HNN-extension of H_3. We show that $H = H_4$ satisfies the requirements of the lemma. First, H_4 is finitely presented over H_3 because G is finitely generated, and hence it is finitely presented over G. Second, if $i \in J$, λ and μ_i are conjugate even as maps into H_1, and ι and λ are conjugate as maps into H_4. Thus ι and μ_i are conjugate as maps into H_4. But the map $\mu_i \colon G \to H_4$ extends to a map $\langle G, S_i \rangle \to H_4$, and hence so does the conjugate map ι. This implies that S_i is soluble in H_4.

This concludes the proof of Lemma 2.5. Obviously, this lemma includes as a special case the implication (ii) $\to$ (i) in Theorem 1.2; so taken with Corollary 2.4 it completes the proof of that theorem.

3. Conclusion

To derive Theorem 1.1 from Theorem 1.2 requires two more lemmas. Both are probably well known, but we give proofs for completeness.

LEMMA 3.1. *Let G, H be finitely generated groups with $G \neq 1$ and* $\mathrm{Rel}(H) \leq_e \mathrm{Rel}(G)$. *Then there exists a sequence $\{S_i, i \in \mathbf{N}\}$ of finite sets of equations over G such that S_i depends recursively on i, and if $J = \{i \mid S_i$ is soluble over $G\}$ then $J \equiv_e \mathbf{N} \setminus \mathrm{Rel}(H)$.*

This depends on the fact that if $g \neq 1$, h are elements of a group G, then the equation $g = y^{-1} h y \cdot z^{-1} h z$ is soluble over G for y and z if and only if $h \neq 1$. We can suppose, by Theorem 2.1, that H is a subgroup of a group $\langle G, S \rangle$, where S is a finite set of equations over G which is soluble over G. Since G is nontrivial, we can find g in G, $g \neq 1$. If $b_1, \dots, b_r$ generate H, we can find a recursive map θ from W_r (the set of words in $x_1, \dots, x_r, x_1^{-1}, \dots, x_r^{-1}$) to the set of words in the generators of G and the variables of S, such that $w\theta = w(b_1, \dots, b_r)$. If we write S_w for the union of S and the single equation $g = y^{-1}(w\theta) y z^{-1}(w\theta) z$, then the set S_w depends recursively on w, and is soluble over G if and only if

$$w \in W_r \setminus \mathrm{Rel}(b_1, \ldots, b_r).$$

Reindexing the set $\{S_w, w \in W_r\}$ by a recursive bijection from W_r to $\mathbf{N}$ gives the result.

LEMMA 3.2. *If G is a finitely generated group, there exists a finitely generated group H such that* $\mathrm{Rel}(H) \leq_e \mathrm{Rel}(G)$ *but* $\mathbf{N} \setminus \mathrm{Rel}(H) \nleq_e \mathrm{Rel}(G)$.

The corresponding set theoretic result, that there exists a subset X of $\mathbf{N}$ such that $X \leq_e \mathrm{Rel}(G)$ but $\mathbf{N} \setminus X \nleq_e \mathrm{Rel}(G)$, can be found in Rogers [4]. Given X, it is sufficient to take $H = \langle a, b, c, d;\ a^{i+1}b^{i+1} = c^{i+1}d^{i+1},\ i \in X\rangle$.

Now let G be any nontrivial finitely generated group. Construct H as in Lemma 3.2, and then the sequence $\{S_i, i \in \mathbf{N}\}$ as in Lemma 3.1. Then the set $J = \{i \mid S_i$ is soluble over $G\}$ satisfies $J \nleq_e \mathrm{Rel}(G)$. By Theorem 1.2, there is no group finitely presented over G in which every set S_i soluble over G can be solved. This proves Theorem 1.1.

We have the following corollaries.

COROLLARY 3.3. *If G, H are finitely generated groups and M is existentially closed, and $G < M < H$, then H is not finitely presented over G.*

We can clearly assume $G \neq 1$, and then this follows from Theorem 1.1 and the fact that every finite set of equations soluble over G is soluble in M.

COROLLARY 3.4. *No existentially closed group is embeddable in a finitely presented group.*

This is immediate from Corollary 3.3. There are, however, many other roads to it. For instance, one can use the theorem of [1], since an existentially closed group is simple, and has finitely generated subgroups with insoluble word problem.

COROLLARY 3.5. *Under the hypotheses of Corollary* 3.3, *G is not isomorphic to H.*

This is the result of Hickin and Macintyre [2] referred to earlier. To prove it, note that if $G < M < H$ with G and H isomorphic, then $G < M < H \leq K$, where K is an HNN-extension making G and H conjugate. Because G is finitely generated, K is finitely presented over H. But H and G are conjugate in K, so K is finitely presented over G. This contradicts Corollary 3.3.

We mention, lastly, that the twist in the derivation of Corollary 3.5 is necessary.

FACT 3.6. *There exist finitely generated groups G, H with $G < H$, G isomorphic to H, but H not finitely presented over G.*

The construction is left to the reader.

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