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GENERATING VARIETIES OF LATTICE-ORDERED GROUPS: APPROXIMATING WREATH PRODUCTS

BY

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In fond memory of Bill Boone

0. Introduction

In this note we will be concerned with varieties of lattice-ordered groups, finitely presented lattice-ordered groups and wreath products of lattice-ordered groups.

The totally ordered group of integers **Z** is finitely presented as a lattice-ordered group: $\mathbf{Z} = \langle x; x \wedge 1 = 1 \rangle$, where 1 denotes the group identity. Moreover, the variety $\mathfrak{A}$ of Abelian lattice-ordered groups is the smallest variety of lattice-ordered groups containing **Z** [14]. So certain "natural" non-trivial varieties of lattice-ordered groups are generated by a single finitely presented lattice-ordered group. Note that "finitely presented" means in the variety of *all* lattice-ordered groups, not in the subvariety being considered. Now if G and H are finitely presented lattice-ordered groups and generate varieties $\mathfrak{U}$ and $\mathfrak{V}$ respectively, then clearly $G \boxplus H$ generates $\mathfrak{U} \vee \mathfrak{V}$, where $G \boxplus H$ is the ordered direct product of G and H where $(g, h) \geq 1$ if and only if $g \geq 1$ (in G) and $h \geq 1$ (in H). Further, $G \boxplus H$ is finitely presented: take as generators the disjoint union of the generating sets $\{g_i \colon i \in I\}$ of G and $\{h_j \colon j \in J\}$ of H, and as defining relations the union of those of G and H together with $|g_i| \wedge |h_j| = 1$ $(i \in I, j \in J)$, where $|x| = x \vee x^{-1}$. Hence the set of varieties of lattice-ordered groups generated by single finitely presented lattice-ordered groups is a join semilattice of the lattice of varieties of lattice-ordered groups.

If $\mathfrak{U}$ and $\mathfrak{V}$ are each generated by a single finitely presented lattice-ordered group, what about $\mathfrak{U} \cap \mathfrak{V}$ and $\mathfrak{U}\mathfrak{V}$? In the case that $\mathfrak{V} = \mathfrak{A}$ which is generated by **Z**, the answer is yes. By [14], $\mathfrak{U} \cap \mathfrak{A} = \mathfrak{A}$ except when $\mathfrak{U}$ is the variety defined by $\forall x \forall y\ (x = y)$. The main result in this paper is therefore:

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THEOREM. *If* $\mathfrak{U}$ *is a variety of lattice-ordered groups generated by a single finitely presented lattice-ordered group, then so is* $\mathfrak{U}\mathfrak{A}$.

Since $\mathfrak{U}\mathfrak{A}$ is generated by $G \operatorname{wr} \mathbf{Z}$ if $\mathfrak{U}$ is generated by G [6, Theorem 4.2], the proof of the theorem requires us in some sense to approximate $G \operatorname{wr} \mathbf{Z}$ by a finitely presented lattice-ordered group.[2] The construction is far more general than that given in [4], where the finitely presented lattice-ordered group constructed from G was a sublattice subgroup of $G \operatorname{wr} \mathbf{Z}$ having word and conjugacy problems of the same degrees as those of G.[3] We will show that the finitely presented lattice-ordered group H obtained in the proof of the theorem from G has word and conjugacy problems of the same degrees as those of G.

Let $\mathfrak{S}_n$ denote the Scrimger n variety, $1 < n \in \omega = \{0, 1, 2, \dots\}$ [13]. As noted in [4], $\mathfrak{S}_n$ is generated by the finitely presented lattice-ordered group G_n for all $1 < n$. Hence, as an immediate corollary to the theorem and the above we have:

COROLLARY. *Let* $m \in \omega$ *and* $1 < n \in \omega$.

(i) $\mathfrak{A}^m$ *is generated by a finitely presented lattice-ordered group with soluble conjugacy problem.*

(ii) $\mathfrak{S}_n\mathfrak{A}^m$ *is generated by a finitely presented lattice-ordered group with soluble conjugacy problem.*

Thus we obtain a countably infinite set of distinct "natural" varieties of lattice-ordered groups each of which is generated by a single finitely presented lattice-ordered group. As there is only a countably infinite set of non-isomorphic finitely presented lattice-ordered groups, this is the maximum number possible. Since there are continuum many varieties of lattice-ordered groups [11], most varieties are not so generated.

1. Background definitions and results

A convex normal sublattice subgroup of a lattice-ordered group is called an *ideal*. Ideals are precisely the kernels of homomorphisms between lattice-ordered groups. (Of course, homomorphism is with respect to both the group and lattice operations.)

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[2] The H constructed is a sublattice subgroup of $[(G \boxplus \mathbf{Z}) \operatorname{Wr} \mathbf{Z}] \boxplus A_2$, where

$$A_2 = \langle x, y;\ x \wedge 1 = 1,\ x \wedge y = x,\ xy = yx \rangle.$$

[3] The finitely presented lattice-ordered group in [4] does not generate $\mathfrak{U}\mathfrak{A}$.

For any lattice-ordered group G, let $\mathbf{v}\{G\}$ denote the intersection of all varieties of lattice-ordered groups that contain G. So $\mathbf{v}\{G\}$ is itself a variety of lattice-ordered groups.

If $\mathfrak{U}$ and $\mathfrak{V}$ are varieties of lattice-ordered groups, then $\mathfrak{U}\mathfrak{V}$ is defined by: $G \in \mathfrak{U}\mathfrak{V}$ if and only if there is an ideal N of G such that $N \in \mathfrak{U}$ and $G/N \in \mathfrak{V}$ (see [6] but contrast with [12] where this definition would give $\mathfrak{V}\mathfrak{U}$). It is indeed a variety of lattice-ordered groups [12]. Now $\mathbf{v}\{G\}\mathfrak{A} = \mathbf{v}\{G \operatorname{wr} \mathbf{Z}\}$ [6, Theorem 4.2] so the theorem states that if G is any finitely presented lattice-ordered group, there is a finitely presented lattice-ordered group H such that $\mathbf{v}\{H\} = \mathbf{v}\{G \operatorname{wr} \mathbf{Z}\}$; i.e., H and $G \operatorname{wr} \mathbf{Z}$ generate the same variety of lattice-ordered groups.

If G is a lattice-ordered group and $g \in G$, then $|g| = g \vee g^{-1} \geq 1$ and $|g| = 1$ if and only if $g = 1$. Moreover, $g = (g \vee 1)(g^{-1} \vee 1)^{-1}$ [1, 1.3.3, 1.3.10 & 1.3.11]. Hence if $\{g_1, \ldots, g_m\}$ generates G, so does

$$\{g_1 \vee 1, g_1^{-1} \vee 1, \ldots, g_m \vee 1, g_m^{-1} \vee 1\};$$

i.e., the generators of any finitely generated lattice-ordered group can be assumed to be greater than or equal to 1. Further, if $r_1(\mathbf{g}), \ldots, r_n(\mathbf{g})$ are any elements of the free lattice-ordered group on these generators, then

$$r_1(\mathbf{g}) = 1 \,\&\, \cdots \,\&\, r_n(\mathbf{g}) = 1$$

if and only if

$$|r_1(\mathbf{g})| \vee \cdots \vee |r_n(\mathbf{g})| = 1.$$

Therefore any finitely presented lattice-ordered group can be written in the form

$$\langle g_1, \ldots, g_m; r(\mathbf{g}) = 1 \rangle, \quad \text{where } g_i \geq 1 \quad (1 \leq i \leq m).$$

Throughout, the following standard notation will be used: a^b for $b^{-1}ab$; a^{-b} for $(a^{-1})^b$; $[a, b]$ for $a^{-1}b^{-1}ab$; $a \ll b$ for $a^m \leq b$ for all $m \in \omega$.

The only way I can prove the theorem is to use some results on order-preserving permutations of totally ordered sets. The following can be found in [2].

Let $A(\mathbf{R}) = \operatorname{Aut}(\langle \mathbf{R}, \leq \rangle)$, the lattice-ordered group of all order-preserving permutations of the real line, the group operation being composition and the order being pointwise ($f \leq g$ if and only if $\alpha f \leq \alpha g$ for all $\alpha \in \mathbf{R}$). The *support* of $g \in A(\mathbf{R})$ is denoted by

$$\operatorname{supp}(g) = \{\alpha \in \mathbf{R}: \alpha g \neq \alpha\}.$$

If for all (any) $\alpha_0 \in \operatorname{supp}(g)$ the convexification of $\{\alpha_0 g^n: n \in \mathbf{Z}\}$ in $\mathbf{R}$ is the entire support of g, then g is said to have *one bump*. Such a g is called a *bump of* f if $\alpha f = \alpha g$ for all $\alpha \in \text{supp}(g)$. More generally, h is said to be a *set of bumps of* f if every bump of h is a bump of f; so $\alpha f = \alpha h$ for all $\alpha \in \text{supp}(h)$ and $[f, h] = 1$ in this case. The following are easy to prove:

LEMMA 1 [2, Lemma 1.9.1]. *If* $1 \leq f, g \in A(\mathbf{R})$ *and*

$$\text{supp}(f) \cap \text{supp}(g) = \varnothing,$$

then $[f, g] = 1$.

LEMMA 2 [2, Lemma 1.9.3]. *If* $1 \leq f, h \in A(\mathbf{R})$, *then* $h \wedge fh^{-1} = 1$ *if and only if* h *is a set of bumps of* f. *Hence* $[f, h] = 1$ *if* $h \wedge fh^{-1} = 1$.

LEMMA 3 [2, Lemma 1.9.4]. *If* $1 \leq f, g \in A(\mathbf{R})$ *and* $f \wedge f^g = 1$, *then* $f \ll g$.

In order to use these results we need a consequence of an analogue of Cayley's theorem for groups:

LEMMA 4 [2, Corollary 2L]. *Any countable lattice-ordered group is isomorphic to a sublattice subgroup of* $A(\mathbf{R})$.

Actually, by [2, Corollary 2L], any countable lattice-ordered group is isomorphic to a sublattice subgroup of the lattice-ordered group of all order-preserving permutations of the rationals. Since this latter lattice-ordered group can clearly be embedded in $A(\mathbf{R})$, the lemma follows.

We will always identify a countable lattice-ordered group with its associated sublattice subgroup of $A(\mathbf{R})$; so Lemmas 1–3 can then be applied to countable lattice-ordered groups.

For the notation, definitions and properties of wreath products, see [2, Section 5.1].

2. Proof of the theorem

Rephrasing the theorem in the notation of §1, we have:

THEOREM. *If* G *is a finitely presented lattice-ordered group, then* $\mathbf{v}\{G\}\mathfrak{A} = \mathbf{v}\{H\}$ *for some finitely presented lattice-ordered group* H.

Proof. Let $G = \langle g_0, \ldots, g_{m_0}; r(\mathbf{g}) = 1 \rangle$ generate $\mathfrak{U}$. As noted above, we may assume that each $g_i \geq 1$. Furthermore, by adding an extra generator and relation we may assume that $g_1 \vee \cdots \vee g_{m_0} = g_0$, and incorporate this into $r(\mathbf{g})$.

Let $H = \langle a, g_0, \ldots, g_{m_0}, h_0;\ r(\mathbf{g}) = 1,\ a \wedge h_0 = h_0,\ g_0 \wedge h_0 = 1,\ h_0 g_0^{-a} \wedge g_0^a = 1,\ h_0 h_0^{-a} \wedge h_0^a = 1\rangle$.

By Lemma 4, we may assume that H is a sublattice subgroup of $A(\mathbf{R})$.

Since $h_0 h_0^{-a} \geq 1$, an easy induction shows that $h_0 \geq h_0^{a^m}$ for all $m \in \omega$. Since $h_0 g_0^{-a} \geq 1$, $h_0^{a^m} \geq g_0^{a^{m+1}}$ for all $m \in \omega$. Hence $h_0 \geq g_0^{a^{m+1}} \geq 1$ for all $m \in \omega$. But $g_0 \wedge h_0 = 1$; so $g_0 \wedge g_0^{a^{m+1}} = 1$ for all $m \in \omega$. Thus $g_0^{a^m} \wedge g_0^{a^n} = 1$ if m and n are distinct integers. By Lemma 1, $[g_0^{a^m}, g_0^{a^n}] = 1$ for all $m, n \in \mathbf{Z}$.

Suppose that $h_0^{a^m}$ is a set of bumps of $h_0^{a^n}$ for some $m \geq n$. Then as $h_0 h_0^{-a} \wedge h_0^a = 1$, $h_0^{a^m} h_0^{-a^{m+1}} \wedge h_0^{a^{m+1}} = 1$. So if $\alpha \in \operatorname{supp}(h_0^{a^{m+1}})$, then $\alpha h_0^{a^{m+1}} = \alpha h_0^{a^m} = \alpha h_0^{a^n}$ by hypothesis. Thus $h_0^{a^n} h_0^{-a^{m+1}} \wedge h_0^{a^{m+1}} = 1$. It follows by Lemma 2 and induction that $h_0^{a^m}$ is a set of bumps of $h_0^{a^n}$ whenever $m \geq n$. Hence $[h_0^{a^m}, h_0^{a^n}] = 1$ for all $m, n \in \mathbf{Z}$ by Lemma 2. Also, by the same argument, $g_0^{a^m}$ is a set of bumps of $h_0^{a^n}$ whenever $m > n$. Hence $[g_0^{a^m}, h_0^{a^n}] = 1$ for all $m, n \in \mathbf{Z}$ by Lemma 2.

Let N be the ideal of G generated by $g_0, \ldots, g_{m_0}, h_0$. Since $g_0 \wedge g_0^a = 1$ & $1 \leq h_0 \leq a$, $g_0 \ll a$ by Lemma 3. Thus $g_i \ll a$ $(0 \leq i \leq m_0)$. Moreover

$$(a^2)^{g_i^\varepsilon} = g_i^{-\varepsilon} a^2 g_i^\varepsilon = a^2 (g_i^{-\varepsilon})^{a^2} g_i^\varepsilon \geq a \quad (\varepsilon = \pm 1) \quad \text{and} \quad g_i^{h_0} = g_i;$$

so the ideal generated by $g_0, \ldots, g_{m_0}$ is very much less than a. As g_0^a is a set of bumps of h_0, $a \notin N$. Thus H/N is generated by a and so $H/N \in \mathfrak{A}$. Therefore to prove that $H \in \mathfrak{U}\mathfrak{A}$ it is enough to show that $N \in \mathfrak{U}$.

Let $\alpha \in \operatorname{supp}(h_0^{a^n}) \setminus \operatorname{supp}(h_0^{a^{n+1}})$. Then α belongs to the support of a bump of $h_0^{a^n}$ that is not a bump of $h_0^{a^{n+1}}$. Hence the same is true of $\alpha h_0^{ra^n}$ for all $r \in \mathbf{Z}$. Thus $\alpha h_0^{ra^n} < \alpha a$ for all $r \in \mathbf{Z}$.

Next let h_1 be the join (in $A(\mathbf{R})$) of the set of bumps of h_0 that are disjoint from their conjugate by a, and h_2 the join (in $\mathfrak{A}(\mathbf{R})$) of the remaining set of bumps of h_0. Note that no claim is made that $h_1, h_2 \in H$. Moreover, if Δ is the support of a bump of h_2, then as $h_0^{a^m}$ is a set of bumps of h_0 for all $m \in \omega$, $h_0^{a^m} | \Delta = h_0 | \Delta$ for all $m \in \omega$. Hence if $m \in \omega$ and $\beta \in \operatorname{supp}(h_2)$, $\beta h_0^{a^m} a = \beta a h_0^{a^{m+1}} = \beta a h_0 = \beta a h_0^{a^m}$ since $\Delta a = \Delta$. Also observe that $g_0^{a^n} \wedge h_2 = 1 = h_1^{a^n} \wedge h_2$ for all $n \in \mathbf{Z}$.

Let f_0 be the join (in $\mathfrak{A}(\mathbf{R})$) of the bumps of $\{g_0^{a^n}\colon n \in \mathbf{Z}\}$ and f_1 the join (in $\mathfrak{A}(\mathbf{R})$) of the remaining set of bumps of h_1. Observe that

$$\operatorname{supp}(f_j) H = \operatorname{supp}(f_j) \quad (j = 0, 1).$$

Furthermore, any element of N when restricted to $\operatorname{supp}(f_0)$ is an element of $\Pi\{G\colon n \in \mathbf{Z}\} \in \mathfrak{U}$ since $N | \operatorname{supp}(f_1) \ll a$ and

$$h_0 | \Delta_n = \begin{cases} g_0^{a^n} | \Delta_n & \text{if } 0 < n \in \mathbf{Z} \\ 1 | \Delta_n & \text{if } 0 \geq n \in \mathbf{Z} \end{cases}$$

where $\Delta_n = \operatorname{supp}(g_0^{a^n})$. Also, on $\operatorname{supp}(f_1) \setminus \operatorname{supp}(f_0)$, any element of N is just a power of f_1. Therefore $N | \operatorname{supp}(f_1) \setminus \operatorname{supp}(f_0) \in \mathfrak{A} \subseteq \mathfrak{U}$. Finally, on $\mathbf{R} \setminus \operatorname{supp}(f_1)$ any element of N agrees with a finite join of a finite meet of $h_2^s a^t$

(s, t integers). Since $[h_2, a] = 1$ as noted in the previous paragraph,

$$N|\mathbf{R}\setminus \operatorname{supp}(f_1) \in \mathfrak{A} \subseteq \mathfrak{U}.$$

Consequently, $N \in \mathfrak{U}$ ($\operatorname{supp}(f_j)\bar{H} = \operatorname{supp}(f_j)$ ($j = 0, 1$)).[4] Thus $\mathbf{v}\{H\} \subseteq \mathfrak{U}\mathfrak{A}$.

Let $\bar{g}_i$, $\bar{h}_0$, $\bar{a} \in G \operatorname{Wr} \mathbf{Z}$ be $(\{g_{i,n}\}, 0)$, $(\{h_{0,n}\}, 0)$ and $(\{0\}, 1)$ respectively, where

$$g_{i,n} = \begin{cases} g_i & \text{if } n = 0 \\ 1 & \text{if } n \neq 0 \end{cases} \quad \text{and} \quad h_{0,n} = \begin{cases} g_0 & \text{if } n > 0 \\ 1 & \text{if } n \leq 0 \end{cases}.$$

Then $\bar{g}_i$, $\bar{h}_0$, $\bar{a}$ $(0 \leq i \leq m_0)$ satisfy the defining relations of H and hence generate a sublattice subgroup A of $G \operatorname{Wr} \mathbf{Z}$ that is a homomorphic image of H. But the sublattice subgroup B of A generated by $\bar{g}_i$, $\bar{a}$ $(0 \leq i \leq m_0)$ is isomorphic to $G \operatorname{wr} \mathbf{Z}$, so as $G \operatorname{wr} \mathbf{Z}$ generates $\mathfrak{U}\mathfrak{A}$ [6, Theorem 4.2], $\mathfrak{U}\mathfrak{A} \subseteq \mathbf{v}\{H\}$.

3. The word and conjugacy problems for H

We now sketch that the word and conjugacy problems for the H constructed in the proof of the theorem are of the same degrees of those of G.

First observe that since $g_0, \ldots, g_{m_0} \ll a$ and g_0^a is a set of bumps of h_0, $w = \bigvee_j \bigwedge_j w_{ij} = 1$ in H with w_{ij} group words in $g_0, \ldots, g_m, h_0, a$ only if for some $i_0 \in I$, $\min_j e(w_{i_0 j}, a) = 0 \geq \min_j e(w_{ij}, a)$ for all $i \in I$, where $a^{e(w_{ij}, a)}$ is the result of replacing each occurrence of $g_0, \ldots, g_{m_0}, h$ by 1 in w_{ij}. If this condition is satisfied then consider what occurs on $\operatorname{supp}(f_0)$ using the algorithm for G. If this is the identity on $\operatorname{supp}(f_0)$, then consider what occurs on $\operatorname{supp}(f_1)$ using (i) h_0^a is a set of bumps of h_0, (ii) the disjointness of any bump of f_1 from its conjugate by a and (iii) $g_i^{a^k}$ is the identity on $\operatorname{supp}(f_1)$ $(0 \leq i \leq m_0,\ k \in \mathbf{Z})$. Clearly we can determine whether or not this is the identity on $\operatorname{supp}(f_1)$ using the technique (but with many cases deleted) in [7]. On $\operatorname{supp}(h_2)$, $g_i^{a^k}$ is the identity $(0 \leq i \leq m_0;\ k \in \mathbf{Z})$ and $[h_0, a] = 1$. Since the universal theory of abelian lattice-ordered groups is decidable [9], we can determine if w is the identity on $\operatorname{supp}(h_2)$. If any of these tests come up with a non-identity permutation of the requisite subset of $\mathbf{R}$, $w \neq 1$ in H; if they all yield the identity permutation on $\mathbf{R}$, $w = 1$ in H.

Since $[g_0^{a^m}, h_0^{a^n}] = 1$ for all $m, n \in \mathbf{Z}$ and g_0^a is a set of bumps of h_0, we can clearly adapt the above argument to determine conjugacy in H given an oracle for G. Hence we have:

COROLLARY. *The finitely presented lattice-ordered group H obtained in the proof of the theorem has word and conjugacy problems of the same degrees as those of G.*

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[4] The proof shows that H is a sublattice subgroup of

$$[(G \boxplus \mathbf{Z})\operatorname{Wr} \mathbf{Z}] \boxplus A_2 \quad \text{where} \quad A_2 = \langle x, y;\ x \wedge 1 = 1,\ x \wedge y = x,\ xy = yx \rangle.$$

4. Concluding remarks

The last paragraph of the proof of the theorem shows that $G \operatorname{wr} \mathbf{Z}$ is a homomorphic image of a sublattice subgroup of H with H and $G \operatorname{wr} \mathbf{Z}$ generating the same variety of lattice-ordered group. So, in some sense, H is a finitely presented approximation to $G \operatorname{wr} \mathbf{Z}$. Furthermore, the map $g_i \mapsto \bar{g}_i$ embeds G in $G \operatorname{wr} \mathbf{Z}$; thus the map $g_i \mapsto g_i$ embeds G in H. If I could prove that H had trivial centre (which I conjecture), there would be an alternative proof of [5, Corollary A4]: Every finitely presented lattice-ordered group can be embedded in one with trivial center. See [3] for other results on embedding finitely presented lattice-ordered groups in nice such.

As we saw, many well known varieties—e.g., $\mathfrak{A}^m$ $(m \in \omega)$—are generated by a single finitely presented lattice-ordered group. None of these varieties is generated by a set of totally ordered groups. Actually, if a lattice-ordered group G is a subdirect product of totally ordered groups, then $f \wedge f^g = 1$ implies $f = 1$ [1, Theorem 4.2.5], cf. Lemma 3. Moreover, if ξ is any irrational real number, then $(m, n) \geq (0, 0)$ if and only if $m + n\xi \geq 0$ gives a total order on $\mathbf{Z} \oplus \mathbf{Z}$; it is hard to imagine any single defining relation between generators that would determine ξ uniquely. For this reason I conjecture:

(1) The only totally ordered groups that are finitely presented as lattice-ordered groups are $\mathbf{Z}$ and $\{1\}$.

More generally:

(2) Is every subdirect product of totally ordered groups that is finitely presented as a lattice-ordered group abelian?

Since every nilpotent lattice-ordered group is a subdirect product of totally ordered groups (see [8] or [10]), a positive answer to (2) would imply that no non-abelian nilpotent lattice-ordered group can be finitely presented as a lattice-ordered group. If this at first seems strange, it should be pointed out that, for example,

$$[x \vee y, z] = \left([x, z] \vee x^{-1}y[y, z]\right) \wedge \left(y^{-1}x[x, z] \vee [y, z]\right)$$

in any lattice-ordered group. Hence there is no guarantee that $[a, b]$ is central implies that $[a^m \vee b^n, b]$ is for all $m, n \in \omega$.

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