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THE VARIATION OF VECTOR MEASURES AND CYLINDRICAL CONCENTRATION

BY

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The mutual relationship between the differentiability of vector measures and the smoothness properties of the associated cylindrical measures has often been used in the study of measure theory on vector spaces; see for example A. Goldman [4]. In [3] it was shown that a lifting is not always the best method for constructing weak densities for vector measures. If the notion of a "density" is relaxed somewhat, then it follows that a vector measure m will be differentiable with respect to a probability λ when the associated (m, λ)-distribution is σ-additive. Conversely, the existence of this weaker type of density suffices to guarantee the σ-additivity of the (m, λ)-distribution. We give a few examples to show the limitations of these techniques.

The construction of a regular density for a vector measure usually involves an argument utilizing its average range with respect to a probability. By appropriately defining the notion of the variation of a vector measure, we point out that the differentiability of a vector measure is an intrinsic property, independent of the associated scalar measure—a simple observation that allows us to complete the cycle relating the variation of m to the average range of m with respect to λ, and the cylindrical concentration of the (m, λ)-distribution.

In a number of locally convex spaces, indefinite integrals have special variational properties. By using the new notion of the variation of a vector measure, we give a number of conditions ensuring that a locally convex space has a form of variation property which has a bearing on the regularity of cylindrical measures defined on the space. As a by-product, a necessary and sufficient condition for the existence of a density for a vector measure with values in a space of regular Borel measures is given; it does not seem to have been stated explicitly previously.

Let E be a locally convex space. Let $\mathscr{Z}(E)$ be the smallest algebra, and $\mathscr{C}(E)$ the smallest σ-algebra for which elements of E' are measurable. The families $\mathscr{Z}(E)$ and $\mathscr{C}(E)$ contain all sets of the form $\varphi^{-1}(B)$ where φ: $E \to \mathbf{R}^k$ is a continuous linear map and B is a Borel subset of $\mathbf{R}^k$.

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An additive set function μ: $\mathscr{Z}(E) \to [0,1]$ is called a *cylindrical probability* if for each continuous linear map φ: $E \to \mathbf{R}^k$, $k \in \mathbf{N}$, the set function $\mu \circ \varphi^{-1}$ defined by $\mu \circ \varphi^{-1}(B) = \mu(\varphi^{-1}(B))$ for each Borel set B in $\mathbf{R}^k$ is a probability on the Borel σ-algebra $B(\mathbf{R}^k)$ of $\mathbf{R}^k$.

Denote by μ_σ the probability on $\mathscr{C}(E'^*)$ canonically induced by the cylindrical probability μ on E (E'^* is the algebraic dual of E', isomorphic to the completion of E in its weak topology). For a given measure ν on a σ-algebra of subsets of a set X, the outer measure on the family of all subsets of X is denoted by ν^*.

Let $\mathscr{F}$ be a family of subsets of E. The cylindrical probability μ is *scalarly concentrated* on $\mathscr{F}$ if, for every $\varepsilon > 0$, there is a set $F \in \mathscr{F}$ for which $(\mu \circ \xi^{-1})^*(\xi(F)) \geq 1 - \varepsilon$ for all $\xi \in E'$. It is *cylindrically concentrated* on $\mathscr{F}$, if for every $\varepsilon > 0$, there is a set $F \in \mathscr{F}$ for which

$$(\mu \circ \varphi^{-1})^*(\varphi(F)) \geq 1 - \varepsilon$$

for all continuous linear maps φ: $E \to \mathbf{R}^k$ and $k \in \mathbf{N}$.

Now let $\mathscr{B}$ be a saturated family of subsets of E [10]. The space E' with the topology of uniform convergence on elements of $\mathscr{B}$ is denoted by $E'_{\mathscr{B}}$. Let E'_ρ be the space E' with the locally convex topology ρ.

Suppose that $p > 0$. The cylindrical probability μ: $\mathscr{Z}(E) \to [0,1]$ has *p-th order moments* if $\mu(|\xi|^p) < \infty$ for every $\xi \in E'$. It has (ρ, p)*-moments* if the map T: $E' \to \mathbf{R}$ defined by $T(\xi) = \mu(|\xi|^p)$, $\xi \in E'$ is continuous on E'_ρ. If T is continuous on $E'_{\mathscr{B}}$ then μ is said to have $(\mathscr{B}, p)$*-moments*. A systematic study of cylindrical probabilities is given in [11].

If $(\Omega, \mathscr{S}, \lambda)$ is a probability space, then the indefinite integral of a function f: $\Omega \to E$ (in Pettis's sense) is denoted by $f\lambda$, with $\lambda(f) = f\lambda(\Omega)$.

Cylindrical measures and vector measures are related in the following manner. Denote by ι_σ the identity map on E'^*. Let τ be the Mackey topology of E. If μ has $(\tau, 1)$-moments, then ι_σ is μ_σ-integrable and

$$\iota_\sigma \mu_\sigma : \mathscr{C}(E'^*) \to E.$$

Conversely suppose that E is sequentially complete and that ι_σ is μ_σ-integrable with $\iota_\sigma \mu_\sigma$ taking values in E. Then μ has $(\tau, 1)$-moments.

Furthermore, if the vector measure m: $\mathscr{S} \to E$ is absolutely continuous with respect to the probability λ: $\mathscr{S} \to [0,1]$, then the (m, λ)*-distribution* μ is defined by

$$\mu \circ \varphi^{-1} = \lambda \circ (d\varphi \circ m/d\lambda)^{-1}$$

on the Borel σ-algebra of $\mathbf{R}^k$, for every continuous linear map φ: $E \to \mathbf{R}^k$, $k \in \mathbf{N}$. If E is sequentially complete, then the (m, λ)-distribution has $(\tau, 1)$-moments, and conversely, if μ has $(\tau, 1)$-moments, then μ is the $(\iota_\sigma \mu_\sigma, \mu_\sigma)$-distribution [4].

The bearing that the regularity of μ has on the existence of densities for m with respect to λ has been previously studied [4], so the question arises of what can be said if μ is only σ-additive or τ-additive. First we point out ways of showing when a cylindrical probability has $(\tau, 1)$-moments.

Let E and F be locally convex spaces. The family of continuous linear maps from E into F is denoted by $\mathscr{L}(E, F)$.

PROPOSITION 1. *Let $u \in \mathscr{L}(E, F)$. If μ has $(\tau, 1)$-moments on E, then $\mu \circ u^{-1}$ has $(\tau, 1)$-moments on F.*

Let μ be the (m, λ)-distribution on E. Then $\mu \circ u^{-1}$ is the $(u \circ m, \lambda)$-distribution on F.

Proof. Suppose that μ has 1st order moments and let $T\colon E' \to \mathbf{R}$ be the map defined by $T(\xi) = \mu(|\xi|)$, $\xi \in E'$. Let $e\colon \mathbf{R} \to \mathbf{R}$ be the identity map on $\mathbf{R}$. Then for each $\xi \in E'$, $T(\xi) = \mu \circ \xi^{-1}(|e|)$.

Now suppose that $\zeta \in F'$. Then

$$\mu \circ u^{-1}(|\zeta|) = \left(\mu \circ u^{-1}\right) \circ \zeta^{-1}(|e|) = \mu \circ (\zeta \circ u)^{-1}(|e|) = T(\zeta \circ u),$$

so the map $\zeta \mapsto \mu \circ u^{-1}(|\zeta|)$, $\zeta \in F'$ is defined and continuous for the Mackey topology on F' whenever μ has $(\tau, 1)$-moments.

If μ is the (m, λ)-distribution and $\varphi\colon F \to \mathbf{R}^k$ is a continuous linear map, then

$$\begin{aligned}\left(\mu \circ u^{-1}\right) \circ \varphi^{-1} &= \mu \circ (\varphi \circ u)^{-1} \\ &= \lambda \circ \left(d(\varphi \circ u) \circ m / d\lambda\right)^{-1} \\ &= \lambda \circ \left(d\varphi \circ (u \circ m)/d\lambda\right)^{-1}.\end{aligned}$$

Consequently, $\mu \circ u^{-1}$ is the $(u \circ m, \lambda)$-distribution. ∎

PROPOSITION 2. *Let μ be the (m, λ)-distribution on E and let $f\colon \Omega \to E'^*$ be a $\mathscr{C}(E'^*)$-measurable function such that $m = f\lambda$. Then $\mu_\sigma = \lambda \circ f^{-1}$ and $\iota_\sigma \mu_\sigma = m \circ f^{-1}$.*

Proof. Let $\varphi\colon E \to \mathbf{R}^k$ be a continuous linear map. Viewing E as a subspace of E'^* in the natural way, φ is the restriction to E of a uniquely defined linear map $\varphi_\sigma\colon E'^* \to \mathbf{R}^k$. Furthermore, $\mu \circ \varphi^{-1} = \mu_\sigma \circ \varphi_\sigma^{-1}$ on the Borel σ-algebra of $\mathbf{R}^k$ and we have

$$\mu \circ \varphi^{-1} = \lambda \circ \left(d\varphi \circ m / d\lambda\right)^{-1} = \lambda \circ \left(\varphi_\sigma \circ f\right)^{-1} = \left(\lambda \circ f^{-1}\right) \circ \varphi_\sigma^{-1},$$

so $\mu_\sigma = \lambda \circ f^{-1}$. Furthermore, for each $A \in \mathscr{C}(E'^*)$,

$$\iota_\sigma \mu_\sigma(A) = \iota_\sigma(\lambda \circ f^{-1})(A) = (f\lambda)\big(f^{-1}(A)\big) = m \circ f^{-1}(A). \qquad \blacksquare$$

PROPOSITION 3. *Let $\mathscr{B}$ be a saturated family of subsets of E for which $E'_{\mathscr{B}}$ is barrelled. Let μ be a cylindrical probability on E. If $p > 1$ and μ has pth order moments and is scalarly concentrated on $\mathscr{B}$, then μ has $(\mathscr{B}, p)$-moments.*

Proof. The proof is essentially that of Proposition 10.2 of [1]. Suppose first that μ has pth order moments. Then the set $S = \{\xi\colon \mu(|\xi|^p) \leq 1\}$ is disked and absorbing in E'. If S is closed in $E'_{\mathscr{B}}$ then S is a barrel, and because $E'_{\mathscr{B}}$ is assumed to be a barrelled space, S is a neighbourhood of zero in $E'_{\mathscr{B}}$. Consequently, μ has $(\mathscr{B}, p)$-moments.

To see that S is closed in $E'_{\mathscr{B}}$, suppose that $\xi_\iota \in S$ is a net converging to $\xi \in E'$ in $E'_{\mathscr{B}}$. Let $M\colon E' \to L^p(\mu_\sigma)$ be the random linear function associated with μ [11]. Then $M(\xi_\iota)$ converges to $M(\xi)$ in μ_σ-probability, because μ is scalarly concentrated on the family $\mathscr{B}$ [11].

Now for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $A \in \mathscr{C}(E'^*)$ with $\mu_\sigma(A^c) < \delta$,

$$\mu_\sigma(|M(\xi)|^p) < |M(\xi)|^p \mu_\sigma(A) + \varepsilon/2.$$

Choose ι such that

$$\mu_\sigma\big\{\big||M(\xi)|^p - |M(\xi_\iota)|^p\big| > \varepsilon/2\big\} < \delta.$$

Let $A = \{x \in E'^*\colon \big||\langle x, \xi\rangle|^p - |\langle x, \xi_\iota\rangle|^p\big| \leq \varepsilon/2\}$. Then

$$\mu(|\xi|^p) < |M(\xi)|^p \mu_\sigma(A) + \varepsilon/2 < |M(\xi_\iota)|^p \mu_\sigma(A) + \varepsilon < 1 + \varepsilon.$$

Therefore $\xi \in S$ and S is closed. $\blacksquare$

COROLLARY 1. *Let X be a Banach space. If the cylindrical probability $\mu\colon \mathscr{Z}(X) \to [0,1]$ has 1st order moments, then it has $(\beta, 1)$-moments if and only if it is scalarly concentrated on the balls of X.*

COROLLARY 2. *Let E be a locally convex space. The following conditions are equivalent.*

(i) *The space E is semireflexive.*

(ii) *A cylindrical probability $\mu\colon \mathscr{Z}(E) \to [0,1]$ has $(\tau, 1)$-moments if and only if it has 1st order moments and is scalarly concentrated on the bounded sets in E.*

A saturated family $\mathscr{B}$ of subsets of E is said to be *complete* if for each closed disked member $B \in \mathscr{B}$, the normed space E_B is complete. A set $A \subset E$

is said to be *$\mathscr{B}$-strictly weakly compact* if there exists a disked set $B \in \mathscr{B}$ such that $A \subset B$ and A is weakly compact in E_B. The saturated hull [10] of the $\mathscr{B}$-strictly weakly compact sets is denoted by $\mathscr{B}\mathscr{W}\mathscr{C}$.

The next theorem is an extension of a result of Schachermayer [9] to the locally convex space setting. It uses an analogue of Grothendieck's completeness theorem proved in [6].

THEOREM 1. *Let $\mathscr{B}$ be a complete saturated family of subsets of the locally convex space E. Let $p > 1$ and let μ: $\mathscr{Z}(E) \to [0,1]$ be a cylindrical probability on E. If*

(i) *μ has $(\mathscr{B}, p)$-moments, and*

(ii) *μ is scalarly concentrated on weakly compact convex subsets of E, then μ has $(\mathscr{B}\mathscr{W}\mathscr{C}, q)$-moments for every $0 < q < p$.*

Proof. Let M: $E' \to L^p(\mu_\sigma)$ be the linear random function associated with μ [11]. As μ has $(\mathscr{B}, p)$-moments, there exists a closed disked set $B \in \mathscr{B}$ such that $M(B^\circ)$ is a subset of the unit ball of $L^p(\mu_\sigma)$.

The image of the unit ball of $L^p(\mu_\sigma)$ in $L^q(\mu_\sigma)$ is uniformly integrable of order q for $1 \leq q < p$, so the topologies of $L^0(\mu_\sigma)$ and $L^q(\mu_\sigma)$ coincide there [2, p. 122]. Since μ is scalarly concentrated on weakly compact convex sets in E, M is $\tau(E', E)$-continuous into $L^0(\mu_\sigma)$ [11], so $M|B^\circ$ is $\tau(E', E)$-continuous into $L^q(\mu_\sigma)$. Now M: $E'_{\mathscr{B}} \to L^q(\mu_\sigma)$ is continuous, so from [6, Corollary 2.4] the map M is $\tau(E', E) - L^q(\mu_\sigma)$-continuous because the family $\mathscr{B}$ is complete.

Furthermore, $M(B^\circ)$ is relatively compact in $L^q(\mu_\sigma)$ because $p > 1$ and $1 \leq q < p$. From [6, Theorem 2.5], the map M: $E' \to L^q(\mu_\sigma)$ is continuous for the topology of uniform convergence on $\mathscr{B}\mathscr{W}\mathscr{C}$ for every $0 < q < p$. ■

COROLLARY 3. *Let E be a sequentially complete locally convex space. Let m: $\mathscr{S} \to E$ be a vector measure absolutely continuous with respect to the probability λ: $\mathscr{S} \to [0,1]$. Let $\mathscr{B}$ be a complete saturated family in E. Suppose that for some $p > 1$, $\lambda(|d\xi \circ m/d\lambda|^p) < \infty$ for every $\xi \in E'$, and the map*

$$\xi \mapsto \lambda(|d\xi \circ m/d\lambda|^p), \quad \xi \in E'$$

is continuous on $E'_{\mathscr{B}}$. Then for some closed disked set $B \in \mathscr{B}$, m: $\mathscr{S} \to E_B$ is a vector measure.

Proof. Let μ be the (m, λ)-distribution on E. Let e be the identity map on $\mathbf{R}$. Then for each $\xi \in E'$,

$$\lambda(|d\xi \circ m/d\lambda|^p) = \lambda \circ (d\xi \circ m/d\lambda)^{-1}(|e|^p) = \mu \circ \xi^{-1}(|e|^p) = \mu(|\xi|^p).$$

Thus μ has $(\mathscr{B}, p)$-moments and $(\tau, 1)$-moments. By Theorem 1, μ has $(\mathscr{B}\mathscr{W}\mathscr{C}, 1)$-moments.

Therefore there exists a disked set $D \in \mathscr{B}\mathscr{W}\mathscr{C}$ such that for every $\xi \in D^\circ$,

$$|\langle m, \xi\rangle|(\Omega) = \lambda(|d\xi \circ m/d\lambda|) = \mu(|\xi|) \leq 1,$$

so that $\overline{\mathrm{co}}(m(S))$ is weakly compact in E_B for some closed disked set $B \in \mathscr{B}$. By the Orlicz-Pettis theorem $m: \mathscr{S} \to E_B$ is σ-additive. ∎

In [12, Theorem 1], conditions related to those of Theorem 1 ensuring that a cylindrical probability has $(\tau, 1)$-moments are given. The completeness of the whole space E is used in a similar manner. However, such an assumption is often inappropriate in the general locally convex space setting, as for example when E is endowed with its weak topology.

The conclusion of Corollary 3 may fail to hold for $p = 1$, as may be seen from the vector measure $m: \mathscr{S} \to (L^\infty(\lambda), \sigma(L^\infty, L^1))$ defined by

$$m(A) = \chi_A, \quad A \in \mathscr{S},$$

which is absolutely continuous with respect to the Lebesgue measure $\lambda: \mathscr{S} \to [0, 1]$ on the unit interval.

To see the bearing that the additivity of the (m, λ)-distribution has on the differentiability of m with respect to λ, we need to introduce a weaker notion of a density for a vector measure.

Let $(\Omega, \mathscr{S}, \lambda)$ be a probability space and $m: \mathscr{S} \to E$ a vector measure such that $m \ll \lambda$. Denote by $\mathscr{S}(m, \lambda)$ the σ-algebra generated by the family

$$\left\{(d\varphi \circ m/d\lambda)^{-1}(B) \colon \varphi \in \mathscr{L}(E, \mathbf{R}^k), B \in \mathscr{B}(\mathbf{R}^k), k \in \mathbf{N}\right\}$$

in $\mathscr{S}$.

A measure space $(\Gamma, \tilde{T}, \tilde{\nu})$ is said to be an *inessential extension* of the measure space (Γ, T, ν) if $T \subset \tilde{T}$, $\tilde{\nu}(A) = \nu(A)$ for every $A \in \mathscr{T}$, and if for every $T \in \tilde{\mathscr{T}}$, there is a set $S \in \mathscr{T}$ such that $\tilde{\nu}(S \Delta T) = 0$.

If $m: \mathscr{T} \to E$ is absolutely continuous with respect to ν, then there exists a unique extension $\tilde{m}$ of m to $\tilde{\mathscr{T}}$ such that $\tilde{m} \ll \tilde{\nu}$.

A typical example of an inessential extension of a measure space $(\Gamma, \mathscr{T}, \nu)$ is where $\mathscr{T}$ is the σ-algebra of Baire subsets of a completely regular topological space Γ, and $\tilde{\nu}$ is the canonical extension of the tight Baire measure ν to the Borel σ-algebra $\tilde{\mathscr{T}}$ of T. Another example is where S is some subset of the set Γ. $\tilde{\mathscr{T}}$ is the σ-algebra generated by $\mathscr{T}$ and S, and $\tilde{\nu}(S)$ takes some value between the ν-inner measure and the ν-outer measure of S [5].

A function $f: \Omega \to E$ is said to be a *virtual density* for m with respect to λ (briefly, (m, λ)) if there is an inessential extension $(\tilde{m}, \tilde{\lambda})$ of (m, λ) such that f is $\tilde{\lambda}$-integrable and $\tilde{m} = f\tilde{\lambda}$.

A function $f: \Omega \to E$ is called a *cylindrical density* for (m, λ) if f is a virtual density for $(m|\mathscr{S}(m, \lambda), \lambda|\mathscr{S}(m, \lambda))$.

If $f\colon \Omega \to E$ is λ-integrable and $m = f\lambda$, then we merely say that f is a *density* for (m, λ).

Viewing E as a subspace of E'^* in the natural way, a componentwise application of the scalar Radon-Nikodym theorem shows that there always exists a density $f\colon \Omega \to E'^*$ for (m, λ). A density $dm/d\lambda$ for (m, λ) with values in the space E may not exist, but the (m, λ)-distribution is a substitute for the distribution of $dm/d\lambda$. The relationship is made precise in the following theorem (see also [7]).

THEOREM 2. *If (m, λ) has a cylindrical density in E, than the (m, λ)-distribution is σ-additive.*

In particular, a cylindrical probability μ with $(\tau, 1)$-moments is σ-additive if and only if $(\iota_\sigma \mu_\sigma, \mu_\sigma)$ has a virtual density.

Proof. Let μ be the (m, λ)-distribution, and suppose that $f\colon \Omega \to E$ is a virtual density for $(m|\mathscr{S}(m, \lambda), \lambda|\mathscr{S}(m, \lambda))$ with an inessential extension $(\tilde{m}, \tilde{\lambda})$.

If $\varphi \in \mathscr{L}(E, \mathbf{R}^k)$, $B \in \mathscr{B}(\mathbf{R}^k)$ and $k \in \mathbf{N}$, then $(d\varphi \circ m/d\lambda)^{-1}(B) \in \mathscr{S}(m, \lambda)$ and

$$\mu \circ \varphi^{-1}(B) = \lambda \circ (d\varphi \circ m/d\lambda)^{-1}(B) = \tilde{\lambda} \circ (d\varphi \circ m/d\lambda)^{-1}(B).$$

Furthermore,

$$d\varphi \circ \tilde{m}/d\tilde{\lambda} = d\varphi \circ m/d\lambda$$

$\tilde{\lambda}$-a.e., so

$$\begin{aligned}\tilde{\lambda} \circ (d\varphi \circ m/d\lambda)^{-1}(B) &= \tilde{\lambda} \circ (d\varphi \circ \tilde{m}/d\tilde{\lambda})^{-1}(B) = \tilde{\lambda}\big((\varphi \circ f)^{-1}(B)\big) \\ &= \tilde{\lambda} \circ f^{-1}\big(\varphi^{-1}(B)\big).\end{aligned}$$

Consequently, $\mu = \tilde{\lambda} \circ f^{-1}|\mathscr{Z}(E)$ and μ is σ-additive.

Now suppose that μ has $(\tau, 1)$-moments on E and is σ-additive. Then $\mu_\sigma^*(E) = 1$. Let $\tilde{\mu}_\sigma$ be the extension of μ_σ to the σ-algebra $\tilde{\mathscr{C}}(E'^*)$ generated by $\mathscr{C}(E'^*)$ and the set E, with $\tilde{\mu}_\sigma(E) = 1$ [5, p. 75].

Define $f\colon E'^* \to E$ by letting $f(x) = x$ for $x \in E$ and $f(x) = 0$ for $x \in E'^* \setminus E$. Then $(E'^*, \tilde{\mathscr{C}}(E'^*), \tilde{\mu}_\sigma)$ is an inessential extension of $(E'^*, \mathscr{C}(E'^*), \mu_\sigma)$ and f is scalarly $\tilde{\mu}_\sigma$-measurable with $\iota_\sigma \tilde{\mu}_\sigma = f\tilde{\lambda}$. Thus f is a virtual density for $(\iota_\sigma \mu_\sigma, \mu_\sigma)$. ∎

To prove that the (m, λ)-distribution is σ-additive, it suffices to show that a cylindrical density for (m, λ) exists. There is no distinction between a cylindrical density and a virtual density for $(\iota_\sigma \mu_\sigma, \mu_\sigma)$. The following example shows that there is a need to distinguish between the two notions in general.

Example 1. Let $\Omega = [0,1]$, let $\lambda\colon \mathscr{B}[0,1] \to [0,1]$ be the Lebesgue measure on $[0,1]$, and let $\mathscr{S} = \mathscr{B}[0,1]$. Define $f\colon \Omega \to L^1(\lambda)$ by $f(t) = \chi_{[0,t]}$ for $t \in [0,1]$. Then $\nu = \lambda \circ f^{-1}$ is a σ-additive cylindrical probability on $L^1(\lambda)$ with $(\tau, 1)$-moments. Its extension to $\mathscr{C}(L^1(\lambda))$ is also denoted by ν.

Let $A \subset [0,1]$ be a set with $\lambda^*(A) = 1$ and the λ-inner measure of A, $\lambda_*(A) = 0$. Give the subspace

$$E = \operatorname{span}\big(\{\chi_{[0,t]}\colon t \in A\} \cup C[0,1]\big)$$

of $L^1(\lambda)$ the relative topology. Then $\nu^*(E) = 1$ and $\nu_*(E) = 0$. If ι is the identity on $L^1(\lambda)$, then the range of the vector measure

$$\iota\nu = (f\lambda)\circ f^{-1}$$

is contained in the space E. The $(\iota\nu, \nu)$-distribution on E has $(\tau, 1)$-moments and it is clearly σ-additive.

Suppose that $(\iota\nu, \nu)$ has a density $h\colon L^1(\lambda) \to E$. Then the separability of $L^1(\lambda)$ implies that $h = \iota\nu$ – a.e.. Consequently,

$$\nu_*(E) = \nu_*\big(\iota^{-1}(E)\big) = \nu_*\big(h^{-1}(E)\big) = \nu\big(h^{-1}(E)\big) = 1,$$

which is a contradiction.

However if $\bar{\nu}$ is the extension of ν to the σ-algebra generated by $\mathscr{B}(L^1(\lambda))$ and E satisfying $\bar{\nu}(E) = 1$, then $\iota\chi_E$ is a density in E for $(\iota\bar{\nu}, \bar{\nu})$, and so a virtual density for $(\iota\nu, \nu)$.

It follows that a pair (m, λ) may have a virtual density but no density. If in Example 1, $\tilde{\nu}$ is the extension of ν to the σ-algebra generated by $\mathscr{B}(L^1(\lambda))$ and E satisfying $\tilde{\nu}(E) = 1/2$, then it follows that $(\iota\tilde{\nu}, \tilde{\nu})$ does not have a virtual density in E, but $\iota\chi_E$ is still a cylindrical density for $(\iota\tilde{\nu}, \tilde{\nu})$; that is, a virtual density exists for

$$\big(\iota\tilde{\nu}|\mathscr{B}(L^1(\lambda)), \tilde{\nu}|\mathscr{B}(L^1(\lambda))\big) = (\iota\nu, \nu),$$

but not for $(\iota\tilde{\nu}, \tilde{\nu})$.

The $(\iota\nu, \nu)$-distribution μ on E is the cylindrical probability satisfying

$$\mu(A \cap E) = \nu(A) \quad \text{for every } A \in \mathscr{Z}\big(L^1(\lambda)\big).$$

Because ν is a regular Borel measure on $L^1(\lambda)$ and $\nu^*(E) = 1$, μ is even τ-additive in the sense that for each net $(V_i)_{i \in I}$ of open cylinder sets with $V_i \uparrow E$, we have $\lim_{i \in I}(V_i) = 1$. Therefore there is still a need to distinguish between virtual densities and cylindrical densities for (m, λ) when the (m, λ)-distribution is τ-additive.

The condition that every cylindrical probability with $(\tau, 1)$-moments is σ-additive is a type of weak Radon-Nikodým property for E.

To guarantee the existence of a density for (m, λ), something more than just the σ-additivity or the τ-additivity of the (m, λ)-distribution is needed, so some additional concepts have to be introduced.

Let $m: \mathscr{S} \to E$ be a vector measure. Let D be a disked subset of E with p_D the gauge of D. The *D-variation*

$$V(m, D): \mathscr{S} \to [0, \infty]$$

is defined by

$$V(m, D)(B) = \sup\left\{ \sum_{A \in \pi} p_D(m(A)) : \pi \in \Pi(B) \right\},$$

where $B \in \mathscr{S}$ and $\Pi(B)$ is the family of all finite partitions of B by elements of $\mathscr{S}$.

Let $\mathscr{F}$ be a family of subsets of E. The vector measure m is said to have *finite $\mathscr{F}$-variation* if there exists a disked set $F \in \mathscr{F}$ such that $V(m, F)(\Omega) < \infty$. It has *σ-finite $\mathscr{F}$-variation* if there exists a partition $(\Omega_k)_{k \in \mathbf{N}}$ of Ω by elements of $\mathscr{S}$ such that $m|\mathscr{S} \cap \Omega_k$ has finite $\mathscr{F}$-variation for every $k \in \mathbf{N}$.

If $m \ll \lambda$, then the *average range* $AR_B(m, \lambda)$ of m with respect to λ on $B \in \mathscr{S}$ is defined by

$$AR_B(m, \lambda) = \{ m(C)/\lambda(C) : C \in \mathscr{S}, C \subset B, \lambda(C) > 0 \}.$$

We say that (m, λ) has *local $\mathscr{F}$-average range* if for each $B \in \mathscr{S}$ with $\lambda(B) > 0$, there exists $C \in \mathscr{S}$ with $C \subset B$, $\lambda(C) > 0$ and $AR_C(m, \lambda) \in F$.

Now let T be a Hausdorff topological space and $\mathscr{G}$ some family of subsets of T. A Borel probability $\nu: \mathscr{B}(T) \to [0, 1]$ is said to be $\mathscr{G}$-regular if

(i) for each $A \in \mathscr{B}(T)$, $\nu(A) = \sup\{ \mu(C) : C \subset A, C$ is closed in $T \}$, and

(ii) $\sup\{ \nu^*(G) : G \in \mathscr{G} \} = 1$.

The next theorem connects the concepts of average range, cylindrical concentration and variation.

THEOREM 3. *Let $(\Omega, \mathscr{S}, \lambda)$ be a probability space and $m: \mathscr{S} \to E$ a vector measure such that $m \ll \lambda$. Let $\mathscr{F}$ be a family of subsets of E such that:*

(i) *any subset of E contained in an element of $\mathscr{F}$ is also in $\mathscr{F}$;*

(ii) *scalar multiples of elements of $\mathscr{F}$ are also in $\mathscr{F}$;*

(iii) *closed disked hulls of finitely many members of $\mathscr{F}$ are in $\mathscr{F}$.*

The following conditions are equivalent.

(a) *(m, λ) has local $\mathscr{F}$-average range.*

(b) *m has σ-finite $\mathscr{F}$-variation.*

(c) *the (m, λ)-distribution is cylindrically concentrated on $\mathscr{F}$.*

Proof. For every disked set $F \in \mathscr{F}$ and linear map $\varphi \in L(E, \mathbf{R}^k)$, $k \in \mathbf{N}$,

$$\overline{\varphi(F)} = \bigcap_{a>1} a\varphi(F),$$

so the equivalence of (a) and (c) follows from [4, Théorème 2.1].

Suppose that the average range of (m, λ) on the set $B \in \mathscr{S}$ is contained in the disked set $D \in \mathscr{F}$. Then for every $A \subset B$, $A \in \mathscr{S}$ with $\lambda(A) > 0$, we have $p_D(m(A)/\lambda(A)) \leq 1$. It follows that m has finite $\mathscr{D}$-variation on $\mathscr{S} \cap B$. Combined with an exhaustion argument, it follows that (a) implies (b).

Suppose that m has finite $\mathscr{F}$-variation. Then for some closed disked set $D \in \mathscr{F}$, $V(m, D)(\Omega) < \infty$. The set function $V = V(m, D)$: $\mathscr{S} \to [0, \infty)$ is a measure (because p_D is lower semicontinuous on E), such that $m \ll V \ll \lambda$.

For some non-negative function g: $\Omega \to [0, \infty)$, $V = g\lambda$. Let

$$\Omega_n = \{\omega \in \Omega : n - 1 \leq g(\omega) < n\} \quad \text{for each } n \in \mathbf{N}.$$

Then

$$p_D(m(A)) \leq V(A) \leq n\lambda(A)$$

for every $A \in \mathscr{S} \cap \Omega_n$, so that (m, λ) has local $\mathscr{F}$-average range. Thus (b) implies (a), and the proof is complete.

COROLLARY 4. *Let (Ω, S, λ) be a complete probability space, m: $\mathscr{S} \to E$ a vector measure such that $m \ll \lambda$, and μ the (m, λ)-distribution on E.*

Let ρ be a topology on E finer than the weak topology $\sigma(E, E')$. Denote by $\mathscr{P}$ the family of ρ-compact convex subsets of E.

The following statements are equivalent.

(i) *(m, λ) has an essentially unique $\mathscr{B}(E_\rho)$-measurable density f for which $\lambda \circ f^{-1}$ is $\mathscr{P}$-regular on E.*
(ii) *μ is cylindrically concentrated on $\mathscr{P}$.*
(iii) *μ is the restriction of a unique $\mathscr{P}$-regular Borel measure on E.*
(iv) *m has σ-finite $\mathscr{P}$-variation.*
(v) *m has locally relatively ρ-compact average range.*

Proof. If (i) holds, then $\mu = \lambda \circ f^{-1}|\mathscr{Z}(E)$ so (ii) follows. Conditions (ii) and (iii) are equivalent by Prokhorov's theorem [11]. From the preceding theorem (iv) follows from (ii), and (iv) implies (v).

As in [4, Proposition 2.6], it follows that (v) implies that (m, λ) has local $\mathscr{F}$-average range for the saturated hull $\mathscr{F}$ of $\mathscr{P}$, of all subsets of elements of $\mathscr{P}$. By Theorem 3 again, μ is cylindrically concentrated on $\mathscr{P}$ and so condition (iii) holds. According to [13, 1.7], there exists a $\mathscr{B}(E)$-measurable function f: $\Omega \to E$ for which $\lambda \circ f^{-1}$ is a $\mathscr{P}$-regular Borel measure on E. Because the topologies ρ and $\sigma(E, E')$ agree on ρ-compact sets, f is even $\mathscr{B}(E_\rho)$-measurable, and $\lambda \circ f^{-1}$ is a $\mathscr{P}$-regular Borel measure on E.

The essential uniqueness of the density f can be established in the following manner. Suppose that $g\colon \Omega \to E$ is $\sigma(E, E')$-Borel measurable and

$$\lambda \circ g^{-1}\colon \mathscr{B}(E_\sigma) \to [0,1]$$

is a Radon measure.

Then as in [4, Théorème 2.1], for any closed cylinder set C containing $AR_\Omega(g\lambda, \lambda)$, $\lambda \circ g^{-1}(C) = 1$. Because Radon measures are continuous for decreasing convergence of closed sets, denoting the $\sigma(E, E')$-closure of $AR_\Omega(g\lambda, \lambda)$ by $\overline{AR}^\sigma_\Omega(g\lambda, \lambda)$, we have

$$\lambda \circ g^{-1}\big(\overline{AR}^\sigma_\Omega(g\lambda, \lambda)\big) = 1.$$

Consequently, if $h\colon \Omega \to E$ is a $\sigma(E, E')$-Borel measurable function such that

$$\lambda \circ h^{-1}\colon \mathscr{B}(E_\sigma) \to [0,1]$$

is a Radon measure and $h\lambda = f\lambda$, then $(h - f)\lambda(A) = 0$ for every $A \in \mathscr{S}$, and so $AR_\Omega((h - f)\lambda, \lambda) = \{0\}$. Moreover, the function $h - f$ is $\sigma(E, E')$-Borel measurable and $\lambda \circ (h - f)^{-1}$ is a Radon measure. Thus, $h = f$ λ-a.e..

The essential uniqueness of densities of the above type does not seem to have been previously noticed. It often has some useful implications.

Note that the topology ρ need not be compatible with the vector space structure of E, let alone locally convex. Corollary 4 may be viewed as an extension of the classical theorem of Phillips for Banach spaces.

An additive set function m with values in the space of bounded additive set functions on an algebra of sets $\mathscr{A}$, will have finite variation-bounded variation if and only if it can be written as $m = m^+ - m^-$ for two additive set functions m^+, m^-, with values in the non-negative bounded additive set functions on $\mathscr{A}$. Let T be a completely regular space, and denote by $M_t(T)$ the space of signed Radon measures on T endowed with the topology of convergence on finite families of bounded continuous functions on T. A subset of $M_t(T)$ is bounded if and only if it is bounded in variation. Thus a vector measure $m\colon \mathscr{S} \to M_t(T)$ can be written as the difference between two vector measures with values in the space $M_t^+(T)$ of non-negative Radon measures on T if and only if m has finite bounded variation in $M_t(T)$.

Suppose now that the vector measure $m\colon \mathscr{S} \to M_t(T)$ has σ-finite bounded variation in the space $M_t(T)$ and $m \ll \lambda$. Then according to Theorem 3, for every $\varepsilon > 0$ there exists a Borel set $S \subset T$ such that $\lambda(S^c) < \varepsilon$ and $m|\mathscr{B}(T) \cap S$ can be written as $m^+ - m^-$ for vector measures $m^+, m^-\colon \mathscr{B}(T) \cap S \to M_t^+(T)$ such that both (m^+, λ) and (m^-, λ) have bounded average range in $M_t(T)$. Applying Théorème 4.2 of Goldman [4], we see that $m|\mathscr{B}(T) \cap S$ has a density with respect to λ. Therefore (m, λ) has a density with values in $M_t(T)$ by patching.

We will also see that $m: \mathscr{S} \to M_t(T)$ has a density with respect to λ only if m has σ-finite bounded variation (respectively, σ-finite uniformly tight variation).

Let $\mathscr{F}$ be a family of subsets of the locally convex space E. We say that E has the *$\mathscr{F}$-variation property* if every indefinite integral with values in E has σ-finite $\mathscr{F}$-variation.

THEOREM 4. *Let $\mathscr{B}$ be a saturated family of subsets of the sequentially complete locally convex space E. Then E has the $\mathscr{B}$-variation property if and only if each σ-additive cylindrical probability with $(\tau, 1)$-moments is cylindrically concentrated on $\mathscr{B}$.*

Proof. Suppose that each σ-additive $\mu: \mathscr{Z}(E) \to [0,1]$ with $(\tau, 1)$-moments is cylindrically concentrated on $\mathscr{B}$.

Let $f: \Omega \to E$ be a λ-integrable function. Then $\lambda \circ f^{-1}|\mathscr{Z}(E) = \mu$ is the $(f\lambda, \lambda)$-distribution, and it has $(\tau, 1)$-moments because E is sequentially complete. Now μ is cylindrically concentrated on $\mathscr{B}$ since it is σ-additive, so by Theorem 3, $f\lambda$ has σ-finite variation.

Conversely, let E have the $\mathscr{B}$-variation property, and assume that the $(\iota_\sigma\mu_\sigma, \mu_\sigma)$-distribution is σ-additive. Then $(\iota_\sigma\mu_\sigma, \mu_\sigma)$ has a virtual density by Theorem 2, and the corresponding inessential extension of $\iota_\sigma\mu_\sigma$ has σ-finite $\mathscr{B}$-variation. From Theorem 3, μ is cylindrically concentrated on $\mathscr{B}$.

The family of bounded subsets of a locally convex space is denoted by $\mathscr{B}_0$. The following facts are easily verified.

PROPOSITION 4. (i) *Every subspace of a locally convex space with the $\mathscr{B}_0$-variation property also has the $\mathscr{B}_0$-variation property.*

(ii) *The projective limit of a sequence of spaces with the $\mathscr{B}_0$-variation property has the $\mathscr{B}_0$-variation property.*

(iii) *The strict inductive limit of a sequence of complete spaces with the $\mathscr{B}_0$-variation property has the $\mathscr{B}_0$-variation property.* ∎

A completely regular space is said to be *strongly measure-compact* if every measure on the Baire σ-algebra is tight; that is, it is the restriction to the Baire sets of a uniquely defined Radon measure. It follows from Theorem 4 that a locally convex space which is strongly measure compact for its weak topology has the weakly compact variation property, provided it is quasi-complete for its Mackey topology. This last condition ensures that the family of relatively weakly compact sets is saturated. In particular, any quasi-complete Souslin locally convex space [11] or separable Fréchet space has the compact variation property, and so the bounded variation property. For these spaces any

σ-additive cylindrical probability is the restriction of a uniquely defined Radon measure to the algebra of cylinder sets.

If the locally convex space E contains a bounded absorbing set, then K. Musial [8, Proposition 1] has shown that E has the bounded variation property. By localization, it follows that if E or E' has a countable covering consisting of bounded sets, then E has the $\mathscr{B}_0$-variation property. In particular this applies to a metrizable locally convex space.

The family of all measures with variation ≤ 1 is a bounded absorbing set in the space $M_t(T)$ of Radon measures on the completely regular space T, so $M_t(T)$ has the $\mathscr{B}_0$-variation property. The discussion is summarized in the following statement.

PROPOSITION 5. *Let* $(\Omega, \mathscr{S}, \lambda)$ *be a complete probability measure space. Let*

$$m: \mathscr{S} \to M_t(T)$$

be a vector measure such that $m \ll \lambda$. *Then* (m, λ) *has a density* $f: \Omega \to M_t(T)$ *if and only if* m *has* σ*-finite bounded variation* (*respectively*, σ*-finite uniformly tight variation*).

In this case, there exists an essentially unique Borel measurable density f *such that* $\lambda \circ f^{-1}$ *is* $\mathscr{F}$*-regular for the family* $\mathscr{F}$ *of uniformly tight subsets of* $M_t(T)$. ■

The uniform tightness property is a consequence of [4, 4.2].

Example 2. Let X and Y be Banach spaces. The space $\mathscr{L}(X, Y)$ of continuous linear operators from X into Y with the strong topology has a bounded absorbing set; namely, those operators $u: X \to Y$ such that $\|u\| \leq 1$. Consequently $\mathscr{L}(X, Y)$ has the bounded variation property. If Y is reflexive, then $\mathscr{L}(X, Y)$ is semi-reflexive, so by Corollary 4, $m: \mathscr{S} \to \mathscr{L}(X, Y)$ has a density with respect to λ if and only if m has σ-finite $\mathscr{B}_0$-variation.

Dual nuclear spaces also have a strong form of variation property. Suppose that E_τ is quasi-complete and E'_τ is nuclear [10]. Then every vector measure $m: \mathscr{S} \to E$ has a Bochner integrable density $f: \Omega \to E_B$ for some weakly compact subset B of E, so E certainly has the compact variation property. The prescribed conditions hold whenever E is quasi-complete with E'_β nuclear; that is, E is dual nuclear. All complete metric and dual metric nuclear spaces are dual nuclear, so this includes all the common spaces of distributions.

It is not sufficient that E be complete and nuclear for it to have the bounded variation property. This is easily seen by taking the space $\mathbf{R}^{[0,1]}$, the Lebesgue measure λ on the unit interval, and the λ-integrable function $f: [0,1] \to \mathbf{R}^{[0,1]}$ defined by

$$f(t)(s) = |t - s|^{-1/2}, \quad s, t \in [0,1], s \neq t,$$

and $f(t)(t) = 0$, $t \in [0,1]$. The vector measure f has infinite $\mathscr{B}_0$-variation on every set of positive measure.

We now return to the situation considered in Theorem 2.

LEMMA 1. *Let E be a subspace of the locally convex space F with the relative topology. Let $(\Omega, \mathscr{S}, \lambda)$ be a probability space with m: $\mathscr{S} \to E$ a vector measure such that $m \ll \lambda$. Suppose that there exists a function f: $\Omega \to F$ such that $m = f\lambda$ and $f(\Omega) \in \overline{\mathscr{C}(F)}$, the $\lambda \circ f^{-1}$-completion of $\mathscr{C}(F)$.*

If the (m, λ)-distribution μ is σ-additive, then (m, λ) has a cylindrical density in E.

Proof. Let j: $E \to F$ be the inclusion map. Let $\bar{\mu}$ be the extension of μ to a σ-additive measure on $\mathscr{C}(E)$. Then $\bar{\mu} \circ j^{-1}$ is σ-additive on $\mathscr{C}(F)$ and $(\bar{\mu} \circ j^{-1})^*(j(E)) = 1$.

Moreover, $\bar{\mu} \circ j^{-1} = \lambda \circ f^{-1}$ so that $(\lambda \circ f^{-1})^*(j(E)) = 1$. Because $f(\Omega) \in \overline{\mathscr{C}(F)}$, $(\lambda \circ f^{-1})^*(j(E) \cap f(\Omega)) = 1$.

Let λ_0 be the restriction of λ to $\mathscr{S}(m, \lambda)$. Then it follows that

$$\lambda_0^*\left(f^{-1}(j(E))\right) = 1.$$

Let λ_1 be the extension of λ_0 to the σ-algebra generated by $\mathscr{S}$ and $f^{-1}(j(E))$ with $\lambda_1(f^{-1}(j(E))) = 1$. Put $f_1(\omega) = f(\omega)$ if $\omega \in f^{-1}(j(E))$ and $f_1(\omega) = 0$ if $\omega \in \Omega \setminus f^{-1}(j(E))$. Then $m|\mathscr{S}(m, \lambda) = f_1\lambda_1|\mathscr{S}(m, \lambda)$ and f_1 is scalarly measurable with respect to the σ-algebra $\mathscr{S}(m, \lambda)$, so f_1 is a cylindrical density for (m, λ). ■

Lemma 1 suggests that the existence of a cylindrical density for a vector measure m with respect to a probability λ is related to the so-called "image measure catastrophe" of L. Schwartz [11, p. 30]. This phenomenon explains why the regularity of the (m, λ)-distribution is so important in this context. If the (m, λ)-distribution is not regular, then the problem is likely to involve subtler measure-theoretic arguments [3].

Example 1 and the argument following it shows that (m, λ) need not have a density, or even a virtual density in E. The following theorem gives conditions guaranteeing the existence of a cylindrical density.

A measure space $(\Gamma, \mathscr{T}, \nu)$ is said to be *perfect* if for every real valued $\mathscr{T}$-measurable function f: $\Gamma \to \mathbf{R}$ there exists a Borel set B such that

$$B \subset f(\Gamma) \quad \text{and} \quad \nu\left(f^{-1}(B)\right) = \nu(\Gamma).$$

For such a measure space, the same holds true for a $\mathscr{T}$-measurable function with values in a Polish space. Every Radon measure is perfect.

Let μ be a σ-additive cylindrical probability on the locally convex space E. Then the image of μ by the natural inclusion of E in E'' defines a probability measure on $\mathscr{C}(E''_\sigma)$. Denote the completion of the σ-algebra $\mathscr{C}(E''_\sigma)$ with respect to this measure by $\overline{\mathscr{C}}^\mu(E''_\sigma)$.

Let E_μ denote the family of bounded subsets of the locally convex space $\mathscr{F}$ with the following two properties:

(i) For each $F \in \mathscr{F}_\mu$, there exists a countable subset of E' separating points of the closure of F in E''_σ;

(ii) The closure of each set $F \in \mathscr{F}_\mu$ in E''_σ belongs to $\overline{\mathscr{C}}^\mu(E''_\sigma)$.

THEOREM 5. *Let $(\Omega, \mathscr{S}, \lambda)$ be a perfect probability space with $m: \mathscr{S} \to E$ a vector measure such that $m \ll \lambda$. Suppose that the (m, λ)-distribution μ is σ-additive.*

If m has σ-finite $\mathscr{F}_\mu$-variation, then (m, λ) has a cylindrical density.

Proof. By Lemma 1, it suffices to show that (m, λ) has a density $f: \Omega \to E''$ such that $f(\Omega) \in \overline{\mathscr{C}}^\mu(E''_\sigma)$.

Let $(\Omega_k)_{k \in \mathbf{N}} \subset \mathscr{S}$ be a partition of Ω for which m has finite B_k-variation on $\mathscr{S} \cap \Omega_k$, $B_k \in \mathscr{F}_\mu$, $k \in \mathbf{N}$. Let H_k be the linear span of a countable subset of E' separating points of the closure $\overline{B}_k$ of B_k in E''_σ. Then H_k also separates points of $E_k = \bigcup_{n \in N} nB_k$.

Let $F_k = E'_{B_k^\circ} = E'/(E_{B_k})^\circ$. Then (E_k, H_k) is a metrizable topology coarser than $\sigma(E_k, F_k)$. Since B_k is bounded, the closure $\overline{B}_k$ of B_k in E'^* is a compact subset of E''_σ. It follows that $\overline{B}_k$ can be identified with the closure of B_k in $(F_k^*, (F_k^*, F_k))$ and $(H_k, (H_k^*, H_k))$, and the two topologies agree and are metrizable on $\overline{B}_k$. Let $\tilde{E}_k = \bigcup_{n \in \hat{\mathbf{N}}} n\overline{B}_k$.

The completion of a perfect measure space is perfect, so it may be assumed from the outset that $(\Omega, \mathscr{S}, \lambda)$ is complete. By Corollary 4, there exists a Borel measurable density $f_k: \Omega_k \to \tilde{E}_k$ for the relative topology of E''_σ. It may be supposed that $f_k(\Omega_k) \subset \overline{B}_k$, and because $\overline{B}_k$ is compact and metrizable in E''_σ and $(\Omega, \mathscr{S}, \lambda)$ is perfect, it can also be assumed that $f_k(\Omega_k) \in \mathscr{B}(E''_\sigma) \cap \overline{B}_k$. Moreover, $\mathscr{B}(E''_\sigma) \cap \tilde{E}_k = \mathscr{C}(E''_\sigma) \cap \tilde{E}_k$ because $\overline{B}_k$ is compact and metrizable in E''.

By assumption $\overline{B}_k \in \overline{\mathscr{C}}^\mu(E''_\sigma)$, so $\tilde{E}_k \in \overline{\mathscr{C}}^\mu(E''_\sigma)$. Thus $f_k(\Omega_k) \in \overline{\mathscr{C}}^\mu(E''_\sigma)$. Put $f(\omega) = f_k(\omega)$ if $\omega \in \Omega_k$, $k \in \mathbf{N}$. Then $f: \Omega \to E''$ is a density in E'' for (m, λ), and $f(\Omega) \in \overline{\mathscr{C}}^\mu(E''_\sigma)$.

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