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PERIODICITY IN THE COHOMOLOGY OF UNIVERSAL G-SPACES

BY

STEFAN WANER

INTRODUCTION

The purpose of this note is to generalize the classical results on periodicity in $H^*(G)$ in the presence of a free G-action on a sphere, and to reinterpret them in terms of global results about equivariant singular cohomology.

Our generalizations proceed in two directions. First, one has a notion of $H^*(G; T)$, where T is a Mackey functor (in the sense of tom Dieck in [2]), generalizing the case T a $\mathbf{Z}G$-module. We show here that classical periodicity continues to hold in this more general setting.

Next, one has the notion of a universal G-space $E\mathscr{F}$, associated with a family $\mathscr{F}$ of subgroups of G. Here, we exhibit periodicity in $H_G^*(E\mathscr{F}; T)$ (for arbitrary G and particular families $\mathscr{F}$), where $*$ is $RO(G)$-grading. (The theory of $RO(G)$-graded equivariant singular cohomology has been announced by Lewis, May, and McClure in [4]. The complete theory will appear in [5], including one of the author's independent formulations, a summary of which appears in §1 below). This periodicity is seen to arise from a "Bott" class $1_V \in H_G^V(\text{point})$ for appropriate representations V, in the sense that $\cup 1_V$ is an isomorphism in a range. Further, we see that this class lies at the source of the classical periodicity results, which emerge as special cases.

Finally, we use the periodicity to extend the computation of $H_G^n(E\mathscr{F}; T)$ carried out in [7] and [8] to that of $H_G^{nV+m}(E\mathscr{F}; T)$ for $m, n \geq 0$ and $\mathscr{F}$ a family of subgroups determined by V. These latter groups (which are also modules over the Burnside ring of G) turn out to be purely algebraic invariants of G and V. (Throughout, G will be a finite group.)

1. Equivariant RO(G)-graded singular cohomology

We recall here in brief some of the theory of equivariant $RO(G)$-graded singular cohomology, developed by Lewis, May, McClure and the author in [5].

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Let $\mathscr{U}$ be the orthogonal G-module $(\mathbf{R}G)^\infty$, $\mathbf{R}G$ being the real group algebra endowed with its natural inner product. We shall write $V < \mathscr{U}$ to signify that V is a finite-dimensional G-invariant submodule of $\mathscr{U}$.

If $V < \mathscr{U}$, then a G-CW(V) complex is a G-space X with a given decomposition $X = \operatorname{colim} X^n$ such that:

(i) X^0 is a disjoint union of G-orbits, $X^0 = \coprod_\gamma G/H_\gamma$ where V is a trivial H_γ-module for each γ;

(ii) X^n is obtained from X^{n-1} by attaching "cells" of the form $G \times_H D(V - m)$, where H is such that V has a trivial m-dimensional summand as an H-module, and where $n = \dim V - m$ (here, $D(W)$ denotes the unit disc in $W < \mathscr{U}$).

In [5], one sees that any G-CW complex (in the sense of Bredon-Illman) has the G-homotopy type of a G-CW(V) complex for every $V < \mathscr{U}$. (One considers $X \times D(V) \sim X$ for a G-CW complex X, where $D(V)$ is seen to have a G-CW(V) structure with cells of dimension $\leq \dim V$). Moreover, cellular approximation and an appropriate version of the Whitehead theorem hold in this context.

G-CW(V) decompositions give rise to cellular chains, which one may use to define $H_G^{V+n}(X)$ for all $n \in \mathbf{Z}$, as follows.

Denote by $[X, Y]_G$ the set of G-equivariant homotopy classes of based G-maps $X \to Y$. If $V < \mathscr{U}$, denote by S^V its one-point compactification, (G acting trivially at the basepoint ∞) and by $\Sigma^V X$ the smash product $X \wedge S^V$ for a based G-space X. Let $\mathscr{O}$ be the category whose objects are the G-spaces G/H for $H \subset G$ and whose morphisms are given by

$$\mathscr{O}(G/H, G/K) = \operatorname*{colim}_{V < \mathscr{U}} \left[\Sigma^V G/H_+, \Sigma^V G/K_+\right]_G,$$

where the subscript $+$ denotes addition of a disjoint basepoint.

A contravariant (resp. covariant) coefficient system (or "Mackey functor") is then a contravariant (resp. covariant) additive functor T: $\mathscr{O} \to \mathscr{A}b$, the category of abelian groups. A map of such systems is then a natural transformation of functors.

If X is G-CW(V), then one has a differentially graded contravariant system given by

$$\overline{C}_{V+n}(X)(G/H) = \operatorname*{colim}_{W \perp V} \left[\Sigma^{V+W} G/H_+, \Sigma^{W-n} X^{v+n}/X^{v+n-1}\right]_G$$

where $v = \dim V$ and W is large enough to contain a trivial n-dimensional summand.

If $\overline{T}$ and $\overline{S}$ are contravariant and $\underline{T}$ is covariant, one has abelian groups $\operatorname{Hom}_{\mathscr{O}}(\overline{T}, \overline{S})$ and $\overline{T} \otimes_{\mathscr{O}} \underline{T}$, given respectively by the group of natural transformations, and by $\sum_{H \subset G} \overline{T}(G/H) \otimes \underline{T}(G/H) \sim$ where, for f: $G/H \to G/K$ in $\mathscr{O}$, one identifies $f^*T \otimes t'$ with $t \otimes f_* t'$.

One then defines $H_G^{V+*}(X; \overline{S})$ and $H^G_{V+*}(X; \underline{T})$ respectively by passage to homology of $\mathrm{Hom}_{\mathcal{O}}(\overline{C}_{V+*}(X), \overline{S})$ and $C_{V+*}(X) \otimes_{\mathcal{O}} \underline{T}$.

Observe that (stable) equivariant self s-duality of the spaces G/H_+ implies that $\mathcal{O} \cong \mathcal{O}^{\mathrm{opp}}$, so that every covariant system may be regarded as contravariant, and vice-versa. One has canonical coefficient systems $\underline{A}$ and $\overline{A}$, analogous to $\mathbf{Z}$-coefficients in the nonequivariant case, given by

$$\underline{A}(G/H) = \mathrm{colim}[S^V, \Sigma^V G/H_+]_G$$

and

$$\overline{A}(G/H) = \mathrm{colim}[\Sigma^V G/H_+, S^V]_G,$$

each isomorphic with the Burnside ring, $A(H)$ of H (Segal, Petrie, tom Dieck).

Following is a list of basic properties of $RO(G)$-graded singular cohomology. (There is also an evident dual list for homology.)

(1) "Dimension Axiom". $H^0_G(G/H; \overline{T}) \cong \overline{T}(G/H)$ for each $H \subset G$; $H^n_G(G/H; \overline{T}) = 0$ if $n \neq 0$;

(2) $H_G^{V+n}(G/H; \overline{T}) = H_G^{-(V+n)}(G/H; \overline{T}) = 0$ if $n > 0$;

(3) $H^\gamma_G(G \times_K X; \overline{T}) \cong H_K^{\gamma|K}(X; \overline{T}|K)$ if $K \subset G$, where $\overline{T}|K$ is $\overline{T}$, regarded naturally as a coefficient system for K-orbits;

(4) "Suspension Isomorphism".

$$\overline{H}^\gamma_G(X; \overline{T}) \overset{\sigma}{\cong} \overline{H}_G^{\gamma+V}(\Sigma^V X; \overline{T}),$$

where the reduced cohomology of a based G-space X is given by the natural construction $H_G((X,*); \overline{T})$ for pairs;

(5) $H^*_G(X; \overline{T})$ has a natural module structure over $A(G)$.

Further, one has for Burnside coefficients $\overline{A}$, and suitable "ring systems" in general, a cup product $\cup$: $H^\gamma_G(X) \otimes H^{\gamma'}_G(X) \to H_G^{\gamma+\gamma'}(X)$ which is unital, associative and commutative up to certain units in the Burnside ring of G. Further, $H^*_G(X; \overline{T})$ is an $H^*_G(\text{point}; \overline{A})$-module for any X and $\overline{T}$.

All of the above properties and more will be developed in detail in [5]. For the purposes of this paper, suffice to say that the theory can be manipulated just as ordinary cohomology ought to be.

The relationship with Bredon cohomology [1] is as follows. Let $\mathcal{G}$ denote the category whose objects are those of $\mathcal{O}$ and whose morphisms are the G-maps $G/H \to G/K$. A contravariant system $\overline{T}$: $\mathcal{O} \to \mathcal{O}b$ is automatically a Bredon contravariant system $\overline{T}|$: $\mathcal{G} \to \mathcal{O}b$ (in the sense of [2]) via the inclusion $\mathcal{G} \to \mathcal{O}$. If a Bredon system $\overline{T}$, extends to a contravariant (Mackey) system $\overline{T}'$, then Bredon cohomology (with $\overline{T}$-coefficients) agrees with $H^n_G(X; \overline{T}')$ for $n \in \mathbf{Z}$ up to natural isomorphism.

2. The V-dimensional Bott class

By (2) of §1, $H_G^V(\text{point}) = 0$ if V has a trivial summand of dimension ≥ 1. When $V^G = 0$, there is an element $1_V \in H_G^V(\text{point}; \underline{A})$ given by $1_V = \iota^*(1_0)$, where 1_0 is the fundamental class in $H_G^0(\text{point}; \underline{A}) \cong \overline{H}_G^V(S^V; \underline{A}) \cong A(G)$, and $\iota\colon S^0 \to S^V$ is inclusion.

The inclusion $S(V) \to D(V)$ of the unit sphere in V gives rise to a long exact sequence

$$(2.1) \qquad H_G^{\gamma - V}(\text{point}) \overset{\mathcal{O}}{\cong} \overline{H}_G^{\gamma}(S^V) \xrightarrow{\mu} H_G^{\gamma}(\text{point}) \to H_G^{\gamma}(S(V))$$

$$\underbrace{\overline{H}_G^{\gamma}(S^V) \quad \longleftarrow \quad H_G^{\gamma}(S(V))}_{\text{degree } -1}$$

(with $\underline{A}$-coefficients suppressed), where μ coincides with $\cup 1_V$.

Consider the case $\gamma = nV + m$, where, for interest, V should have no trivial summand.

LEMMA 2.2. *Let $n > 1$ and $m \geq 0$. Then $\cup 1_V$: $H_G^{(n-1)V+m}(\text{point}) \to H_G^{nV+m}(\text{point})$ is an isomorphism. (By (2) of §1, the case $m > 0$ is, of course, immediate).*

Proof. By the sequence (2.1), one need only show that

$$H_G^{nV}(S(V)) = H_G^{nV-1}(S(V)) = 0.$$

$S(V)$, however, is made up of G-cells of the form $G \times_H D(V - i)$ with $1 \leq i \leq \dim V^H$. If $n \geq 1$, $H_G^{nV}(S(V))$ is computed by giving $S(V) \times D((n - 1)V)$ the product G-CW(nV) structure, so that $C_{nV}(S(V)) = 0$, the top dimensional cells being in dimension $nV - 1$. Similarly, $H_G^{(n+1)V-1}(S(V)) = 0$. Indeed, if X denotes the $(v - 2)$-skeleton of $S(V)$), then $H_G^{(n+1)V-1}(X) = 0$ by the argument above, and the inclusion of X in $S(V)$ gives a long exact sequence

$$\begin{array}{l} \bigoplus_{K_i} \overline{H}_{K_i}^{(n+1)V-1}(S^{V-i}) \to H_G^{(n+1)V-1}(S(V)) \to H_G^{(n+1)V-1}(X) = 0 \\ \quad \wr\| \\ \bigoplus_{K_i} H_{K_i}^{nV}(\text{point}) \end{array}$$

where $K_i \subset G$ are proper subgroups such that V has a trivial summand as a K_i-module. Hence $H_{K_i}^{nV}(\text{point}) = 0$ if $n \geq 1$. $\square$

For the case $n = 1$, one has the sequence

$$\begin{array}{l} \cdots \to H_G^{V-1}(S(V)) \xrightarrow{\xi} H_G^0(\text{point}) \to H_G^V(\text{point}) \to H_G^V(S(V)) = 0. \\ \qquad\quad \wr\| \\ \qquad\quad A(G) \end{array}$$

To compute ξ, one has, by equivariant Poincare Duality [9],

$$H_G^{V-1}(S(V)) \cong H_0^G(S(V))$$

(where for the latter, one regards $\overline{A}$ as a covariant system via the canonical equivalence $\mathcal{O} \cong \mathcal{O}^{\text{opp}}$). When V has no one-dimensional fixed-set, it is easy to compute $H_0^G(S(V))$.

Let $\mathscr{F}(V)$ be the family of subgroups given by $H \in \mathscr{F}(V)$ iff $V^H \neq 0$. Then

$$S(V^\infty) = \operatorname{colim} S(V^n)$$

is a universal G-space of the form $E\mathscr{F}(V)$. Such a space has the property that $E\mathscr{F}(V)^K$ is empty if $K \notin \mathscr{F}(V)$ and is contractible if $K \in \mathscr{F}(V)$. In [8] it is shown that $H_G^P(E\mathscr{F}(V); T) \cong \operatorname{Ext}^P_{\mathscr{B}(\mathscr{F}(V))}(\overline{\mathbf{Z}}, T)$, where the ext groups are given as follows. $\mathscr{B}(\mathscr{F})$ is the category of Bredon coefficient systems (see [1]) T: $\mathscr{G}(\mathscr{F}) \to \mathscr{A}b$, where $\mathscr{G}(\mathscr{F})$ has objects G/H with $H \in \mathscr{F}$ and morphisms the equivariant maps. $\overline{\mathbf{Z}}$ is the constant coefficient system, $\overline{\mathbf{Z}}(G/H) = \mathbf{Z}$; $\overline{\mathbf{Z}}(f) = 1$, and the ext groups are constructed in the category $\mathscr{B}(\mathscr{F})$. In particular, $H_G^0(E\mathscr{F}(V); \overline{T}) \cong \operatorname{Hom}_{\mathscr{B}(\mathscr{F})}(\overline{\mathbf{Z}}, \overline{T})$. Dually, one has

$$H_0^G(E\mathscr{F}(V); \underline{T}) \cong \overline{\mathbf{Z}} \otimes_{\mathscr{B}(\mathscr{F}(V))} \underline{T},$$

whence $H_0^G(S(V^\infty); \underline{A}) \cong \overline{\mathbf{Z}} \otimes_{\mathscr{B}(\mathscr{F}(V))} \underline{A}$, where one regards $\underline{A}$ as a Bredon coefficient system.

LEMMA 2.3. *Let V be such that V^H does not have dimension* 1 *for $H \subset G$. Then*

$$H_0^G(S(V)) \cong H_0^G(S(V^\infty)).$$

Proof. Under the hypothesis, and by uniqueness of universal G-spaces, one can obtain a G-homotope X of $S(V^\infty)$ by attaching G-cells of the form $G/H \times D^i$ with $i \geq 2$ to $S(V)$, giving the result by the associated exact sequences in Bredon-Illman homology. $\square$

By naturality of Poincare duality, one obtains

$$\xi\colon H_G^{V-1}(S(V)) \cong \overline{\mathbf{Z}} \otimes_{\mathscr{B}(\mathscr{F}(V))} \underline{A} \to A(G) \cong H_G^0(\text{point})$$

coinciding with $\xi(n \otimes a) \cong nf_*(a)$ for $n \in \overline{\mathbf{Z}}(G/H)$, $a \in \underline{A}(G/H)$ and $f\colon G/H \to G/G$ the only possible map. One then has $H_G^V(\text{point}) \cong A(G)/\operatorname{Im}\xi$. For general V, this remains true, except that ξ must be computed explicitly from the 0- and 1- dimensional geometry of V. In either case, the unit $1 \in H_G^0(\text{point})$ goes to the Bott class $1_V \in H_G^V(\text{point})$ under the quotient $A(G) \to A(G)/\operatorname{Im}\xi$.

LEMMA 2.4. *If $m \geq 0$ and $n > m$, then*

$$\cup 1_V\colon H_G^{m-(n-1)V}(\text{point}) \to H_G^{m-nV}(\text{point})$$

is an isomorphism.

Proof. By the sequence (2.1) with $\gamma = m - nV$, it suffices to show that

$$H_G^{m-nV}(S(V)) = H_G^{m-1-nV}(S(V)) = 0.$$

One has contributions to $\overline{H}_G^{m-nV}(S(V))$ of the form

$$\overline{H}_G^{m-nV}\left(G_+ \wedge_K S^{V-i}\right) \cong \overline{H}_K^{m-nV}\left(S^{V-i}\right) \cong \overline{H}_K^{m+i}\left(S^{(n+1)V}\right) = 0,$$

since $m + i < n + \dim V^K \leq (n+1)\dim V^K$, ($\dim V^K$ being ≥ 1 for a G-CW(V) decomposition of $S(V)$). Similarly, $H_G^{m-1-nV}(S(V)) = 0$. □

Note that in particular the lemma implies that

$$H_G^{-V}(\text{point}) \cong \cdots \cong H_G^{-nV}(\text{point}) \cong \cdots \quad \text{for } n \geq 1.$$

One may compute $H_G^{-V}(\text{point})$ explicitly for nice V just as we computed $H_G^V(\text{point})$. This is done in [5].

When G is $\mathbf{Z}_p$ with p prime, Stong has computed $H_G^{nV+m}(\text{point})$ for all n and m. The data $H_G^{nV-m}(\text{point})$ for $m, n \geq 0$ and general G are not known, although the author has machinery for grinding these out in general, as well as explicit formulations of $H_G^{n(V-v)}(\text{point})$ in [5].

3. Periodicity in $H_G^*(E\mathscr{F})$

We are now ready to prove the main result. Let $V < \mathscr{U}$ and $\mathscr{F} = \mathscr{F}(\mathscr{V})$, the family determined by V as above.

THEOREM 3.1. *Let $\gamma = nV + m$ with $n, m \geq 0$. Then*

$$\cup 1_V\colon H_G^{\gamma}(E\mathscr{F}; \overline{T}) \to H_G^{\gamma+V}(E\mathscr{F}; \overline{T})$$

is an isomorphism for each $\overline{T}$ if $n > 0$ or $m > 0$ and an epimorphism if $n = 0$ and $m = 0$.

Proof. Let $N > m + nv + 1$, and consider the commutative diagram, obtained from (2.1):

$$\begin{array}{ccccccc}
\cdots \to & \overline{H}_G^{nV+m}(S^{NV}) & \to & \overline{H}_G^{nV+m}(S^0) & \to & H_G^{nV+m}(S(NV)) \\
& \downarrow e & & \downarrow f & & \downarrow g \\
\cdots \to & \overline{H}_G^{(n+1)V+m}(S^{NV}) & \to & \overline{H}_G^{(n+1)V+m}(S^0) & \to & H_G^{(n+1)V+m}(S(NV))
\end{array}$$

$$\begin{array}{ccc}
\to & \overline{H}_G^{nV+m+1}(S^{NV}) & \to 0 \\
& \downarrow h & \\
\to & \overline{H}_G^{(n+1)V+m+1}(S^{NV}) & \to 0
\end{array}$$

Here, the vertical maps are all multiplication by $\cup 1_V$ and coefficients are in $\overline{T}$. Lemma 2.4 implies that e and h are isomorphisms by our choice of N, and Lemma 2.2 implies that f is an isomorphism if $n > 0$ or $m > 0$ and an epimorphism if n and m are 0. The five lemma now implies that the same is true for g.

Finally, one has the inclusion $\iota\colon S(NV) \to E\mathscr{F}$. One sees that

$$\iota^H\colon S(NV)^H \to E\mathscr{F}^H$$

is an $(nv + m + 1)$-equivalence for each $H \subset G$, so that E may be obtained from $S(NV)$ by attaching G-cells of the form $G/H \times D^r$ with $r > nv + m$. For such cells, $\overline{H}_G^{nV+m}(G/H_+ \wedge S^r) \cong H_H^{nV+m-r}(\text{point}) = 0$, since $nv + m - r < 0$, and we are done. $\square$

Remarks 3.2. (i) This general periodicity is the source of periodicity in $H^*(G)$ under special circumstances, as we shall see in §4.

(ii) There is no class 1_{-V} such that $\cup 1_{-V}$ is inverse to 1_V. Indeed, if $\overline{T}$ is the coefficient system arising from a G-module M, (as will be explained below), then $H_G^{-V}(\text{point}; T) = 0$. This also shows that one cannot expect a periodic class in $H_G^{-V}(\text{point})$.

Consider the following diagram in the case $m = n = 0$, with coefficients in $\overline{T}$:

$$\begin{array}{ccccccccccc}
 & 0 \to & \overline{H}_G^0(S^{NV}) & \to & \overline{H}_G^0(S^0) & \to & H_G^0(S(NV)) & \to & \overline{H}_G^{-1}(S^{NV}) & \to 0 \\
 & & \downarrow e & & \downarrow f & & \downarrow g & & \downarrow h & \\
H_G^{V-1}(S(NV)) & \to & \overline{H}_G^V(S^{NV}) & \xrightarrow{\eta} & \overline{H}_G^V(S^0) & \to & H_G^V(S(NV)) & \to & \overline{H}_G^{V+1}(S^{NV}) & \to 0.
\end{array}$$

Since f and g are epic, one has a short exact sequence

$$(3.3)\qquad \begin{array}{c} 0 \to \ker(\eta e) \to \ker f \to \ker g \to 0 \\ \wr\| \\ \ker f \cap H_G^{-NV}(\text{point}). \end{array}$$

If V has no one-dimensional fixed-sets, one has

$$\ker f = \operatorname{Im}\big(\xi\colon \overline{\mathbf{Z}} \otimes_{\mathscr{B}(\mathscr{F})} \underline{T} \to \underline{T}(G/G)\big),$$

and

$$H_G^{-NV}(\text{point}) = \ker\big(\mu\colon \underline{T}(G/G) = \overline{T}(G/G) \to \operatorname{Hom}_{\mathscr{B}(\mathscr{F})}(\overline{\mathbf{Z}}, \overline{T})\big)$$

by the results in [5] (or by arguments dual to those preceeding Lemma 2.3). Thus

$$\begin{aligned} \ker g &\cong \ker f/\ker f \cap H_G^{-NV}(\text{point}) \\ &\cong \operatorname{Im}\xi/(\operatorname{Im}\xi \cap \ker\mu). \end{aligned}$$

It follows that

$$\ker(\cup 1_V)\colon H_G^0(E\mathscr{F}; \overline{T}) \to H_G^V(E\mathscr{F}; \overline{T}) \cong \operatorname{Im}\xi/(\operatorname{Im}\xi \cap \ker\mu).$$

One now obtains

$$\begin{aligned} H_G^V(E\mathscr{F}; \overline{T}) &\cong H_G^0(E\mathscr{F}; \overline{T})/\operatorname{Im}(\mu\xi) \\ &\cong \operatorname{Hom}_{\mathscr{B}(\mathscr{F})}(\overline{\mathbf{Z}}, T)/\operatorname{Im}(\mu\xi), \end{aligned}$$

with $\cup 1_V$ coinciding with the natural quotient. We therefore conclude, by the above, Theorem 3.1 and [7] (or its generalization in [8]).

THEOREM 3.4. *Let V have no one-dimensional fixed-sets, and let m and $n \geq 0$. Let $\mathscr{F} = \mathscr{F}(V)$. Then*

$$H_G^{nV+m}(E\mathscr{F}; \overline{T}) \cong \begin{cases} \operatorname{Hom}_{\mathscr{B}(\mathscr{F})}(\overline{\mathbf{Z}}, T) & \text{if } n = m = 0, \\ \operatorname{Hom}_{\mathscr{B}(\mathscr{F})}(\overline{\mathbf{Z}}, T)/\operatorname{Im}(\mu\xi) & \text{if } m = 0,\ n \geq 1, \\ \operatorname{Ext}^m_{\mathscr{B}(\mathscr{F})}(\overline{\mathbf{Z}}, T) & \text{if } m \neq 0. \end{cases} \qquad \square$$

As an example, we compute $H_{\mathbf{Z}_2}^{n\rho+m}(E\mathbf{Z}_2; \overline{A})$, where ρ is the non-trivial one-dimensional $\mathbf{Z}_2$-module, and $\mathscr{F} = \{e\}$, so that $E\mathscr{F} = E\mathbf{Z}_2$. Since $A(e) =$

$\mathbf{Z}$, one has

$$\mathrm{Hom}_{\mathscr{B}(\mathscr{F})}(\overline{\mathbf{Z}}, \overline{A}) \cong \mathbf{Z},$$

and ξ: $\overline{\mathbf{Z}} \otimes_{\mathscr{B}(\mathscr{F})} \underline{A} \cong \mathbf{Z} \otimes \mathbf{Z} \to A(\mathbf{Z}_2)$ coinciding with $1 \otimes 1 \to [\mathbf{Z}_2]$, the class of the free $\mathbf{Z}_2$-set, $\mathbf{Z}_2$. Finally,

$$\mu\colon A(\mathbf{Z}_2) \to \mathrm{Hom}_{\mathscr{B}(\mathscr{F})}(\overline{\mathbf{Z}}, \overline{A}) = \mathrm{Hom}(\mathbf{Z}, \mathbf{Z}) \cong \mathbf{Z}$$

takes $[\mathbf{Z}_2]$ to 2, whence $\mathrm{Hom}_{\mathscr{B}(\mathscr{F})}(\overline{\mathbf{Z}}, \overline{A})/\mathrm{Im}(\mu\xi) \cong \mathbf{Z}/2\mathbf{Z}$. $\mathrm{Ext}^m_{\mathscr{B}(\mathscr{F})}(\overline{\mathbf{Z}}, \overline{A})$ will be computed in §4. One therefore has

$$H_G^{n\rho}(E\mathbf{Z}_2; \overline{A}) \cong \begin{cases} \mathbf{Z} & \text{if } n = 0, \\ \mathbf{Z}/2\mathbf{Z} & \text{if } n > 0 \end{cases},$$

with $A(\mathbf{Z}_2)$ acting everywhere via the forgetful map $A(\mathbf{Z}_2) \to A(e) \cong \mathbf{Z}$.

4. Free actions

Now, suppose that G acts freely on $S(V)$, so that $\mathscr{F} = \{e\}$ and $E\mathscr{F} = EG$. We relate $H_G^{nV+m}(E\mathscr{F})$ to $H_G^{nv+m}(E\mathscr{F})$ and deduce classical periodicity results in a more general setting (in that we allow arbitrary Mackey functor coefficients).

PROPOSITION 4.1. *Let X be a free G-*CW *complex, and let $V < \mathscr{U}$ be any G-module such that the action on V by each $g \in G$ is orientation-preserving. Then there exists a natural isomorphism*

$$\phi\colon H_G^{nV+m}(X; \overline{T}) \to H_G^{nv+m}(X, \overline{T})$$

for $n \geq 0$ and any m. (Recall that $v = $* dim *$V$.)

Proof. $H_G^{nV+m}(X; \overline{T})$ may be computed cellularly via a G-CW(V) decomposition of X. Since X is free, the given G-CW decomposition is automatically a G-CW(V) decomposition, so that $\overline{C}_{nV+m}(X) \cong \overline{C}_{nv+m}(X)$ for each m (as contravariant systems). Further, in both cases, X^{nv+m}/X^{nv+m-1} is a wedge of G-spaces of the form $G_+ \wedge S^{nV+m} \cong G_+ \wedge S^{nv+m}$. To compute the boundary homomorphisms for $\overline{C}_{nV+*}(X)$, one orients each summand of $G_+ \wedge S^{nV+m} \cong \bigvee_{g \in G} S^{nV+m}$ by first suspending by a large enough trivial G-module to make S^{nV+m} G-invariant, orienting the identity summand arbitrarily, and then using translation by elements of G to orient the remaining cells. By the hypothesis on V, this coincides with the orientation of cells in $\overline{C}_{nv+*}(X)$, whence the resulting chain complexes are isomorphic. (Naturality follows easily by cellular approximation.)

COROLLARY 4.2. *Let G act freely on $S(V)$ through orientation-preserving maps. Then $\cup 1_V$ induces isomorphisms $H_G^i(EG; \overline{T}) \to H_G^{i+v}(EG; \overline{T})$ for any $i > 0$, and an epimorphism if $i = 0$.*

Remark 4.3. The hypothesis of Proposition 4.1 explains the failure of period 1 periodicity in $H^*(\mathbf{Z}_2)$, and the presence of a period of 2 (by choosing $V = \rho \oplus \rho$ where ρ is the one-dimensional irreducible $\mathbf{Z}_2$-vector space). One still retains, however, *equivariant* period 1 periodicity of the form

$$H_G^{\gamma}(EG; \overline{T}) \cong H_G^{\gamma+\rho}(EG; \overline{T}),$$

where ρ has dimension 1.

One may now compute $\mathrm{Ext}^m_{\mathscr{B}(\mathscr{F})}(\hat{\mathbf{Z}}; \overline{A})$ for $G = \mathbf{Z}_2$ and $V = \rho$, as promised in §3. By the above, $\mathrm{Ext}^{2m}_{\mathscr{B}(\mathscr{F})}(\overline{\mathbf{Z}}, \overline{A}) \cong H_G^{2m}(E\mathbf{Z}_2; \overline{A}) \cong H_G^{2\rho}(E\mathbf{Z}_2; \overline{A}) = \mathbf{Z}/2\mathbf{Z}$, and it remains to compute $\mathrm{Ext}^{2m+1}_{\mathscr{B}(\mathscr{F})}(\overline{\mathbf{Z}}, \overline{A})$. Since

$$\begin{aligned}\mathrm{Ext}^1_{\mathscr{B}(\mathscr{F})}(\overline{\mathbf{Z}}, A) &\cong H_G^1(E\mathbf{Z}_2; \overline{A}) \cong H_G^{2m+1}(E\mathbf{Z}_2; \overline{A}) \cong H_G^{2m+1}(E\mathbf{Z}_2; \overline{A}) \\ &\cong \mathrm{Ext}^{2m+1}_{\mathscr{B}(\mathscr{F})}(\mathbf{Z}, A),\end{aligned}$$

it therefore suffices to compute the first Ext group. By (2.1),

$$\mathrm{Ext}^1 \cong H_G^1(S(NV)) \cong H_G^{2-NV}(\text{point}) = 0,$$

by Stong's calculation in [5].

5. Relationship with Classical Results

In order to specialize Corollary 4.2 to classical results about $H^*(G; A)$ for a $\mathbf{Z}G$-module A, we recall some material from [8].

Let $\mathscr{H}(\mathscr{F})$ denote the category whose objects are the spaces G/H with $H \in \mathscr{F}$, and whose morphisms $G/H \to G/K$ are the $\mathbf{Z}G$-module homomorphisms $\mathbf{Z}G/H \to \mathbf{Z}G/K$, where $\mathbf{Z}G/J$ denotes the free $\mathbf{Z}$-module on G/J. A Hecke functor (based on $\mathscr{F}$) is then an additive functor

$$T\colon \mathscr{H}(\mathscr{F}) \to \mathscr{A}b.$$

If A is a $\mathbf{Z}G$-module, then the assignment $\overline{A}\colon G/H \to \mathrm{Hom}_{\mathbf{Z}G}(\mathbf{Z}G/H, A)$ gives a contravariant Hecke functor, while $G/H \to \mathrm{Hom}_{\mathbf{Z}G}(A, \mathbf{Z}G/H)$ gives a covariant one.

If $\mathscr{O}(\mathscr{F})$ is the full subcategory of $\mathscr{O}$ with objects G/H for $H \in \mathscr{F}$, then one has a forgetful functor $\mathscr{O}(\mathscr{F}) \to \mathscr{H}(\mathscr{F})$ by [8]. This turns every Hecke functor into a Mackey functor.

One now has, by results in [8].

$$H^i(G; A) \cong H_G^i(EG; \bar{A}),$$

so that all the classical periodicity results follow from §4, and continue to hold in the more general form of 4.2.

REFERENCES

1. G.E. BREDON, *Equivariant cohomology theories*, Springer Lecture Notes in Mathematics, vol. 34, Springer-Verlag, New York, 1967.
2. T. TOM DIECK, *Transformation groups and representation theory*, Springer Lecture Notes in Mathematics, vol. 766, Springer-Verlag, 1979.
3. G. LEWIS, *Green functors*, mimeographed notes, Syracuse University, Syracuse, New York, 1983.
4. G. LEWIS, J.P. MAY, AND J. MCCLURE, *Ordinary equivariant $RO(G)$-graded cohomology*, Bull. Amer. Math. Soc., vol. 4 (1981), pp. 208–212.
5. G. LEWIS, J.P. MAY, J. MCCLURE AND S. WANER, *Ordinary equivariant $RO(G)$-graded cohomology*, in preparation. University of Chicago, Chicago, Illinois.
6. R.E. STONG, *The cohomology of a point*, Preprint, University of Virginia, 1981.
7. S. WANER, *A generalization on the cohomology of groups*, Proc. Amer. Math. Soc., vol. 85 (1982), pp. 469–474.
8. ______, *Mackey functors and G-cohomology*, Proc. Amer. Math. Soc., vol. 90 (1984), pp. 641–648.
9. ______, *Oriented G-mainfolds*, Preprint, Hofstra University, 1981.

HOFSTRA UNIVERSITY
HEMPSTEAD, NEW YORK