ILLINOIS JOURNAL OF MATHEMATICS
Volume 30, Number 3, Fall 1986

MEROMORPHIC SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS WHICH ARE NOT SOLUTIONS OF FIRST ORDER ALGEBRAIC DIFFERENTIAL EQUATIONS

BY

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1. Introduction

In [11], Siegel proved that no nontrivial solution of Bessel's equation

$$z^2y'' + zy' + (z^2 - \alpha^2)y = 0$$

can satisfy a first order algebraic differential equation (ADE) with coefficients in the field of rational functions, provided that α is not one-half of an odd integer. This result was extended by Bank [1], who proved that no nontrivial solution of the above equation will satisfy a first order ADE with coefficients in the field of meromorphic functions of order less than one, the condition on α remaining the same.

Here we consider equations of the form

$$w'' + Pw' + Qw = 0, \tag{1.1}$$

where P and Q are meromorphic functions of finite order. We assume that all solutions of (1.1) are meromorphic, and find conditions under which solutions do not satisfy first order ADE with suitable coefficients. We require a lemma due to Siegel [11], which we state in a less general form.

DEFINITION. *A differential field L of meromorphic functions is a field of meromorphic functions which contains derivatives of all its elements.*

If α is any nonnegative real number we denote by L_α the field of all meromorphic functions of order less than or equal to α. Clearly L_α is a differential field of meromorphic functions.

Siegel's lemma is as follows:

LEMMA 1. *Let L be a differential field of meromorphic functions and let P and Q in* (1.1) *belong to L. Let the solutions of* (1.1) *be meromorphic. Suppose*

Received September 27, 1983.

w_0 is a solution which is not algebraic over L and which satisfies a first order ADE with coefficients in L. Then there is a nontrivial solution w_1 such that w_1'/w_1 is algebraic over L.

We need also the following result (see [6]).

LEMMA 2. *If a meromorphic function f satisfies an algebraic equation*

$$f^n + \phi_1 f^{n-1} + \cdots + \phi_n = 0 \tag{1.2}$$

where the ϕ_i are also meromorphic, then

$$T(r,f) \leq \sum_{j=0}^{n} T(r,\phi_j) + O(1).$$

2. Theorem 1 and examples

For any meromorphic function f we use the following notation: $\sigma(f)$ is the growth order of f; $\lambda(f)$ is the exponent of convergence of the zeros of f; $\mu(f)$ is the exponent of convergence of the poles of f. Let $n(r, a)$ denote the number of a points in $|z| \leq r$ $(0 \leq |a| \leq \infty)$ of f, the roots being counted with multiplicity and $\bar{n}(r, a)$ the number of distinct a points in $|z| \leq r$. Then $\bar{n}(r, a) \leq n(r, a)$ and

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log^+ n(r,0)}{\log r};$$

$$\mu(f) = \limsup_{r\to\infty} \frac{\log^+ n(r,\infty)}{\log r}.$$

Write

$$\bar{\lambda}(f) = \limsup_{r\to\infty} \frac{\log^+ \bar{n}(r,0)}{\log r}, \quad \bar{\mu}(f) = \limsup_{r\to\infty} \frac{\log^+ \bar{n}(r,\infty)}{\log r}.$$

Clearly $\bar{\lambda}(f) \leq \lambda(f) \leq \sigma(f)$; $\bar{\mu}(f) \leq \mu(f) \leq \sigma(f)$. We now state:

THEOREM 1. *Suppose all solutions of* (1.1) *are meromorphic and $P, Q \in L_\alpha$. If every nontrivial solution w is such that* $\max(\bar{\lambda}(w), \bar{\mu}(w)) > \alpha$, *then no nontrivial solution satisfies a first order ADE with coefficients in L_α.*

Remark. $\bar{\lambda}(w)$ *and* $\bar{\mu}(w)$ *may not be finite.*

Proof. Suppose w_0 is a solution that satisfies a first order ADE with coefficients in L_α. If w_0 is algebraic over L_α and satisfies an equation of the form (1.2) with $\phi_i \in L_\alpha$, then by Lemma 2 we conclude that

$$T(r, w_0) \leq \sum_{j=0}^{n} T(r, \phi_j) + O(1).$$

This inequality implies that $\sigma(w_0) \leq \alpha$, but by hypothesis $\sigma(w_0) \geq \max(\bar{\lambda}(w_0), \bar{\mu}(w_0)) > \alpha$. This contradiction proves that w_0 is not algebraic over L_α and we may apply Siegel's lemma. Thus there is a nontrivial solution w_1 such that w_1'/w_1 is algebraic over L_α. As before, we may deduce that $\sigma(w_1'/w_1) \leq \alpha$, which in turn implies $\bar{\mu}(w_1'/w_1) \leq \alpha$. Note that the poles of w_1'/w_1 are simple and are exactly at the zeros and poles of w_1 and we have $\max(\bar{\mu}(w_1), \bar{\lambda}(w_1)) \leq \alpha$. This contradiction proves the theorem.

We need the following definition to state a corollary of Theorem 1. Suppose that a meromorphic function f has zeros at z_i $(i = 1, 2, \dots)$ of multiplicity m_i $(i = 1, 2, \dots)$. We shall then say that the zeros of f are of *bounded multiplicity* if there is a constant M such that $m_i \leq M$ for all i. If there is no such M we shall say that the zeros are of *unbounded multiplicity*. In the same manner, we speak of poles of bounded or unbounded multiplicity. Further if $0 < \sigma(f) \equiv \sigma < \infty$ then

$$\tau(f) = \limsup_{r \to \infty} T(r, f)/r^\sigma$$

will denote the type of f.

COROLLARY 1. *Suppose w_1 and w_2 are linearly independent meromorphic solutions of* (1.1) *such that* (i) $\sigma(w_1)$ *and* $\sigma(w_2)$ *are finite non-integers and* (ii) $\min(\sigma(w_1), \sigma(w_2)) > \alpha$; *and either* (iii) $\sigma(w_1) \neq \sigma(w_2)$ *or* (iv) $\sigma(w_1) = \sigma(w_2)$ *and* $\tau(w_1) \neq \tau(w_2)$. (v) *If the zeros and poles of all nontrivial solutions are of bounded multiplicity then no nontrivial solution satisfies a first-order ADE with coefficients in L_α, provided $P, Q \in L_\alpha$.*

Proof. *Case* 1. $\sigma(w_1) \neq \sigma(w_2)$. We may suppose $\sigma_1 \equiv \sigma(w_1) > \sigma(w_2) \equiv \sigma_2$. Since w_1, w_2 are linearly independent, we may denote by $w = c_1 w_1 + c_2 w_2$ $(c_1 c_2 \neq 0)$ any other solution of (1.1). Then

$$T(r, w) \leq T(r, w_1) + T(r, w_2) + O(\log r) \tag{2.1a}$$

$$T(r, w_1) \leq T(r, w) + T(r, w_2) + O(\log r) \tag{2.1b}$$

This shows that $\sigma(w) = \sigma_1$.

Case 2. $\sigma(w_1) = \sigma(w_2)$. From (2.1b) and hypothesis (iii), we can again conclude that $\sigma(w) = \sigma(w_1)$.

Now using hypothesis (i) and (ii) and the Hadamard factorization theorem, we get

$$\alpha < \max\{\lambda(w), \mu(w)\} = \max\{\bar{\lambda}(w), \bar{\mu}(w)\}.$$

The corollary now follows from Theorem 1.

COROLLARY 2. *Suppose $Q(z)$ is an entire function with $\lambda(Q) < \sigma(Q) < \infty$. At least one nontrivial solution of the equation*

$$w'' - Q(z)w = 0 \tag{2.1}$$

satisfies a first order ADE with coefficients in $L_{\sigma(Q)}$ if and only if there is a solution w of (2.1) *for which $\lambda(w) = \sigma(Q)$.*

Proof. By Theorem 2(B) of [2(ii)] all nontrivial solutions w of (2.1) satisfy $\lambda(w) \geq \sigma(Q)$. Suppose $\lambda(w) > \sigma(Q)$ for all w. Since the solutions of (2.1) have simple zeros, $\bar{\lambda}(w) = \lambda(w) > \sigma(Q)$. The hypothesis of Theorem 1 is satisfied and no nontrivial solution of (2.1) satisfies a first order ADE with coefficients in $L_{\sigma(Q)}$.

Now suppose there is a solution w of (2.1) for which $\lambda(w) = \sigma(Q)$. Clearly $\mu(w'/w) = \lambda(w) = \sigma(Q)$. Thus

$$N\left(r, \frac{w'}{w}\right) = O(r^{\sigma(Q)+\varepsilon}). \tag{2.2}$$

To estimate $m(r, w'/w)$ we use a result of Valiron [13, p. 105]; see also [14]. Thus

$$\log^+\left|\frac{w''(z)}{w(z)}\right| = 2\log^+\left|\left(\{1 + o(1)\}\frac{\nu(r)}{z}\right)\right|$$

outside a set of finite logarithmic measure. But $w''(z)/w(z) = Q(z)$, which is entire and of finite order, so that

$$\log^+\left|\frac{\nu(r)}{z}\right| = O(r^{\sigma(Q)+\varepsilon}). \tag{2.3}$$

Again, we have

$$\frac{w'(z)}{w(z)} = \frac{\nu(r)}{z}(1 + o(1)), \tag{2.4}$$

outside of a set of finite logarithmic measure. This relation together with (2.3) gives

$$m\left(r, \frac{w'}{w}\right) = O(r^{\sigma(Q)+\varepsilon}). \tag{2.4}$$

Putting together (2.2) and (2.4) we have

$$T\left(r, \frac{w'}{w}\right) = O\left(r^{\sigma(Q)+\varepsilon}\right). \tag{2.5}$$

We conclude from (2.5) that $(w'/w) \in L_{\sigma(Q)}$. Thus $w'(z)/w(z) = R(z)$ (say) $\in L_{\sigma(Q)}$ and w satisfies the first order ADE $w' - R(z)w = 0$ with coefficients in $L_{\sigma(Q)}$.

Remark 1. One could derive (2.4) by the method of [4, Lemma 3.3] as well or (2.5) directly from [5, Theorem 1].

Remark 2. Here $\sigma(Q)$ must be a positive integer. For a related result of Strelitz on algebraic differential equations $P(z, w, w') = 0$, see [10, Theorem 2.9] and [12].

Example 1. Consider the equation

$$w'' - \tan zw' + (z-1)w = 0. \tag{2.6}$$

We show that this equation and its solutions satisfy all the hypotheses of Theorem 1. Let $1 \leq \alpha < 3/2$. Since $\tan z$ is a meromorphic function of order 1, the coefficients of (2.6) belong to L_1. The solutions of (2.6) are of the form $w(z) = E(z)/\sin z$, where $E(z)$ is entire and satisfies $E'' + zE = 0$. The Wiman-Valiron theory implies that $\sigma(E) = 3/2 = \lambda(E)$. (See also [10, p. 249] or [3, p. 426] for the orders of the solutions of the DE $w'' + Pw' + Qw = 0$ where P and Q are polynomials or rational functions in z, and for various references.) It is easy to check that $E(z)$ has simple zeros and since $\sin z$ has simple zeros too, we have

$$\max(\bar{\lambda}(w), \bar{\mu}(w)) = 3/2 > \alpha.$$

All the conditions of Theorem 1 are fulfilled and so no nontrivial solution of (2.6) can satisfy a first order ADE with coefficients in L_α.

We note that equation (2.6) illustrates that Theorem 5.4.3 and 5.4.4 of [7] cannot be extended to meromorphic functions.

Example 2. We show by an example that if, for some solutions w in Theorem 1,

$$\max(\bar{\lambda}(w), \bar{\mu}(w)) = \alpha,$$

then all the solutions may satisfy first order ADEs with coefficients in L_α. It is not difficult to see that the general solution of the equation

$$w'' + \left\{\frac{2e^z}{e^z - 1} - \frac{1}{z}\right\}w' + \left\{\frac{e^z(z-1)}{z(e^z-1)} - 4z^2\right\}w = 0 \tag{2.7}$$

is $w(z) = (c_1 \cosh z^2 + c_2 \sinh z^2)/(e^z - 1)$. The coefficients of the equation (2.7) belong to L_1. For $c_1 \neq \pm c_2$, $\bar{\lambda}(w) = 2$, and $\max(\bar{\lambda}(w), \bar{\mu}(w)) = 2 > 1$, when $c_1 = \pm c_2$,

$$w(z) = ce^{\pm z^2}/(e^z - 1)$$

so that $\bar{\mu}(w) = 1$ but $\bar{\lambda}(w) = 0$. Thus in this case $\max(\bar{\lambda}(w), \bar{\mu}(w)) = 1$, and all solutions of (2.7) satisfy a first order ADE with coefficients in L_1. For, in general we may write the solution of (2.7) as

$$w = \frac{c_1 e^{z^2} + c_2 e^{-z^2}}{(e^z - 1)}$$

and this satisfies

$$\left(w' + \frac{e^z}{e^z - 1} w\right)^2 = 4z^2 w^2 - \frac{16 c_1 c_2 z^2}{(e^z - 1)^2}.$$

Example 3. In Theorem 1 it is not enough to assume that $\max(\lambda(w), \mu(w)) > \alpha$. This is because w may have zeros or poles of unbounded multiplicity so that

$$\max(\bar{\lambda}(w), \bar{\mu}(w))$$

may become less than or equal to α. We give an example of such a situation:

Let $F(z)$ be the canonical product with zeros at $z = 1, 2, \ldots, n, \ldots,$ where the zero at $z = n$ is of multiplicity n. Thus the zeros of F are of unbounded multiplicity. Let $f(z) = 1/F(z)$, so that $\mu(f) = 2$. Set $Q(z) = f''(z)/f(z)$. Evidently $Q(z)$ is meromorphic with double poles at $z = 1, 2, \ldots, n, \ldots$. We have

$$m(r, Q) = m(r, f''/f) = O(\log r),$$

and

$$N(r, Q) = \int_0^r \frac{n(t, Q)}{t}\, dt \sim 2r,$$

since $n(t, Q) \sim 2t$. Thus

$$T(r, Q) \sim 2r \quad \text{and} \quad Q \in L_1.$$

The function $f(z)$ is a meromorphic solution of the DE

$$w'' - Q(z) w = 0. \tag{2.8}$$

It can be verified by direct substitution that

$$g(z) = f(z)\int_0^z F^2(t)\,dt$$

is a linearly independent meromorphic solution of (2.8). Thus all solutions of (2.8) are meromorphic and of the form

$$h(z) = f(z)\left(c_1 + c_2\int_0^z F^2(t)\,dt\right).$$

It is possible that the function $c_1 + c_2\int_0^z F^2(t)\,dt$ has zeros at some or all of the points $z = 1, 2, \ldots, n, \ldots$. Let the zeros be at $\{n_k\}$, which could be an infinite sequence. Clearly the zero at n_k will be of multiplicity $2n_k + 1$ so that $h(z)$ will have a zero multiplicity $n_k + 1$. It can be seen that in the three cases (i) $c_1 = 0$, (ii) $c_2 = 0$, or (iii) $c_1c_2 \neq 0$,

$$\max(\lambda(h), \mu(h)) \geq 2$$

but

$$\max\{\bar{\mu}(h), \bar{\lambda}(h)\} = 1.$$

Now consider $f(z)$. By an argument similar to the one used for computing $T(r, Q)$ we can show that $T(r, f'/f) \sim r$. Hence $f'/f \in L_1$, that is, $f'(z)/f(z) = R(z)$ (say), where $R(z) \in L_1$. But then $f(z)$ satisfies the first order ADE $f'(z) - R(z)f(z) = 0$, with coefficients in L_1.

Example 4. We now show that the solutions of Mathieu's equation

$$w'' + (a + b\cos 2z)w = 0 \quad \text{where } b \neq 0,$$

do not satisfy a first order ADE with coefficients in L_F, the field of all meromorphic functions of finite order. For suppose that a solution w satisfies an ADE

$$\sum_{i,j} a_{ij}(z) w^i (w')^j = 0,$$

where the $a_{ij}(z)$ are meromorphic functions of finite order. Let

$$\rho = \max_{i,j}\{1, \sigma(a_{ij})\}$$

so that the coefficients of the ADE and of Mathieu's equation belong to L_ρ. Bank and Laine [2] have proven that $\lambda(w) = \bar{\lambda}(w) = \infty$; thus $\bar{\lambda}(w) > \rho$.

Theorem 1 implies that w does not satisfy a first order ADE with coefficients in L_ρ. This contradiction proves our contention.

Example 5. The fact that all solutions of an equation have zeros or poles of unbounded multiplicity does not necessarily imply that the hypothesis of Theorem 1 will not hold. We use Mathieu's equation to demonstrate this. Let $F(z)$ be the function of Example 3. Straightforward computation shows that the solutions of the equation

$$w'' - \frac{2F'}{F}w' + \left(Q + \frac{2F'^2}{F^2} - \frac{F''}{F}\right)w = 0 \tag{2.9}$$

where $Q(z) = a + b\cos 2z$, are of the form $w(z) = F(z)E(z)$, where $E(z)$ is a solution of Mathieu's equation. The coefficients of (2.9), according to the computations in Example 3, are of order at most 1. The zeros of $w(z)$ are of unbounded multiplicity, because of the factor $F(z)$. Since $\lambda(E) = \infty$, we have $\bar{\lambda}(w) = \infty$. Thus, as before, no nontrivial solution of (2.9) satisfies a first order ADE with coefficients in L_F.

Example 6. There are DE's with all solutions of non-integral order though not all are of the same order. We construct an equation with one solution of order $3/2$ and another one of order $1/2$.

Consider the equation $w'' - zw = 0$. By the Wiman-Valiron theory, all non-zero solutions have order $3/2$. Let P_1 and P_2 be two linearly independent (1.i.) solutions. We claim that P_1 and P_2 have no common zeros. For these are 1.i. and so their Wronskian $P_1(t)P_2'(t) - P_2(t)P_1'(t)$ is non-zero. Also since the zeros of P_1 and P_2 are simple, $P_1'(a) \neq 0$ at a zero a of P_1. Hence at a,

$$-P_2(a)P_1'(a) \neq 0$$

and so

$$P_2(a) \neq 0.$$

Now P_1 and P_2 are functions of order $3/2$ and hence the exponent of convergence of zeros of these two functions is also $3/2$. Let $\{a_n\}$ be the sequence of zeros of P_1. Let $\{a_{n_k}\}$ be a subsequence whose exponent of convergence is $1/2$. Let $\{a_n\}'$ be the sequence $\{a_n\}$ with the elements $\{a_{n_k}\}$ removed. Denote by $\tilde{P}_1(z)$ the canonical product with zeros at $\{a_n\}'$. Then the functions $P_1(z)/\tilde{P}_1(z)$, $P_2(z)/\tilde{P}_1(z)$ are functions of order $1/2$ and $3/2$ respectively, and are solutions of the DE

$$w'' + 2\frac{\tilde{P}_1'(z)}{\tilde{P}_1(z)}w + \left(\frac{\tilde{P}_1''(z)}{\tilde{P}_1(z)} - z\right)w = 0.$$

3. Theorem 2

Example 3 shows that meromorphic solutions of the equation $w'' + Pw' + Qw = 0$ may have zeros and poles of unbounded multiplicity though P and Q themselves have poles of bounded multiplicity. We now determine conditions on P and Q that will imply that the zeros and poles of the solution are of bounded multiplicity.

Suppose f is a meromorphic function with poles at z_i $(i = 1, 2, \ldots)$ of multiplicity m_i $(i = 1, 2, \ldots)$. Let $f(z) = \sum_{i=-m_k}^{\infty} c_{i,k}(z - z_k)^i$ be the Laurent expansion of $f(z)$ about z_k $(k = 1, 2, \ldots)$. Suppose that

$$|c_{-m_k,k}| \leq K \quad \text{for } k = 1, 2, \ldots, \tag{3.1}$$

where K is a constant independent of k. We say that $f(z)$ has *bounded highest coefficients at the poles* if (3.1) is satisfied. Let $\{z_{n_j}\}$ be a subsequence of poles of $f(z)$. We say that $f(z)$ has the *highest coefficient bounded away from zero for the sequence of poles* $\{z_{n_j}\}$ if there is a $\delta > 0$ such that

$$|c_{-m_k,k}| \geq \delta \quad \text{for } k \in \{n_j\}, \tag{3.2}$$

and δ independent of k.

THEOREM 2. *Consider the DE*

$$w'' + Pw' + Qw = 0 \tag{3.3}$$

where P and Q are meromorphic and have bounded highest coefficients at the poles and P has highest coefficients bounded away from zero for the subsequence of multiple poles of P. Any meromorphic solution of such an equation has zeros and poles of bounded multiplicity.

(Note that we are not assuming that all solutions of (3.3) are meromorphic.)

Remark. No condition is required on any entire coefficient of (3.3) for the theorem to hold.

Proof. Suppose w is a meromorphic solution with a zero of multiplicity n at x. Then

$$w(z) = (z - x)^n g(z),$$

where $g(x) \neq 0$ and $g(z)$ is regular in a neighborhood of x. Suppose $n > 1$ and

$$P(z) = (z - x)^s p(z) \quad \text{and} \quad Q(z) = (z - x)^t q(z),$$

where $p(x) \neq 0 \neq q(x)$. Substituting these expressions in (3.3) we arrive at

$$\begin{aligned}&\left\{n(n-1)(z-x)^{n-2}g(z)+2n(z-x)^{n-1}g'(z)+(z-x)^n g''(z)\right\}\\&\quad+\left\{n(z-x)^{n-1}g(z)+(z-x)^n g'(z)\right\}(z-x)^s p(z)\\&\quad+(z-x)^n g(z)(z-x)^t q(z)=0.\end{aligned}\tag{3.4}$$

If $s \geq 0$ and $t \geq 0$, we divide across by $(z-x)^{n-2}$ and set $z = x$ to get the contradiction $g(x) = 0$. Thus, in this case, $n = 1$. Now consider the case where $P(z)$ has a simple pole. It can be verified that in this case $Q(z)$ can have at most a double pole. Assume that Q has a double pole. (The other cases can be done similarly.) Now $p(x)$ and $q(x)$ are the highest coefficients of $P(z)$ and $Q(z)$ respectively at x. By our hypothesis $|p(x)| \leq K$ and $|q(x)| \leq K$. Multiplying across by $(z-x)^{-n+2}$ in (3.4) and setting $z = x$, we get

$$n(n-1)g(x)+np(x)g(x)+q(x)g(x)=0.$$

Since $g(x) \neq 0$,

$$n(n-1)+np(x)+q(x)=0.$$

The condition on $p(x)$ and $q(x)$ implies that n is bounded by an absolute constant and hence the zeros are of bounded multiplicity.

Now consider the situation where P has an infinite number of nonsimple poles. Let x be one of these. Let $-s$ be the multiplicity of the pole of P (recall that we are writing $P(z) = (z-x)^s p(z)$ so that s is negative) and $-t$ that of Q. Clearly we must have $s = t + 1$. The terms in (3.4) with the lowest power of $z - x$ are

$$n(z-x)^{n-1}g(z)(z-x)^s p(z) \quad \text{and} \quad (z-x)^n g(z)(z-x)^t q(z).$$

Multiplying (3.4) by $(z-x)^{-n-t}$ and setting $z = x$, we have

$$np(x)+q(x)=0.$$

Since $|p(x)| \geq \delta$ and $|q(x)| \leq K$, we get $n \leq K/\delta$, and once again, the zeros are of bounded multiplicity.

The case for the poles of w can be done similarly. We write $w(z) = (z - x)^{-n}g(z)$, $g(x) \neq 0$ and proceed as before.

Remark. Note that equation (2.9) demonstrates that the solutions of (3.3) may not have poles at the singularities of the coefficients of the equation but instead may have zeros of unbounded multiplicity.

4. Rational coefficients

We now turn to the particular case where the differential field is L_R, the field of rational functions.

THEOREM 3. *If all the nontrivial solutions of*

$$w'' + P(z)w' + Q(z)w = 0, \quad \text{where } P, Q \in L_R, \tag{4.1}$$

are meromorphic functions with infinite number of zeros then no nontrivial solution of (4.1) *satisfies a first order ADE with coefficients in L_α for any $\alpha < \frac{1}{2}$.*

Proof. Any solution w has the form $w = (G/T)e^q$ where q, T are polynomials. By hypothesis $(G/T)e^q$ satisfies a DE with rational coefficients and hence so does Ge^q. Suppose then Ge^q satisfies $w'' + P_1w' + Q_1w = 0$. Then G satisfies

$$G'' + (2q' + P_1)G' + \left(q'' + q'^2 + Pq' + Q_1\right)G = 0.$$

Note that G is a transcendental entire function and satisfies a second-order DE with rational coefficients. Hence $\sigma(G) \geq \frac{1}{2}$. Further G has all but a finite number of zeros simple and so G'/G is meromorphic, and since the exponent of convergence of the poles of G'/G is $\geq \frac{1}{2}$, $\sigma(G'/G) \geq \frac{1}{2}$. Now $w'/w = G'/G - T'/T + q'$, and so $\sigma(w'/w) \geq \frac{1}{2}$. But if w satisfies a first-order ADE with coefficients in L_α, $\alpha < \frac{1}{2}$, then by Siegel's lemma $\sigma(w'/w) < \frac{1}{2}$ and we have a contradiction. This completes the proof.

We now write (4.1) in the equivalent form

$$P_0w'' + P_1w' + P_2w = 0, \tag{4.1$'$}$$

where P_j are polynomials.

COROLLARY 1. *Let P_j be of degree α_j ($j = 0, 1, 2$) in* (4.1)$'$. *If this equation has all solutions entire and one solution of non-integral order, then no nontrivial solution satisfies a first-order* ADE *with coefficients in L_ρ for*

$$\rho < \frac{\alpha_2 - \alpha_0 + 2}{2}.$$

In particular, if also $\alpha_0 \neq \alpha_2 + 1$, then no nontrivial solution satisfies a first-order ADE *with coefficients in L_ρ for $\rho < 3/2$.*

Proof. (a) Let $K = \max(\alpha_1 - \alpha_0, \alpha_2 - \alpha_0)$ and $m_k = K + 2 + \alpha_0 - \alpha_k - k$ for $k = 0, 1, 2$. Then (k, m_k) are the points from which Newton's polygon of Wiman-Valiron theory is constructed. Since there is a solution of non-integral order, the Newton's polygon is formed by joining the points $(0, m_0)$ with $(2, m_2)$, and the order of the solution is given by

$$\frac{K+2-(K+2+\alpha_0-\alpha_2-2)}{2}=\frac{\alpha_2+2-\alpha_0}{2}.$$

In particular, we must have (i) $\alpha_2+2>\alpha_0$. Moreover the coordinate $(1, m_1)$ lies on or above the line joining $(0, m_0)$ and $(2, m_2)$. Thus $m_1 \geq (m_0+m_2)/2$. Hence we must have (ii) $\alpha_0 \geq 2\alpha_1-\alpha_2$.

We now show that there cannot be a polynomial solution. Suppose there is a solution $Q(z)$ which is a polynomial of deg n. Then $\deg(P_0Q'')=\alpha_0+n-2$, $\deg(P_1Q')=\alpha_1+n-1$, $\deg(P_2Q)=\alpha_2+n$. By (i) $\alpha_2+n>\alpha_0+n-2$. Therefore if Q does satisfy the DE we must have $\alpha_1+n-1=\alpha_2+n$ for cancellation to take place. But this equality implies $\alpha_1=\alpha_2+1$. From (ii) we get

$$\alpha_0 \geq 2(\alpha_2+1)-\alpha_2,$$

that is

$$\alpha_0-2 \geq \alpha_2.$$

But this contradicts (i). Thus there cannot exist a polynomial solution.

(b) We now complete the proof. We have just proven that all the solutions are transcendental. Since the orders of the transcendental solutions are given by the negative of the slopes of the Newton's polygon, it follows that if one solution is of non-integral order, then all solutions are of non-integral order. Moreover, the order of any solution $w \not\equiv 0$ is $\frac{1}{2}(\alpha_2+2-\alpha_0)$ as in (a). Write $w=(G/T)e^q$ where G is the canonical product and q, T are polynomials. Now G is of non-integer order and as in the proof of the theorem, all but a finite number of zeros of G are simple, and so

$$\sigma(G'/G)=\frac{\alpha_2+2-\alpha_0}{2}.$$

Now if w satisfies a first-order ADE with coefficients in L_ρ, $\rho<\frac{1}{2}(\alpha_2-\alpha_0+2)$, then by Siegel's lemma, there is a solution $w_1 \not\equiv 0$ of the second-order DE for which

$$\sigma(w_1'/w_1) \leq \rho < \tfrac{1}{2}(\alpha_2-\alpha_0+2).$$

But $\sigma(G'/G)=\sigma(w_1'/w_1)$ since w_1 is also of the same non-integral order as w. Thus

$$\sigma(w_1'/w_1)=\tfrac{1}{2}(\alpha_2-\alpha_0+2).$$

This is a contradiction and therefore w cannot satisfy a first-order ADE with coefficients in L_ρ for $\rho < \frac{1}{2}(\alpha_2 - \alpha_0 + 2)$. Finally, if $\alpha_0 \neq \alpha_2 + 1$, then $\frac{1}{2}(\alpha_2 - \alpha_0 + 2) \geq 3/2$ and this proves the last statement of the corollary.

Remark. Several questions about meromorphic solutions of DEs with rational coefficients may be reduced to entire solutions of such equations. (See [8].)

Example 7. Consider the equation

$$w'' - z^2 w = 0.$$

For any solution $w \not\equiv 0$, $\sigma(w) = 2$ [10, p. 249] and also $\bar{\lambda}(w) = 2$ [2(ii), Theorem 1(c)]. Hence no solution $w \not\equiv 0$ of this equation satisfies a first-order ADE with coefficients in L_ρ for any $\rho < 2$.

Example 8. If all the solutions of an equation of the form (4.1) do not have an infinity of zeros, the solutions may satisfy a first order ADE with rational coefficients. For example, $w'' + w = 0$ has the general solution $w = c_1 \cos z + c_2 \sin z$. When $c_1 = \pm\sqrt{-1}\, c_2$, $w = ce^{\pm iz}$, which has no zeros. In all other cases, w has an infinite number of zeros. The solution $w = \cos z$, for example, satisfies the first order ADE $w'^2 + w^2 - 1 = 0$, with rational coefficients.

5. Constant coefficients

It may happen that some solutions of a second order DE satisfy a first order ADE with appropriate coefficients and some do not. We consider this situation in the simplest case $w'' + aw' + bw = 0$ where $a, b \in \mathbf{C}$, the field of complex numbers.

In the following, K denotes the field of meromorphic functions of order less than one together with functions of order equal to one and of minimal type. We now state:

THEOREM 4. *Consider the DE $w'' + aw' + bw = 0$, where $a, b \in \mathbf{C}$, and let λ_1, λ_2 be the roots of $\lambda^2 + a\lambda + b = 0$.*

(i) *If one or both roots are zero then obviously all solutions of the DE satisfy a first order ADE with constant coefficients.*

(ii) (a) *If λ_1/λ_2 is a rational number not equal to* 1, *then all solutions of the DE satisfy a first order ADE with constant coefficients.*

(b) *If $\lambda_1/\lambda_2 = 1$, then all solutions satisfy a first order ADE with rational coefficients.*

(iii) *Suppose λ_1/λ_2 is either an irrational or a (proper) complex number (that is, $Imag(\lambda_1/\lambda_2) \neq 0$). A solution of the DE will not satisfy a first order ADE with coefficients in K if and only if it has a zero.*

Proof. (i) This case is obvious.

(ii) (a) Suppose λ_1/λ_2 is a rational number not equal to 1. There is a number λ such that $\lambda_1 = (p/q)\lambda$ and $\lambda_2 = (r/s)\lambda$ where p, q, r, s are integers and $ps \neq qr$. A general solution of the DE is

$$w = c_1e^{\lambda_1 z} + c_2e^{\lambda_2 z} = c_1e^{p\lambda z/q} + c_2e^{r\lambda z/s},$$
$$w' = p\lambda/qc_1e^{p\lambda z/q} + r\lambda c_2/se^{r\lambda z/s}.$$

Write $ps - qr = d \neq 0$. Then

$$\left(\frac{qw'}{p\lambda} - w\right)^{sp} = \left(\frac{-dc_2}{ps}\right)^{sp} e^{pr\lambda z}. \tag{5.1}$$

Similarly

$$\left(\frac{sw'}{r\lambda} - w\right)^{rq} = \left(\frac{dc_1}{rq}\right)^{rq} e^{pr\lambda z}. \tag{5.2}$$

Combining (5.1) and (5.2), we obtain

$$\left(\frac{qw'}{p\lambda} - w\right)^{sp} = \left(\frac{-dc_2}{ps}\right)^{sp}\left(\frac{rq}{dc_1}\right)^{rq}\left(\frac{sw'}{r\lambda} - w\right)^{rq}.$$

Since $sp \neq rq$, the equation is not an identity and w satisfies a first order ADE with constant coefficients.

(ii) (b) If $\lambda_1 = \lambda_2$, the general solution is $w = (c_1 + c_2z)e^{\lambda_1 z}$ and this satisfies

$$zw' - (\lambda_1 z + 1)w = -c_1/c_2[w' - \lambda_1 w],$$

a first order ADE with rational coefficients.

(iii) Suppose λ_1/λ_2 is not rational. Consider the general solution

$$w(z) = c_1e^{\lambda_1 z} + c_2e^{\lambda_2 z}.$$

If w has no zeros then either $c_1 = 0$ or $c_2 = 0$. In this case, w clearly satisfies a first-order algebraic ADE with constant coefficient. Now suppose that $c_1c_2 \neq 0$ so that w has a zero; in fact, w has an infinite number of zeros. We show that w does not satisfy a first-order algebraic ADE with coefficients in K. We show first that w cannot satisfy a homogeneous equation. For suppose w satisfies a first-order ADE $P(w, w') = 0$, where P is homogeneous and has coefficients in K. Then w'/w is algebraic over K and by Lemma 2 we have $w'/w \in K$. However,

$$\frac{w'}{w} = \lambda_1 + \frac{c_2(\lambda_2 - \lambda_1)}{c_1e^{(\lambda_1 - \lambda_2)z} + c_2} \notin K.$$

Thus, P cannot be homogeneous. Now, consider $P(w, w')$ where P is not homogeneous. Suppose

$$P(w, w') = \sum_{k=l}^{m} P_k(w, w'),$$

where P_k is homogeneous of degree k. Substitute $w \equiv c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z}$ in the expression for P. Now $P_k(w, w')$ will consist of expressions of the form $a(z)e^{(s\lambda_1 + t\lambda_2)z}$ where $a(z) \in K$ and $s + t = k$. Then

$$P_k(w, w') = \sum a(z) e^{(s\lambda_1 + t\lambda_2)z}. \tag{5.3}$$

We assume that all the terms with the same exponential have been combined. Since w cannot satisfy a homogeneous P_k, therefore, at least one $a(z)$ in (5.3) is not zero. In (5.3) retain only those $a(z) \not\equiv 0$. Clearly

$$P(w, w') = \sum_{l} a(z) e^{(s\lambda_1 + t\lambda_2)z} + \cdots + \sum_{m} r(z) e^{(s'\lambda_1 + t'\lambda_2)z}; \tag{5.4}$$

$a(z) \not\equiv 0$, $r(z) \not\equiv 0$ and $a(z), r(z) \in K$ ($s + t = l, \dots, s' + t' = m$). Expressions in different sums cannot be combined, that is, if

$$A(z) e^{(s_1\lambda_1 + t_1\lambda_2)z} \in \sum_{i} a(z) e^{(s\lambda_1 + t\lambda_2)z} \quad (\text{so that } s_1 + t_1 = i)$$

and

$$B(z) e^{(s_2\lambda_1 + t_2\lambda_2)z} \in \sum_{j} b(z) e^{(s'\lambda_1 + t'\lambda_2)z} \quad (\text{so that } s_2 + t_2 = j)$$

and $i \neq j$, then the two expressions cannot be combined into one. For this to happen the two exponentials must be identical, that is, $s_1\lambda_1 + t_1\lambda_2 = s_2\lambda_1 + t_2\lambda_2$. But this implies that λ_1/λ_2 is a rational number. Thus, since expressions in different sums of (5.4) cannot be combined and $a(z) \not\equiv 0, \dots, r(z) \not\equiv 0$, therefore, $P(w, w') \not\equiv 0$, and $w = c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z}$ cannot satisfy a first-order algebraic ADE with coefficients in K.

6. Remark

Finally, we refer to [9] for an interesting collection of problems on ADEs. Also, we sincerely thank the referee for many helpful suggestions, and in particular for the present form of Theorem 3.

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