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ON BROWNIAN MOTION WITH IRREGULAR DRIFT

BY

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Dedicated to the memory of Theophil Henry Hildebrandt

1. Introduction

Our objective is to describe some class of diffusion processes on the line generated by Feller's generalized second order differential operator $D_m D_p^+$ as strong solutions of stochastic equations. In contrast to earlier papers [6]–[8], which investigated the case of nonsmooth speed measure m, we are now concerned with irregularities of the natural scale p.

For a classical second order differential operator

$$\frac{1}{2}\frac{d^2}{dx^2} + b\frac{d}{dx}$$

with unit diffusion coefficient and a locally square integrable drift coefficient b satisfying some growth condition at the boundaries, the situation is well known. Here the scale functions p_b and m_b are defined by the formulae

$$dp_b(x) = \exp\left\{-2\int_0^x b(y)\,dy\right\}dx,$$

$$dm_b(x) = \exp\left\{2\int_0^x b(y)\,dy\right\}2\,dx, \quad x \in \mathbf{R};$$

cf. the paper of Orey [14]. We observe that the derivative π_b of the natural scale p_b uniquely solves the linear integral equation

$$\pi_b(x) = 1 - 2\int_0^x \pi_b(y)b(y)\,dy, \quad x \in \mathbf{R}.$$

For a given standard Brownian motion W, to each initial state $x \in \mathbf{R}$ there

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exists a strong unique solution of the stochastic differential equation

$$dX_t = b(X_t)\,dt + dW_t, \quad t \geq 0,$$
$$X_0 = x,$$

which can be regarded as a diffusion X generated by the differential operator

$$\frac{1}{2}\frac{d^2}{dx^2} + b\frac{d}{dx} = D_{m_b}D_{p_b}^+.$$

If the set $\mathscr{D} = \{x \in \mathbf{R} : b(x) \neq 0\}$ is of positive Lebesgue measure, the diffusion X is a semimartingale consisting of the Brownian motion W and an additional non-vanishing process of bounded variation $\int b\,dt$; the martingale property of W will be perturbed if X_t moves within the set $\mathscr{D}$.

A different kind of process is the so-called skew Brownian motion, introduced by Itô and McKean [11, p. 115], and recently investigated in the framework of stochastic analysis by Walsh [17], and Harrison and Shepp [9]. Intuitively, skew Brownian motion is a process which coincides with standard Brownian motion up to the passage times of zero. The process leaves the origin more easily in one direction than the other. This motion can be regarded as a diffusion with a very irregular drift coefficient, which is in fact a distribution, as shown by Portenko [15], or as the result of a limit procedure applied to a sequence of diffusions with regular coefficients (cf. Rosenkrantz [16]).

In terms of its scale functions, skew Brownian motion is defined by

$$p_\alpha(x) = \begin{cases} (1-\alpha)x, & x \geq 0, \\ \alpha x, & x < 0, \end{cases}$$
$$m_\alpha(x) = \begin{cases} 2(1-\alpha)^{-1}x, & x \geq 0, \\ 2\alpha^{-1}x, & x < 0, \end{cases}$$

with the (non-degenerate) skew coefficient $0 < \alpha < 1$. Away from zero the generator $D_{m_\alpha}D_{p_\alpha}^+$ is just the Laplacian $\frac{1}{2}d^2/dx^2$; but in the origin of the axis for all functions f in the domain of definition of $D_{m_\alpha}D_{p_\alpha}^+$ there arises a transmission condition of the form

$$(1-\alpha)^{-1}f'(0+) = \alpha^{-1}f'(0-).$$

It is shown in [9] that, with respect to a given Brownian motion W, such a process X generated by $D_{m_\alpha}D_{p_\alpha}^+$ is a strong solution to the stochastic equation

$$X_t = W_t + \delta L_t(X), \quad t \geq 0,$$

where $\delta = \frac{1}{2}(2\alpha - 1) \cdot (1-\alpha)^{-1}$, and $L_t(X)$ is the local time of the unknown process at zero,

$$L_t(X) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t I_{\{-\varepsilon \le X_s \le 0\}}\, ds.$$

For the process X the chance to leave an arbitrary interval $(-a, a)$, $a > 0$, say, at the right end point a is

$$\frac{p_\alpha(0) - p_\alpha(-a)}{p_\alpha(a) - p_\alpha(-a)} = \frac{(1-\alpha)0 - \alpha(-a)}{(1-\alpha)a - \alpha(-a)} = \alpha,$$

(cf. Dynkin [2, Chapt. XV]). At all other points $x \in \mathbf{R}$, this chance is—as in the case of an ordinary Brownian motion—one-half, for all intervals $(x - a, x + a)$ not containing the origin.

The aim of the present paper is to give a unified approach to processes with such regular and irregular drift behaviour as described above. First we define a class of suitable pairs (m, p) of speed measures and natural scales, and construct the corresponding diffusion processes from some underlying Brownian motion process B° via the method of Itô and McKean [11] by a successive transformation of time and space. Then we derive a stochastic equation which has, with respect to any given Brownian motion (different from B°), just the diffusion X generated by the infinitesimal operator $D_m D_p^+$ as strong unique solution.

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2. Diffusions with nonsmooth scale

Let β be a function of locally bounded variation on the axis $\mathbf{R}$, which we assume to be right continuous, and vanishing at the origin. Moreover, concerning the jumps $\Delta^-\beta(x) = \beta(x) - \beta(x-0)$, $x \in \mathbf{R}$, of the function β, we assume the condition

$$(1) \qquad 1 + 2\Delta^-\beta(x) > 0, \quad x \in \mathbf{R}.$$

Then, to each initial value $\pi_0 > 0$ there exists a unique solution to the equation

$$(2) \qquad \pi(x) = \pi_0 - 2\int_0^x \pi(y)\, d\beta(y), \quad x \in \mathbf{R},$$

which can be expressed explicitly by the following formula (cf. Hildebrandt [10], and [5]):

$$\pi(x) = \begin{cases} \pi_0 e^{-2\beta(x)} \prod_{0<\tau\leq x} [1 + 2\Delta^-\beta(\tau)]^{-1} e^{2\Delta^-\beta(\tau)}, & x \geq 0 \\ \pi_0 e^{-2\beta(x)} \prod_{x<\tau\leq 0} [1 + 2\Delta^-\beta(\tau)] e^{-2\Delta^-\beta(\tau)}, & x < 0. \end{cases}$$

Because of our assumptions on β the solution π is right continuous, bounded and uniformly positive on each compact interval.

Now we regard the functions π and $1/\pi$ as densities of some scale functions p and m respectively,

$$p(x) = \int_0^x \pi(y)\, dy, \quad m(x) = \int_0^x \pi(y)^{-1} 2\, dy, \quad x \in \mathbf{R}.$$

Let us also assume that

$$(3) \qquad p(x) \quad \text{and} \quad \int_0^x p(y)\, dm(y)$$

go to infinity if x tends to $+\infty$ or $-\infty$. Then $\int_0^x m(y)\, dp(y)$ certainly diverges for $x \to \pm\infty$, and both boundaries $-\infty$, $+\infty$ of the axis $\mathbf{R}$ are natural in Feller's terminology [3]. This is, of course, an assumption on the (signed) measure generating function β.

Under these conditions, the generalized second order differential operator $D_m D_p^+$ is the infinitesimal generator of a conservative continuous strong Markov process $X = (X_t, \mathscr{F}_t, \mathbf{P}_x)$ in the sense of Dynkin [2].

Such a process X can be constructed explicitly from a Brownian motion $B^\circ = (B_t^\circ, \mathscr{F}_t^\circ, \mathbf{P}_x^\circ)$ by successive application of a random time change and a state space transformation; see Itô and McKean [11]. For a short introduction in this technique see the article of Orey [14]. The time change T is defined by the formula

$$t = \int_0^{T(t)} [\pi \circ q(B_s^\circ)]^{-2}\, ds, \quad t \geq 0,$$

where q is the inverse of the (strictly increasing and continuous) scale p. The time changed Brownian motion $Y = (Y_t, \mathscr{F}_t, \mathbf{P}_x^\circ)$ with

$$Y_t = B_{T(t)}^\circ, \quad \mathscr{F}_t = \mathscr{F}_{T(t)}^\circ, \quad t \geq 0,$$

is a continuous strong Markov process on its natural scale $p^*(x) = x$ and with speed measure $m^* = m \circ q$, which can be expressed explicitly as follows. First observe that for all points $x \in \mathbf{R}$ there exists the right derivative $D_x^+ q$ of the inverse scale q. Using the change of variable formula we can write, for all $x, y \in \mathbf{R}$,

$$\begin{aligned} q(x) - q(y) &= \int_{q(y)}^{q(x)} \pi(z)^{-1}\, dp(z) \\ &= \int_y^x [\pi \circ q(z)]^{-1} d(p \circ q)(z) \\ &= \int_y^x [\pi \circ q(z)]^{-1}\, dz. \end{aligned}$$

Because of the right continuity of π this implies [2; p. 237] the relation

$$(D_x^+ q)(x) = (\pi \circ q)(x)^{-1}, \quad x \in \mathbf{R}. \tag{4}$$

Now it follows that

$$\begin{aligned} m^*(x) &= (m \circ q)(x) \\ &= \int_0^{q(x)} \pi(y)^{-1} 2\, dy = \int_0^x [\pi \circ q(y)]^{-1} 2\, dq(y) \\ &= \int_0^x [\pi \circ q(y)]^{-2} 2\, dy, \quad x \in \mathbf{R}. \end{aligned}$$

Observing the relations

$$\int_0^x m^*(y)\, dy = \int_0^{q(x)} m(y)\, dp(y), \quad x \in \mathbf{R},$$

$$\int_0^x y\, dm^*(y) = \int_0^{q(x)} p(y)\, dm(y), \quad x \in \mathbf{R},$$

and $q(\pm\infty) = \pm\infty$, we can see that both boundaries $+\infty$, $-\infty$ are natural also with respect to the process Y. Thus, the infinitesimal generator of the diffusion Y is determined; we have

$$D_{m^*} D_x^+ = \frac{d}{dm^*} \frac{d^+}{dx} = \frac{1}{2} (\pi \circ q)^2 \frac{d}{dx} \frac{d^+}{dx}.$$

Finally, the process $X = (X_t, \mathscr{F}_t, \mathbf{P}_x)$ results from the state space transformation $q : \mathbf{R} \to \mathbf{R}$ by setting

$$X_t = q(Y_t),\ t \geq 0, \quad \mathbf{P}_x = \mathbf{P}^{\circ}_{p(x)},\ x \in \mathbf{R}$$

(cf. [2, Chapter X]).

3. Stochastic differential equations

In this section we describe diffusion processes X with generator $D_m D_p^+$ as solutions of stochastic equations. As in section 2, at first the martingale diffusion $Y = p(X)$ is considered. After introducing a stochastic differential equation for this process, applying the generalized Itô formula we determine the differential of the process X and, finally, the desired stochastic equation.

Because we are interested in strong solutions, let us introduce first a complete probability space $(\Omega, \mathscr{F}, \mathbf{P})$ and an increasing family of sub-sigma-fields $\{\mathscr{F}_t, t \geq 0\}$. Further, let $W = \{W_t, t \geq 0\}$ be a standard Brownian motion adapted to $\{\mathscr{F}_t\}$ and such that the increments $W(t+u) - W(t)$, $0 \leq t \leq t + u < \infty$, are independent of $\mathscr{F}_t$. Moreover, $\mathscr{F}_0$ contains all $\mathbf{P}$-null sets in $\mathscr{F}$. To simplify matters, at first all processes in this section start at time zero in the origin of the axis.

We consider the stochastic equation

$$Y_t = \int_0^t (\pi \circ q)(Y_s)\, dW_s, \quad t \geq 0. \tag{5}$$

The solution π to the integral equation (2) is of bounded variation and strictly positive on each compact interval of the axis. Because of the monotonicity of q the compositum $\pi \circ q$ has the same property. Therefore, following a theorem of Nakao [13], there exists a strong solution $Y = \{Y_t, t \geq 0\}$, adapted to $\{\mathscr{F}_t\}$. More detailed, there exists a weak solution to equation (5), which, according to [13], is pathwise unique. Now we use a result of Yamada and Watanabe [19], saying that the existence of a pathwise unique solution implies the existence and uniqueness of a strong solution.

For the local martingale Y (which is in fact even a martingale, because both of the boundaries $-\infty$ and $+\infty$ are natural—see Arbib [1]) there exists at each point $a \in \mathbf{R}$, a local time $L_t^a(Y)$, characterized $\mathbf{P}$ – a.s. by the relations [21]

$$L_t^a(Y) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t I_{\{a - \varepsilon \leq Y_s \leq a\}}\, d\langle Y, Y\rangle_s, \quad t \geq 0,$$

or

$$(Y_t - a)^+ = (Y_0 - a)^+ + \int_0^t I_{\{Y_s \geq a\}}\, dY_s + \frac{1}{2} L_t^a(Y), \quad t \geq 0,$$

where f^+ denotes the positive part of the function f.

Observe the inequality $Y_s \geq a$ arising in the indicator, instead of the usual $Y_s > a$. Because all functions defined over the state space $\mathbf{R}$ are assumed to be right continuous, it is useful to choose the "left" version of local times, which really makes a difference for a (true) semimartingale; cf. Yor [21] and Youerp [20]. Consequently, as in Walsh [17] and Harrison and Shepp [9] we deal with a version of local time which are not necessarily continuous in the space variable, in contrast to the one used in Itô and McKean [11] which is continuous in both variables.

For an application of Itô's formula we need the first two derivatives of the inverse scale function q. The first one we know already (4). Concerning the second derivative (in the sense of distributions) we observe firstly that the reciprocal function $1/\pi$ is the unique (locally bounded and strictly positive) solution to the linear equation

$$\pi(x)^{-1} = \pi_0^{-1} + 2\int_0^x [\pi(y-0)]^{-1}\, d\beta(y), \quad x \in \mathbf{R}. \tag{6}$$

One can prove the validity of this equation by direct computation [5], or by application of partial integration [2, p. 234] to the functions π and $1/\pi$; for all $x \in \mathbf{R}$ we have

$$\begin{aligned} 0 &= \int_0^x \pi(y)\, d(1/\pi)(y) + \int_0^x (1/\pi)(y-0)\, d\pi(y) \\ &= \int_0^x \pi(y)\left[d(1/\pi)(y) - 2(1/\pi)(y-0)\, d\beta(y)\right]. \end{aligned}$$

Now we deduce from (6) an equation characterizing the right hand derivative of q; for all $x \in \mathbf{R}$ we have

$$\begin{aligned} (D_x^+ q)(x) &= (\pi \circ q)(x)^{-1} \\ &= \pi_0^{-1} + 2\int_0^{q(x)} [\pi(y-0)]^{-1}\, d\beta(y) \\ &= \pi_0^{-1} + 2\int_0^x [\pi \circ q(y-0)]^{-1}\, d(\beta \circ q)(y). \end{aligned} \tag{7}$$

Here we have used $q(0) = 0$ and the relation

$$\pi(q(x) - 0) = (\pi \circ q)(x-0), \quad x \in \mathbf{R},$$

which follows from the continuity and monotonicity of the function q. Consequently, the function $x \to (D_x^+ q)(x)$ is right continuous and of locally bounded variation, and the inverse scale q admits the representation

$$q(x) = \pi_0^{-1}x + 2\int_0^x \int_0^y [\pi \circ q(z-0)]^{-1}\, d(\beta \circ q)(z)\, dy, \quad x \in \mathbf{R}. \tag{8}$$

After these preparations we are able to apply the generalized Itô formula [20], [21] and calculate the differential of the process $X = q(Y)$:

$$\begin{aligned} q(Y_t) &= q(Y_0) + \int_0^t (D_x^+ q)(Y_s)\, dY_s + \frac{1}{2}\int_{\mathbf{R}} L_t^a(Y)\, d(D_x^+ q)(a) \\ &= \int_0^t [\pi \circ q(Y_s)]^{-1} (\pi \circ q)(Y_s)\, dW_s \\ &\quad + \int_{\mathbf{R}} L_t^a(Y)[\pi \circ q(a-0)]^{-1}\, d(\beta \circ q)(a) \\ &= W_t + \int_{q(\mathbf{R})} L_t^{p(a)}(Y)[\pi \circ q(p(a)-0)]^{-1}\, d\beta(a); \end{aligned}$$

this is

$$X_t = W_t + \int_{\mathbf{R}} L_t^{p(a)}(Y)[\pi(a-0)]^{-1}\, d\beta(a), \quad t \geq 0. \tag{9}$$

To show that the expression

$$L_t^{p(a)}(Y)[\pi(a-0)]^{-1}, \quad a \in \mathbf{R},$$

is the local time for X, a stochastic equation for this process is derived. For this reason for each $a \in \mathbf{R}$ we define the function

$$x \to \psi(x) = (q(x) - a)^+, \quad x \in \mathbf{R},$$

and calculate its first two derivatives. From the identity

$$(x-a)^+ = (0-a)^+ + \int_0^x I_{\{y \geq a\}}\, dy, \quad x \in \mathbf{R},$$

we deduce an integral representation of the function ψ:

$$\begin{aligned} \psi(x) &= (q(x) - a)^+ \\ &= (0-a)^+ + \int_0^{q(x)} I_{\{y \geq a\}}\, dy \\ &= (q(0) - a)^+ + \int_0^x I_{\{q(y) \geq a\}}\, dq(y) \\ &= (q(0) - a)^+ + \int_0^x I_{\{q(y) \geq a\}} (D_x^+ q)(y)\, dy \\ &= \psi(0) + \int_0^x I_{\{q(y) \geq a\}} [\pi \circ q(y)]^{-1}\, dy. \end{aligned}$$

Consequently, for the right derivative of ψ we have

$$(D_x^+ \psi)(x) = I_{\{q(x) \geq a\}} [\pi \circ q(x)]^{-1}, \quad x \in \mathbf{R}.$$

Concerning the second derivative of ψ, using integration by parts and formula (7), we can write

$$\begin{aligned}(D_x^+\psi)(x) - (D_x^+\psi)(0) &= I_{\{q(x)\geq a\}}(D_x^+q)(x) - I_{\{q(0)\geq a\}}(D_x^+q)(0)\\ &= \int_0^x I_{\{q(y)\geq a\}}\, d(D_x^+q)(y)\\ &\quad + \int_0^x (D_x^+q)(y-0)\, d\left[I_{\{q(y)\geq a\}}\right]\\ &= 2\int_0^x I_{\{q(y)\geq a\}}[\pi\circ q(y-0)]^{-1}\, d(\beta\circ q)(y)\\ &\quad + \int_0^x [\pi\circ q(y-0)]^{-1}\, d\left[I_{\{q(y)\geq a\}}\right].\end{aligned}$$

Now we are ready to calculate the differential of the process $(X-a)^+$. Because of Itô's formula, for all $t\geq 0$ we have

$$\begin{aligned}(X_t-a)^+ &= (q(Y_t)-a)^+\\ &= \psi(Y_t)\\ &= \psi(Y_0) + \int_0^t (D_x^+\psi)(Y_s)\, dY_s + \frac{1}{2}\int_{\mathbf{R}} L_t^y(Y)\, d(D_x^+\psi)(y)\\ &= (q(Y_0)-a)^+ + \int_0^t I_{\{q(Y_s)\geq a\}}[\pi\circ q(Y_s)]^{-1}\, dY_s\\ &\quad + \int_{\mathbf{R}} L_t^y(Y) I_{\{q(y)\geq a\}}[\pi\circ q(y-0)]^{-1}\, d(\beta\circ q)(y)\\ &\quad + \frac{1}{2}\int_{\mathbf{R}} L_t^y(Y)[\pi\circ q(y-0)]^{-1}\, d\left[I_{\{q(y)\geq a\}}\right]\\ &= (X_0-a)^+ + \int_0^t I_{\{X_s\geq a\}}\, dW_s\\ &\quad + \int_{q(\mathbf{R})} L_t^{p(y)}(Y) I_{\{y\geq a\}}[\pi(y-0)]^{-1}\, d\beta(y)\\ &\quad + \frac{1}{2}\int_{q(\mathbf{R})} L_t^{p(y)}(Y)[\pi(y-0)]^{-1}\, d\left[I_{\{y\geq a\}}\right]\\ &= (X_0-a)^+ + \int_0^t I_{\{X_s\geq a\}}\, dW_s\\ &\quad + \int_0^t I_{\{X_s\geq a\}}\left[\int_{\mathbf{R}} L_{ds}^{p(y)}(Y)[\pi(y-0)]^{-1}\, d\beta(y)\right]\\ &\quad + \frac{1}{2} L_t^{p(a)}(Y)[\pi(a-0)]^{-1}\\ &= (X_0-a)^+ + \int_0^t I_{\{X_s\geq a\}}\, dX_s + \frac{1}{2} L_t^{p(a)}(Y)[\pi(a-0)]^{-1}.\end{aligned}$$

Here we have used Fubini's theorem and equations (5), (9). Consequently, the process

$$L_t^a(X) = L_t^{p(a)}(Y)[\pi(a-0)]^{-1}, \quad t \geq 0,$$

is the local time for the semimartingale X at the point a, and we have showed that X solves the stochastic equation

$$(10) \qquad \begin{aligned} X_t &= W_t + \int_{\mathbf{R}} L_t^a(X)\, d\beta(a), \quad t \geq 0, \\ L_t^a &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t I_{\{a-\varepsilon \leq X_s \leq a\}}\, ds, \quad a \in \mathbf{R}. \end{aligned}$$

Now let $X = \{X_t, t \geq 0\}$ be an arbitrary solution to this equation, adapted to $\{\mathscr{F}_t\}$. Then X is a semimartingale over the probability space $(\Omega, \mathscr{F}, \mathbf{P})$ and we can apply Itô's formula again to calculate the differential of the process $Y = p(X)$. For all $t \geq 0$ we have

$$\begin{aligned} p(X_t) &= p(X_0) + \int_0^t (D_x^+ p)(X_s)\, dX_s + \frac{1}{2}\int_{\mathbf{R}} L_t^a(X)\, d(D_x^+ p)(a) \\ &= \int_0^t \pi(X_s)\, dX_s + \frac{1}{2}\int_{\mathbf{R}} L_t^a(X)\, d\pi(a) \\ &= \int_0^t \pi(X_s)\, dX_s - \int_{\mathbf{R}} L_t^a(X)\pi(a)\, d\beta(a) \\ &= \int_0^t \pi(X_s)\, dX_s - \int_0^t \pi(X_s)\left[\int_{\mathbf{R}} L_{ds}^a(X)\, d\beta(a)\right] \\ &= \int_0^t \pi(X_s)\, dW_s \\ &= \int_0^t (\pi \circ q)(Y_s)\, dW_s. \end{aligned}$$

Thus, the process X satisfies equation (10) if and only if $Y = p(X)$ is a solution to (5). As we have observed already, equation (5) has a unique strong solution, adapted to $\{\mathscr{F}_t\}$, and so X has the same property.

The above consideration does not essentially depend on the choice of the starting point $x \in \mathbf{R}$. It is standard to construct Markovian families of processes $Y = (Y_t, \mathscr{F}_t, \mathbf{P}_{p(x)})$ and $X = (X_t, \mathscr{F}_t, \mathbf{P}_x)$ (cf. the textbook of Wentzell [18, p. 194]) such that to each initial states $y = p(x)$, $x \in \mathbf{R}$ the processes Y_t and X_t are solutions to the respective equations

$$Y_t = p(x) + \int_0^t (\pi \circ q)(Y_s)\, dW_s, \quad t \geq 0, \mathbf{P}_x - \text{a.s.},$$

$$X_t = x + W_t + \int_{\mathbf{R}} L_t^a(X)\, d\beta(a), \quad t \geq 0, \mathbf{P}_x - \text{a.s.}$$

Now, Y is a diffusion process with infinitesimal generator $D_{m*}D_x^+$; remember that $m^*(x) = \int_0^x [\pi \circ q(y)]^{-2} 2\, dy$, $x \in \mathbf{R}$. From this it follows that $X = q(Y)$ is a diffusion with speed measure $m = m^* \circ p = \int \pi^{-1} 2\, dx$ and natural scale $p = \int \pi\, dx$. Thus, under the conditions formulated in Section 2 we have proved the following result.

THEOREM. *To each initial state $x \in \mathbf{R}$ there exists a strong unique solution to the stochastic equation*

$$X_t = x + W_t + \int_{\mathbf{R}} L_t^a(X)\, d\beta(a), \quad t \geq 0,$$

$$L_t^a(X) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t I_{\{a-\varepsilon \leq X_s \leq a\}}\, ds, \quad t \geq 0,$$

which forms a diffusion process with infinitesimal generator $D_m D_p^+$.

Added in proof For a somewhat broader class of diffusions characterized as weak solutions of stochastic equations see my recent paper *Feller's one-dimensional diffusions as weak solutions to stochastic differential equations*, Math. Nachrichten, vol. 122 (1985), pp. 157–165.

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