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MEROMORPHIC AND RATIONAL FACTORS OF AUTOMORPHY

BY

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I. Introduction

Henri Cartan [2] illustrated the appeal of factors of automorphy as a general approach to automorphic forms, and Gunning [12] gave other applications of these factors. Several contributions to the area of factors of automorphy were recently summarized in [14], but to these must be added the work of Rankin [17, p. 70 ff.] and of Christian [3], [4], [5], [6] in the Siegel upper half plane of degree $n > 1$, and of Gunning [11].

A factor of automorphy $v(z, \phi)$ on $D \times \Gamma$ satisfies the consistency condition

$$v(z, \phi \circ \Psi) = v(z, \Psi)v(\Psi z, \phi)$$

for all ϕ and Ψ in a group Γ of homeomorphisms of D onto itself. In this paper we consider the specific case in which D is the complex plane and $\Gamma \subset SL(2, R)$. For each M in Γ there is associated the homeomorphism

$$Mz = (az + b)/(cz + d).$$

The consistency condition becomes

$$v(z, MN) = v(z, N)v(Nz, M) \quad \text{for all } M, N \text{ in } \Gamma.$$

Two familiar factors of automorphy are v identically equal to one, and

$$v(z, M) = u(M)(cz + d)^k \quad \text{for all } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ in } \Gamma.$$

Given a factor of automorphy v, there is customarily an associated function $f(z)$ with the property that $f(Mz) = v(z, M)f(z)$ for all M in Γ. In the first case, $f(Mz) = f(z)$ so f is an (unrestricted) automorphic function, and, in the

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second, $f(Mz) = u(M)(cz + d)^k f(z)$ so f is an (unrestricted) automorphic form of weight $k/2$.

Siegel [18, p. 39] has pointed to the deficiency of knowledge concerning factors of automorphy other than the two just mentioned. For instance, in the specific case $\Gamma \subset SL(2, R)$ and z in the complex plane, results in several papers apply, but only when stringent hypotheses are imposed. In [16] the only group Γ allowed is so large that it fails at each point to be discontinuous, a situation not calculated to permit a corresponding $f(z)$ with $f(Mz) = v(z, M)f(z)$. In [9], smaller Γ are permitted, but the only factors of automorphy $v(z, M)$ considered are those extendable to a much larger group while maintaining certain properties for v (e.g., v analytic). Such extensions can fail to exist if these properties are incompatible with the consistency conditions on the large group. Results in [10] and [12] can be applied to subgroups Γ of $SL(2, R)$, however only for nonvanishing analytic factors of automorphy. The analysis also requires the fundamental region D/Γ to be compact in the quotient topology. Finally, in [17] the focus has been considerably narrowed by requiring $|v(z, M)| = |cz + d|^k$.

In this paper, we consider general meromorphic v and make almost no requirements upon the subgroup Γ of $SL(2, R)$—only that it possess a certain minimum number of matrices. We determine all possible factors of automorphy and find the corresponding $f(z)$.

II. $v(z, M)$ meromorphic

We examine factors of automorphy $v(z, M)$ which for all M in $\Gamma \subset SL(2, R)$ are meromorphic on the complex plane.

THEOREM 1. *Suppose Γ contains*

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \textit{with } c \neq 0.$$

Then any factor of automorphy $v(z, M)$ meromorphic on the complex plane must be rational in z for all M in Γ.

Proof. If $v(z, L)$ were not rational, it would have an essential singularity at infinity, but then the condition $v(z, L^2) = v(z, L)v(Lz, L)$ would force $v(z, L^2)$ to have an essential singularity at $z = -d/c$, a contradiction. So $v(z, L)$ is rational.

Let $A, B, C, \ldots$ generate Γ. Without loss of generality, none of these has lower left entry zero, since for example if A did, it could be replaced in the list of generators by L and LA. Thus $v(z, A), v(z, B), \ldots$ are all rational. Taking $N = I$ in the consistency condition yields $v(z, I) = 1$, so that $1 = v(z, AA^{-1})$

$= v(z, A^{-1})v(A^{-1}z, A)$. Therefore $v(z, A^{-1})$ and similarly $v(z, B^{-1}), \ldots$ are all rational. Any matrix M can be written as a product of the generators and their inverses, so repeated use of the consistency condition shows every $v(z, M)$ is rational.

A *fixed point* for

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a solution to $Mz = z$. When $c = 0$ and $a \neq \pm 1$ the finite fixed point is

$$x = ab/(1 - a^2).$$

For a given

$$M = \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}$$

and a given $v(z, M)$, we form $w(z, M) = cE(z)/F(z)$ where $E(z)$ is the Weierstrass product for the zeros of $v(z, M)$ and $F(z)$ is that for the poles, and c is selected to make $w(x, M) = v(x, M)$, where x is the fixed point. Extend w inductively via

$$w(z, M^k) = w(z, M)w(Mz, M)w(M^2z, M) \cdots w(M^{k-1}z, M)$$

and via $w(z, M^{-k}) = 1/w(M^{-k}z, M^k)$ and $w(z, I) = 1$. Then w is a factor of automorphy for the cyclic group generated by M, and for that group, v/w is an entire nonvanishing factor of automorphy.

THEOREM 2. *Suppose Γ is cyclic, generated by*

$$M = \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \quad \text{with } a \neq \pm 1.$$

For any factor of automorphy v meromorphic on the complex plane, there is an entire nonvanishing $h(z)$ such that

$$v(z, N) = w(z, N)\frac{h(Nz)}{h(z)} \quad \text{for all } N \text{ in } \Gamma$$

(where w is the auxilliary function above).

Proof. Since v/w is nonvanishing entire, it equals $\exp(H(z, M))$ for some entire H. Let $H(z, M) = \sum_{k=0}^{\infty} a_k(z - x)^k$ where x is the fixed point of M. Since $v(x, M) = w(x, M)$ it follows that $a_0 = 0$. Suppose $h(z) =$

$\exp(\sum_{k=0}^{\infty} b_k(z-x)^k)$ so

$$\frac{h(Mz)}{h(z)} = \exp\left(\sum_{k=0}^{\infty} b_k(a^2z+ba-x)^k - \sum_{k=0}^{\infty} b_k(z-x)^k\right)$$
$$= \exp\left(\sum_{k=0}^{\infty} b_k a^{2k}(z-x)^k - \sum_{k=0}^{\infty} b_k(z-x)^k\right)$$

where we have used $x = Mx$, that is, $x = a^2x + ba$. To make $h(Mz)/h(z)$ equal $\exp(H(z, M))$, we define $b_k = a_k/(a^{2k}-1)$ for $k = 1, 2, \ldots$ (b_0 can be arbitrary.) Thus for M,

$$\frac{v(z, M)}{w(z, M)} = \frac{h(Mz)}{h(z)}.$$

Since

$$v(z, M^k) = v(z, M)v(Mz, M)\cdots v(M^{k-1}z, M),$$
$$v(z, I) = 1,$$
$$v(z, M^{-k}) = 1/v(M^{-k}z, M^k),$$

and the analogous three relations hold for $w(z, M)$, it follows that

$$\frac{v(z, M^k)}{w(z, M^k)} = \frac{h(M^kz)}{h(z)} \quad \text{for all integers } k.$$

Theorem 2 can also be proved by applying results in [12] to v/w.

One might hope to omit the $w(z, N)$ term in Theorem 2 by allowing $h(z)$ to have zeros and poles. But suppose such a meromorphic $h(z)$ did exist in the case where $v(z, M) = z - x$ and $v(z, M^k)$ is found via the consistency conditions. So

$$h(Mz)/h(z) = z - x.$$

Write

$$h(z) = (z-x)^k\hat{h}(z) \quad \text{where } \hat{h}(x) \neq 0, \infty.$$

Then at $z = x$, $v(z, M) = 0$ but $h(Mz)/h(z) = a^{2k}$, a contradiction.

THEOREM 3. *Suppose Γ is cyclic, generated by*

$$M = \pm\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

For any factor of automorphy v meromorphic on the plane, there is an entire nonvanishing $h(z)$ such that

$$v(z, N) = w(z, N)\frac{h(Nz)}{h(z)} \quad \text{for all } N \text{ in } \Gamma$$

Proof. For such a Γ, Appell [1, Chapter 1, Section 4] proved that to any entire nonvanishing factor of automorphy u, there corresponds an entire nonvanishing $h(z)$ such that $u(z, N) = h(Nz)/h(z)$ for all N in Γ. We apply this result to v/w.

Theorems 2 and 3 show if a cyclic Γ contains only matrices with lower left entries zero, then any factor of automorphy for Γ has a particularly nice form. If Γ is not cyclic, no such theorem exists, because if M and N are unrelated generators for Γ, then $v(z, M)$ and $v(z, N)$ are independent of one another, so that $h(z)$ for M need have no relation to the $h(z)$ for N. However for discrete Γ we can prove:

THEOREM 4. *Suppose all matrices in Γ have lower left entries zero and Γ is discrete. Then corresponding to any factor of automorphy v meromorphic on the plane, there is an entire nonvanishing $h(z)$ such that*

$$v(z, N) = w(z, N)\frac{h(Nz)}{h(z)} \quad \textit{for all } N \textit{ in } \Gamma.$$

Proof. By Theorem 2H of [15], Γ must be cyclic when considered as a linear fractional transformation group, so either $\Gamma = \langle M \rangle$ or $\langle -I, M \rangle$. The former case is covered by Theorems 2 and 3. In the latter case, let $h(z)$ be the function produced for the subgroup $\langle M \rangle$ so the theorem holds for powers of M. For powers of $-M$ we must extend the definition of w by

$$w(z, -I) = v(z, -I) \quad \text{and} \quad w(z, -M^k) = w(z, -I)w((-I)z, M^k).$$

Note that $(-I)z = z$ so $v(z, -I) = \pm 1$. Therefore

$$\begin{aligned} v(z, -M^k) &= v(z, -I)v((-I)z, M^k) = v(z, -I)v(z, M^k) \\ &= w(z, -I)w(z, M^k)\frac{h(M^k z)}{h(z)} = w(z, -M^k)\frac{h((-M^k)z)}{h(z)} \end{aligned}$$

since $(-M^k)z = M^k z$.

For discrete Γ, Theorem 4 completely characterizes factors of automorphy when all matrices have lower left entries zero, and Theorem 1 shows all other factors of automorphy must be rational.

III. $v(z, M)$ rational

DEFINITION. $f(z, M)$ is *proportional* to $g(z, M)$, denoted $f(z, M) \sim g(z, M)$, means $f(z, M) = cg(z, M)$ for some nonzero constant $c = c(M)$ independent of z.

DEFINITION. By a *rational factor of automorphy* with respect to Γ, we mean, as Petersson did [16], a $v(z, M)$ with the *consistency condition*

$$v(z, MN) = v(z, N)v(Nz, M) \quad \text{for all } M, N \text{ in } \Gamma,$$

and satisfying:

$(*)$ $v(z, M) \sim \prod_{i=1}^{\alpha}(z - x_i)/\prod_{i=1}^{\beta}(z - y_i)$ where α, β do not exceed some fixed interger n. The complex x_i and y_i as well as the α and β may depend on M, but n does not.

Temporarily we also assume:

$(**)$ No x_i or y_i is in $\Gamma\infty$, the orbit of infinity.

Theorem 5 will show that $(**)$ implies $\alpha = \beta$ for each M, so that $(*)$ can be replaced by:

$(*)'$ $v(z, M) \sim \prod^{\alpha}((z - x_i)/(z - y_i))$ with $\alpha = \alpha(M) \le n$ where n is the largest occurring $\alpha(M)$ in Γ.

THEOREM 5. (a) *Every rational factor of automorphy has the form*

$$v(z, M) \sim \prod^{\alpha}\left((z - x_i)/(z - y_i)\right) \quad \textit{where } \alpha = \alpha(M) \le n$$

(b) *If further*, $v(z, N) \sim \prod^{\gamma}((z - z_i)/(z - \xi_i))$, *then*

$$v(z, MN) \sim \prod^{\gamma}\left((z - z_i)/(z - \zeta_i)\right)\prod^{\alpha}\left((z - N^{-1}x_i)/(z - N^{-1}y_i)\right).$$

LEMMA 1. *Let*

$$N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If $v(z, M) \sim \prod^{\alpha}(z - x_i)/\prod^{\beta}(x - y_i)$ *and* $v(z, N) \sim \prod^{\gamma}(z - z_i)/\prod^{\delta}(z - \zeta_i)$, *then*

$$v(z, MN) \sim \left[\prod^{\gamma}(z - z_i)\Big/\prod^{\delta}(z - \zeta_i)\right] \times\left[\prod^{\alpha}\left(z - N^{-1}x_i\right)\Big/\prod^{\beta}\left(z - N^{-1}y_i\right)\right](cz + d)^{\beta - \alpha}.$$

Proof. Assumption $(**)$ implies $a - cx_i \neq 0$, so

$$Nz - x_i = \left[(a - cx_i)z + (b - dx_i)\right](cz + d)^{-1}$$
$$\sim \left(z - N^{-1}x_i\right)(cz + d)^{-1}$$

and similarly for y_i in place of x_i. The lemma follows from the consistency condition.

DEFINITION. x is in the *M-orbit* of ω if $x = M^k\omega$ for any integer k.

LEMMA 2. *If $v(z, M) \sim \Pi^\alpha(z - x_i)/\Pi^\beta(z - y_i)$, then $\alpha = \beta$. Specifically, given any ω, there are the same number of x_i as y_i which are in the M-orbit of ω.*

Proof. Assume false. Without loss of generality, let ω be x_1 and let $v(z, M)$ have at least one more numerator term in the M-orbit of x_1 than denominator terms in the M-orbit of x_1. Repeated use of Lemma 1 with $M = N$ shows $v(z, M^{n+1})$ has at least $n + 1$ more numerator terms than denominator terms with this property. This violates $(*)$.

Proof of Theorem 5. The second lemma gives (a), and the first with $\alpha = \beta$ and $\gamma = \delta$ gives (b).

DEFINITION. $v(z, M)$ has *size* α means $v(z, M) \sim \Pi^\alpha((z - x_i)/(z - y_i))$ where no x_i is a y_j, and $\alpha \leq n$. The size $\alpha = \alpha(M)$ varies with M, and $v(z, M)$ achieves *full size* means $\alpha(M) = n$, the largest occurring size in Γ.

IV. Restrictions on the form of $v(z, M)$

In this and the next section, $v(z, M)$ satisfies $(*)'$ and $(**)$ for every M in Γ. It is proved that most $v(z, M)$ have a nice form.

DEFINITION. The (matrix) *order* g of M is the smallest positive g for which $M^g = I$. The *transformation order* g' of M is the smallest positive g' for which g'-fold composition $M \circ M \circ \cdots \circ Mz$ is the identity transformation. Thus the matrix $M^{g'}$ must be I or $-I$, so that $g' =$ either g or $g/2$.

THEOREM 6. *Suppose v is a rational factor of automorphy with property $(**)$. If M has infinite order then*:

(i) *There are disjoint lists of complex numbers $\{z_i\}_{i=1}^{\alpha}$ and $\{\zeta_i\}_{i=1}^{\beta}$ and positive integers $\{k(i)\}_{i=1}^{\alpha}$ and $\{l(i)\}_{i=1}^{\beta}$ with $\Sigma^\alpha k(i) + \Sigma^\beta l(i) = n' \leq n$, such that*

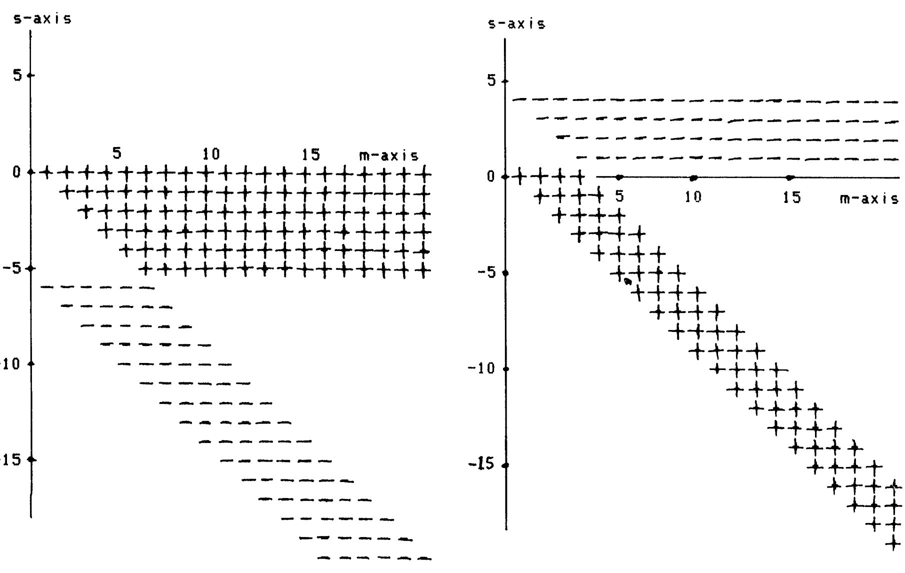
$$v(z, M) \sim \prod^{\alpha}\left((z - z_i)/\left(z - M^{-k(i)}z_i\right)\right)\prod^{\beta}\left(\left(z - M^{-l(i)}\zeta_i\right)/(z - \zeta_i)\right).$$

(ii) *There are disjoint lists of complex numbers* $\{z_i\}_{i=1}^{\alpha'}$ *and* $\{\zeta_i\}_{i=1}^{\beta'}$ *with* $\alpha' + \beta' = n' \leq n$ *such that*

$$v(z, M) \sim \prod^{\alpha'}\left((z - z_i)/(z - M^{-1}z_i)\right)\prod^{\beta'}\left((z - M^{-1}\zeta_i)/(z - \zeta_i)\right).$$

Further, no z_i *or* ζ_i *in either* (i) *or* (ii) *is a fixed point of* M.

Proof. The simplest possible form for $v(z, M)$ is $(z - x)/(z - y)$. By Theorem 5(b) with $M = N$, $v(z, M^2)$ has 2 numerators and 2 denominators. Reapplication shows $v(z, M^{k+1}) = v(z, MM^k)$ has k numerators and denominators. So the size of v for M^{n+1} would exceed the fixed bound n unless either $x = M^k y$ for some $0 < k \leq n$, or $y = M^l x$ for some $0 < l \leq n$, which proves the theorem in this simplest case. Figure 1(a) illustrates the case $k = 6$



CASE (a) $k(1) \geq l(1)$ CASE (b) $l(1) \geq k(1)$

FIG. 1 Numerators $(+)$ and denominators $(-)$ of $v(z, M^m)$ for $m = 1, 2, 3, \ldots$, given that

$$v(z, M) = \frac{z - x}{z - M^{-6}x} \text{ (Case a)) or } v(z, M) = \frac{z - M^{-4}x}{z - x} \text{ (Case b)).}$$

To find $v(z, M^{15})$, for example, construct a vertical line at $m = 15$. A "+" at height s indicates $(z - M^s x)$ is an uncancelled numerator term, and a "$-$" at height s indicates $(z - M^s x)$ appears as an uncancelled denominator term.

and Figure 1(b) illustrates $l = 4$. To find $v(z, M^{15})$, for instance, one draws a vertical line through $m = 15$. A "+" at height s indicates $(z - M^s x)$ is an uncancelled numerator term, and a "−" at that height signifies that $(z - M^s x)$ appears as an uncancelled denominator term.

Another simple case is $v(z, M) \sim \prod^{\alpha}((z - x_i)/(z - y_i))$ where all x_i's and y_i's are M-equivalent to a single x_i (not a fixed point of M); that is,

$$v(z, M) \sim \prod_{i=1}^{\alpha} \left(\left(z - M^{k(i)}x_1\right)\big/\left(z - M^{l(i)}x_1\right)\right).$$

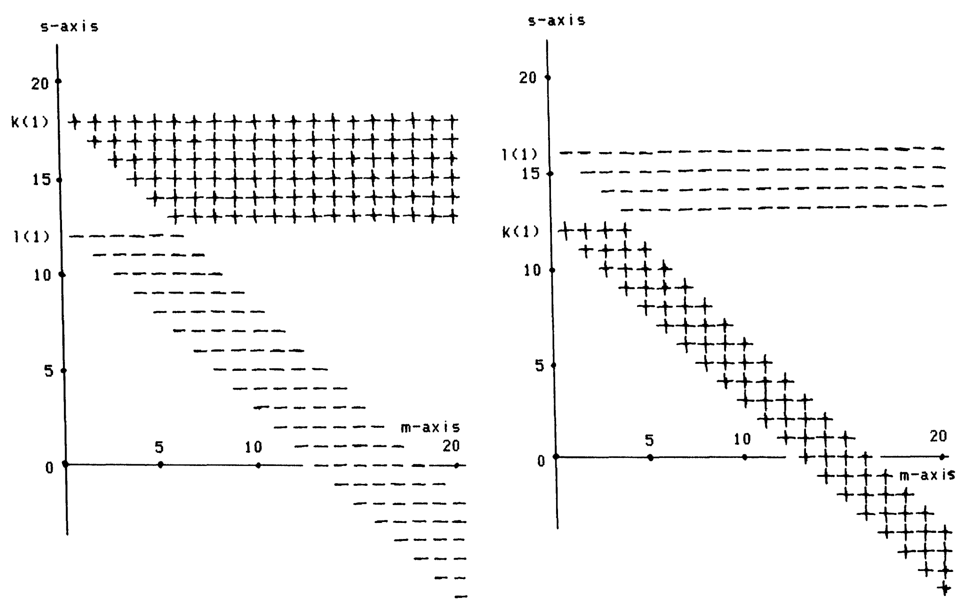
We arrange terms so that $k(i) \geq k(i + 1)$ and $l(i) \geq l(i + 1)$. For such v we have:

LEMMA. *For large enough m, $v(z, M^m)$ has size $\Sigma_{i=1}^{\alpha}|k(i) - l(i)|$.*

Proof. If $\alpha = 1$, the situation corresponds to a vertical translation of Figs. 1(a) and (b) above, and the lemma is obvious. See Figs. 2(a) and (b).

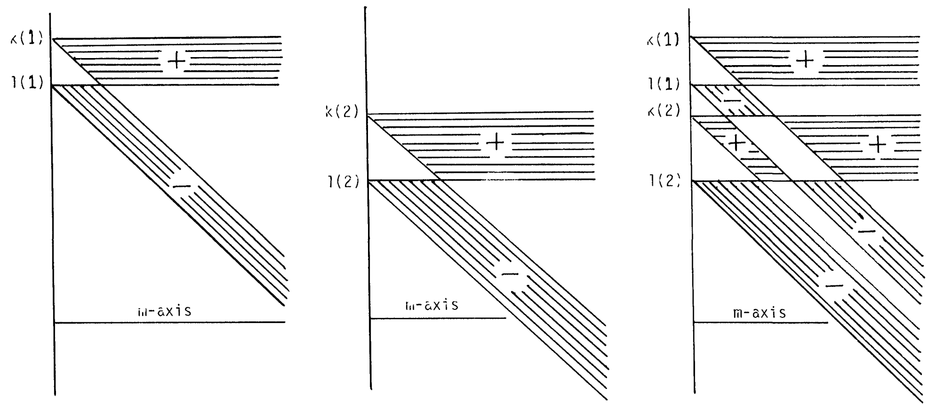
If $\alpha = 2$, then

$$v(z, M) \sim \left(z - M^{k(1)}x_1\right)\left(z - M^{k(2)}x_1\right)/\left(z - M^{l(1)}x_1\right)\left(z - M^{l(2)}x_1\right).$$



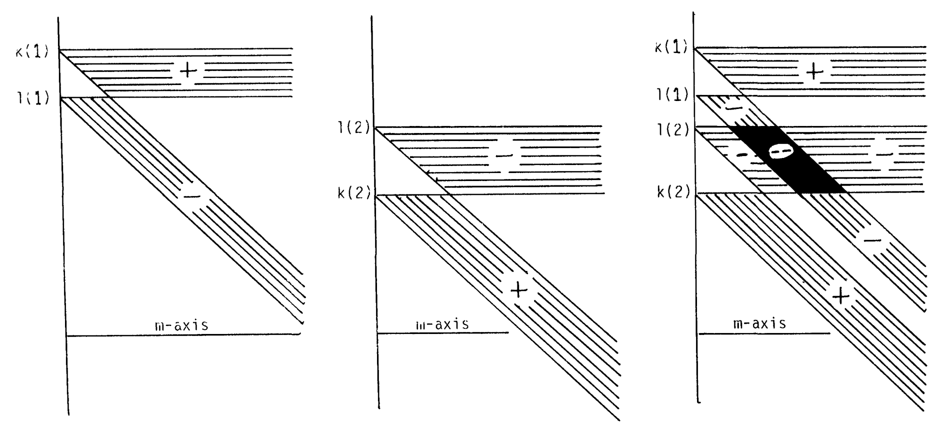
CASE (a) $k = 6$ CASE (b) $l = 4$

FIG. 2



$k(1) \geq l(1) \geq k(a) \geq l(2)$

SUPERPOSITION PRINCIPLE



$k(1) \geq l(1) \geq l(2) \geq k(2)$

SUPERPOSITION PRINCIPLE

FIG. 3 The rightmost figure for $v(z, M^m)$ for $m = 1, 2, 3 \ldots$ illustrates the effect of combining the two simpler factors of automorphy shown in the left and center figures. (Numerators $(+)$ and denominators $(-)$). The left figure corresponds to

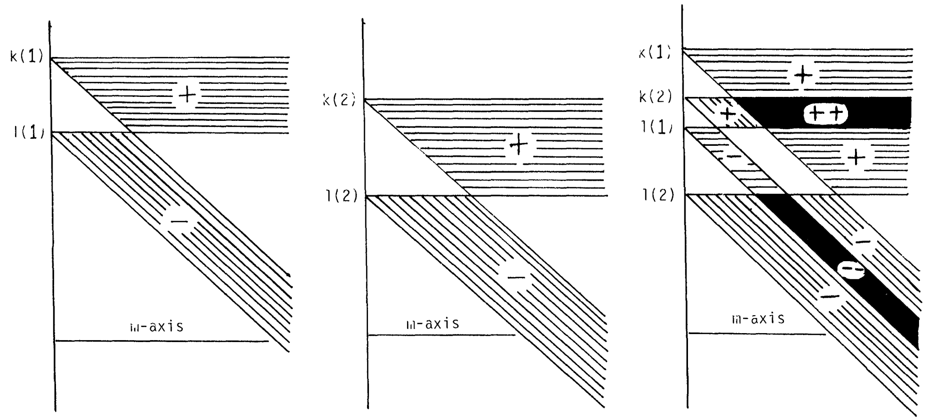
$$v(z, M) = \left(z - M^{k(1)}x_1\right) / \left(z - M^{l(1)}x_1\right).$$

The center figure corresponds to

$$v(z, M) = \left(z - M^{k(2)}x_1\right) / \left(z - M^{l(2)}x_1\right).$$

And the rightmost figure corresponds to the combination of the two,

$$v(z, M) = \left(z - M^{k(1)}x_1\right)\left(z - M^{k(2)}x_1\right) / \left(z - M^{l(1)}x_1\right)\left(z - M^{l(2)}x_1\right).$$



$k(1) \geq k(2) \geq l(1) \geq l(2)$
SUPERPOSITION PRINCIPLE

FIG. 3 (*Continued*)

To find $v(z, M^m)$ as m varies, we superimpose the contributions which stem from

$$\left(z - M^{k(1)}x_1\right)/\left(z - M^{l(1)}x_1\right)$$

upon those which stem from

$$\left(z - M^{k(2)}x_1\right)/\left(x - M^{l(2)}x_1\right).$$

Since $l(1) \geq l(2)$ and $k(1) \geq k(2)$, there are six relative arrangements for these four integers. The three cases where $k(1) \geq l(1)$, illustrated in Figs. 3(a), 3(b), 3(c), show how two diagrams combine. The other three cases are similar.

If the point (m, s) lies in a region of "+", this indicates that $v(z, M^m)$ has, after all cancellation, a numerator term $(z - M^s x_1)$.

If (m, s) lies in a region of "+ +", then $(z - M^s x_1)^2$ appears in the numerator. The regions of "−" and "− −" refer similarly to denominators.

The interference between two contributing parts "stabilizes" at the m-value which corresponds to the rightmost crossing of a horizontal line by a slanting line. To the right of this point, the behavior of $v_1(z, M^m)$ is very simple, and the lemma is obvious.

For $\alpha > 2$ the situation is similar, and the behavior of $v(z, M^m)$ is very simple for m to the right of where the interference "stabilizes." The lemma follows immediately.

Proof of Theorem 6. $v(z, M)$ can be dissected into products

$$v_1(z, M)v_2(z, M) \cdots v_r(z, M)$$

where each v_j contains all original x_i and y_i which are M-equivalent to a given x_j. Each v_j itself must be a factor of automorphy, at least to the extent that $v_j(z, MN) \sim v_j(z, N)v_j(Nz, M)$ and this is sufficient hypothesis to apply Theorem 5(a), so

$$(*) \qquad v_j(z, M) \sim \prod_{i=1}^{\alpha(j)} \left(z - M^{k(i,j)}x_j\right)/\left(z - M^{l(i,j)}x_j\right)$$

where indices are relabelled so that $k(i, j) \geq k(i+1, j)$ and $l(i, j) \geq l(i+1, j)$. By the lemma, for large enough m, $v_j(z, M^m)$ has size $\sum_{i=1}^{\alpha(j)} |k(i, j) - l(i, j)|$. Since there can be no cancellation between the various v_j, the $v(z, M^m)$ has size

$$\sum_{j=1}^{r} \sum_{i=1}^{\alpha(j)} |k(i, j) - l(i, j)|,$$

which by assumption, does not exceed n. We call each

$$\left(z - M^{k(i,j)}x_j\right)/\left(z - M^{l(i,j)}x_j\right)$$

an atom of v, and note that each atom can be expressed both as

$$(z - z_i)/\left(z - M^{l(i,j)-k(i,j)}z_i\right) \quad \text{for some } z_i,$$

and as

$$\left(z - M^{k(i,j)-l(i,j)}\zeta_i\right)/(z - \zeta_i) \quad \text{for some } \zeta_i.$$

If $k(i, j) \geq l(i, j)$, choose the first, and if $k(i, j) < l(i, j)$ choose the second. This produces the z_i and ζ_i in the theorem. For each z_i, the corresponding $k(i)$ is $k(i, j) - l(i, j)$, and for each ζ_i, the corresponding $l(i)$ is $l(i, j) - k(i, j)$. If $k(i, j) = l(i, j)$ the atom disappears. Finally, the sum of all the $k(i)$ and $l(i)$ equals the double sum mentioned above and thus does not exceed n. This proves (i), and clearly (ii) is equivalent.

THEOREM 7. *If $M^2 = -I$ or M has finite order $g \geq 4n^4$ then the two conclusions of Theorem 6 are valid.*

Proof. Let $M^2 = -I$ and $v(z, M) \sim \prod^{\alpha}(z - x_i)/(z - y_i)$. Since $M^2 z = z$,

$$\begin{aligned} \pm 1 = v(z, M^2) &= v(z, M)v(Mz, M) \\ &\sim \prod^{\alpha}(z - x_i)/(x - y_i) \prod^{\alpha}\left(z - M^{-1}x_i\right)/\left(z - M^{-1}y_i\right). \end{aligned}$$

Thus each $M^{-1}x_i$ equals a y_j, and each $M^{-1}y_i$ equals an x_j. The theorem follows.

For M with finite matrix order $g \geq 4n^4$, we know the transformation order is $g' \geq 2n^4$. As in the proof to Theorem 6,

$$v(z, M) = v_1(z, M)v_2(z, M) \cdots v_r(z, M)$$

where each v_j satisfies $(*)$. Figs. 1, 2, and 3 are no longer planar; instead they are wrapped on a torus, because both axes are periodic with period $g' \geq 2n^4$. For v_1 we locate all $2\alpha(1) \leq 2n$ of the $k(i,1)$ and $l(i,1)$ on the s-axis. Trivially, there must be a segment of length at least $(n^3 - 1)$ on the s-axis containing no k or l values. See Fig. 4.

We relabel if necessary so $k(1,1)$ and $l(1,1)$ are the smallest k and l values lying above the $(n^3 - 1)$ gap, and $k(2,1)$, $l(2,1)$ are the next smallest, and so forth. All k's and l's lie within g' units (actually $g' - n^3 + 1$) above the gap.

LEMMA. *$|k(i,1) - l(i,1)| \leq n$ for all $i = 1, 2, \ldots, \alpha(1)$.*

Proof. The $n = 1$ case is (6) and (7) of Theorem 2 in (14). Consider $n > 1$. If the lemma were false, there would be an h such that $k(h,1)$ and $l(h,1)$ are more than n units apart on the s-axis. Select the smallest such h. Without loss of generality $k(h,1) > l(h,1)$. We have observed (Fig. 2(a)) that the contribution to $v_1(z, M^m)$ from the $(z - M^{k(h,1)}x_1)/(z - M^{l(h,1)}x_1)$ term grows with m, and by $m = n + 1$, it would produce more than n denominators (and numerators), a contradiction unless there is interference, i.e., cancellation between numerator and denominator terms. (See Figs. 3 and 4.) In general, to keep the slanting "$-$" region which begins on the s-axis at height $l(h,1)$ from attaining its potential width of more than n units, the k's below $l(h,1)$ must be spaced not more than $2n$ units apart, and furthermore $k(h-1,1)$ must be not more than n units below $l(h,1)$. But below the smallest positive $k = k(1,1)$ there can be no such k within $2n$ units, due to the $(n^3 - 1)$ gap, so for $m = l(h,1) - k(1,1) + (n+1)$, $v_1(z, M^m)$ has at least $n + 1$ denominators, a contradiction. Thus the lemma cannot be false.

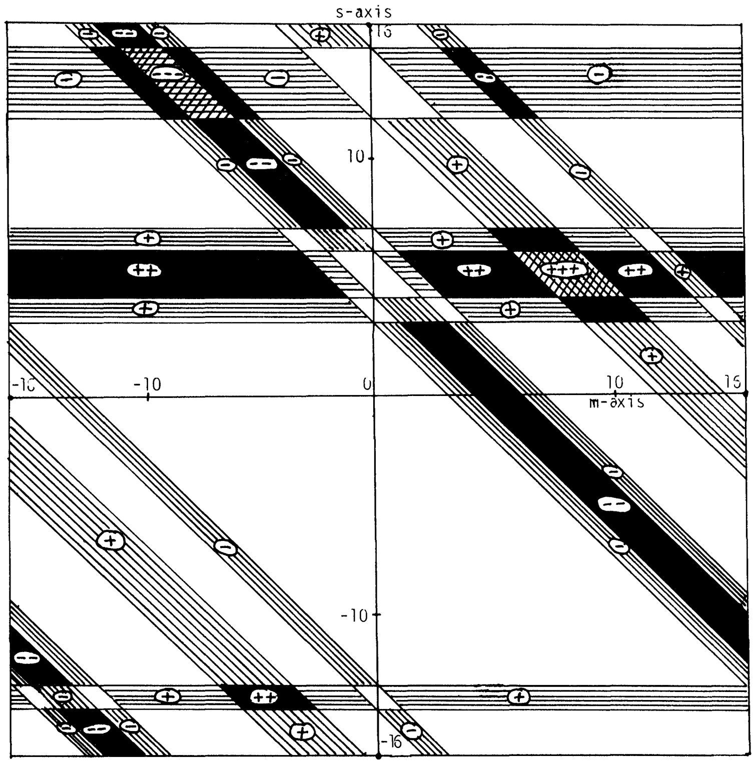
LEMMA. *For $m = 1, 2, \ldots, 2n^4$, the $v_1(z, M^m)$ has size less than*

$$\sum_{j=1}^{\alpha(1)} |k(j,1) - l(j,1)|$$

at most $n^3 + n^2$ times.

Proof. When there is no interference between "+" and "$-$" regions, the collection of terms which stem from a single

$$\left(z - M^{k(i,1)}x_1\right)/\left(z - M^{l(i,1)}x_1\right)$$



$v_1(z, M^m)$ for finite order M

FIG. 4 In this typical example both m and s have period $g' = 32$. Numerator $(+)$ and denominator $(-)$ regions which are truncated by the right edge in the chart above reenter on the left at the same height, and regions leaving the bottom edge reappear at the top. To find $v_1(z, M^{42})$, for instance, construct a vertical line through $m = 10 \equiv 42 \pmod{32}$ to read off which numerators $(+)$ and denominators $(-)$ are present.

in $v_1(z, M^m)$ is of size $|k(i,1) - l(i,1)|$. So to prove the lemma we need show there is interference for a total of at most $n^3 + n^2$ different values of m as m goes through one period of length g'. Since there are at most n horizontal strips (each of height $\leq n$) and since each such strip interfers with any given slanting strip (including its own companion slanting strip) for at most $2n$ values of m, there are at most

$$\binom{n+1}{2}2n = n^3 + n^2$$

values of m for which interference can take place.

To prove (i) of Theorem 7 for M of finite order, the two lemmas are applied to each $v_j(z, M)$. Since $v(z, M^m)$ is the product of $v_j(z, M^m)$ for $1 \leq j \leq r$, and the v_j do not interfere with one another, $v(z, M^m)$ has size less than

$$\sum_{j=1}^{r} \sum_{i=1}^{\alpha(j)} |k(i, j) - l(i, j)|$$

for at most $r(n^3 + n^2)$ of the $m = 1, 2, \ldots, 2n^4$. Now $2n^4 \geq r(n^3 + n^2)$ since $r \leq n$, so m can be chosen such that $v(z, M^m)$ has size at least

$$\sum_{j=1}^{r} \sum_{i=1}^{\alpha(j)} |k(i, j) - l(i, j)|.$$

By (*) of Section III, this is at most n. We choose z_i, ζ_i, $k(i)$ and $l(i)$ in (i) of Theorem 7 in precisely the same method as in the case that M had infinite order. Trivially (ii) follows from (i).

V. Uniformity within groups

Given a matrix, we have lists $\{z_i\}$ and $\{\zeta_i\}$, and α, β guaranteed by Theorems 6 and 7.

THEOREM 8. *Suppose v is a rational factor of automorphy with property (**). If M and N have identical lists $\{z_i\}, \{\zeta_i\}$, α, and β, then any L in the group $\langle M, N\rangle$ also has the same lists and same α, β.*

Proof. Let

$$v(z, M) \sim \prod^{\alpha}(z - z_i)/(z - M^{-1}z_i)\prod^{\beta}(z - M^{-1}\zeta_i)/(z - \zeta_i)$$

and the same hold for N in place of M. It is sufficient to show the same lists are retained under products MN and inverses M^{-1}. Apply Theorem 5(b), simplify, and use $N^{-1}M^{-1} = (MN)^{-1}$ to verify

$$v(z, MN) \sim \prod^{\alpha}(z - z_i)\Big/\left(z - (MN)^{-1}z_i\right)\prod^{\beta}\left(z - (MN)^{-1}\zeta_i\right)\Big/(z - \zeta_i).$$

For inverses, $1 = v(z, MM^{-1}) = v(z, M^{-1})v(M^{-1}z, M)$. In the proof of the first lemma to Theorem 5, we found for any L that $(Lz - x_i) \sim$

$(z - L^{-1}x_i)(cz + d)^{-1}$, so

$$v(M^{-1}z, M) \sim \prod^{\alpha}\left(M^{-1}z - z_i\right)/\left(M^{-1}z - M^{-1}z_i\right)$$

$$\times \prod^{\beta}\left(M^{-1}z - M^{-1}\zeta_i\right)/\left(M^{-1}z - \zeta_i\right)$$

$$\sim \prod^{\alpha}(z - Mz_i)/(z - z_i)\prod^{\beta}(z - \zeta_i)/(z - M\zeta_i).$$

Thus

$$v(z, M^{-1}) \sim \prod^{\alpha}(z - z_i)/(z - Mz_i)\prod^{\beta}(z - M\zeta_i)/(z - \zeta_i).$$

COROLLARY. *If*

$$v(z, L) \sim \prod^{\alpha}(z - z_i)/(z - L^{-1}z_i)\prod^{\beta}(z - L^{-1}\zeta_i)/(z - \zeta_i)$$

then

$$v(z, L^k) \sim \prod^{\alpha}(z - z_i)/(z - L^{-k}z_i)\prod^{\beta}(z - L^{-k}\zeta_i)/(z - \zeta_i) \quad \textit{for all } k.$$

By (8), every subgroup of $SL(2, R)$ is in one of four classes:

Class 1. Groups generated by hyperbolic matrices, not all of which have the same pair of fixed points.

Class 2. Groups which consist of at most two types: hyperbolics all of which have identical fixed points x and y, and elliptics with trace zero which interchange x and y and which when squared equal $-I$.

Class 3. Groups which consist of parabolics, all of which have the same fixed point.

Class 4. Groups which consist of elliptics, all of which have the same two fixed points.

THEOREM 9. *Suppose v is a rational factor of automorphy with property* $(**)$. *Suppose Γ is in class* 1, 3, *or* 4, *and contains a matrix of infinite order. Then Γ has a set of generators each of which has* (1) *infinite order and* (2) *factor of automorphy achieving full size n.*

Proof. For some N, $v(z, N)$ has full size. By Theorem 4 of [8], a set of generators for Γ is selected, all having infinite orders. For each generator M, Lemma 1 below couples with either Lemma 2 or 3 to produce a k so both M^kN and $M^{k+1}N$ have (1) infinite orders and (2) factors of automorphy of full size. In the list of generators, M can be replaced by $M^{k+1}N$ and M^kN since M is generated by these two. When this process is repeated for each generator, the theorem is proved.

There is a minor problem in the proof if a generator M is in class (iii) of Lemma 3; however replacing M in the list of generators by the M_1 and M_2 given in the lemma and beginning again bridges this problem.

LEMMA 1. *If M has infinite order and $v(z, N)$ has full size, then for all but at most $2n^2$ values of k, $v(z, M^kN)$ has full size.*

Proof. The corollary above gives $v(z, M^k)$, so by Theorem 5(b)

$$\begin{aligned} v(z, M^kN) &= v(z, N)v(Nz, M^k) \\ &\sim \prod^{n}(z - x_i)/(z - y_i)\prod^{\alpha}\left(z - N^{-1}z_i\right)/\left(z - N^{-1}M^{-k}z_i\right) \\ &\quad \times \prod^{\beta}\left(z - N^{-1}M^{-k}\zeta_i\right)/\left(z - N^{-1}\zeta_i\right). \end{aligned}$$

Let k be chosen so there is no cancellation involving any term in which k appears; then $v(z, M^kN)$ has full size n.

LEMMA 2. *Let Γ be in class 3 or 4. For M of infinite order, M^kN has infinite order for all but at most one value of k.*

Proof. In class 3, every matrix has infinite order. For class 4, select T so in the conjugate group $T\Gamma T^{-1}$, the two fixed points common to all matrices are i and $-i$. Then

$$TNT^{-1} = \begin{pmatrix} \cos 2\pi\phi & \sin 2\pi\phi \\ -\sin 2\pi\phi & \cos 2\pi\phi \end{pmatrix}$$

which has order m exactly when $\phi = k/m$ with $(k, m) = 1$. Order is invariant under conjugation, and M has infinite order, so

$$TMT^{-1} = \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}$$

with θ irrational. So

$$TM^kNT^{-1} = (TMT^{-1})^kTNT^{-1} = \begin{pmatrix} \cos 2\pi(k\theta + \phi) & \sin 2\pi(k\theta + \phi) \\ -\sin 2\pi(k\theta + \phi) & \cos 2\pi(k\theta + \phi) \end{pmatrix}$$

which has infinite order except when $k\theta + \phi$ is rational, which can happen for at most one k.

LEMMA 3. *Let Γ be in class 1. For hyperbolic M, at least one of the following holds*:

(i) *M^kN has infinite order for all sufficiently large positive k*;
(ii) *M^kN has infinite order for all sufficiently large negative k*;
(iii) *$(M^kN)^2 = -I$ for all k.*

For M satisfying (iii), *there is an L and i such that $M_1 = M^iL$ and $M_2 = M^{i+1}L$ are hyperbolic and satisfy either* (i) *or* (ii).

Proof. Select T so $\hat{M} = TMT^{-1}$ has fixed points 0 and ∞, so

$$\hat{M} = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$$

with real $\lambda \neq \pm 1$. If

$$TNT^{-1} = \hat{N} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

then

$$\operatorname{trace}(M^kN) = \operatorname{trace}(\hat{M}^k\hat{N}) = \lambda^k\alpha + \lambda^{-k}\delta.$$

If not both α and δ are zero, then either all large positive k make $|\operatorname{tr}(M^kN)| > 2$, or all large negative k do. For such k, M^kN is hyperbolic and thus of infinite order. If however $\alpha = \delta = 0$ then $\gamma = -1/\beta$ and $(\hat{M}^k\hat{N})^2 = -I$, so $(M^kN)^2 = -I$ for all k.

For M satisfying (iii), select a hyperbolic L having different fixed points than M. Then, as above,

$$\hat{M} = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \quad \text{and} \quad \hat{N} = \begin{pmatrix} 0 & \beta \\ -1/\beta & 0 \end{pmatrix},$$

and

$$\hat{L} = TLT^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

nondiagonal. A necessary and sufficient condition that the square of a matrix in $SL(2, R)$ be $-I$ is that its trace be zero. Now

$$\operatorname{tr}(M^kL) = \operatorname{tr}(\hat{M}^k\hat{L}) = \lambda^k a + d/\lambda^k$$

and

$$\operatorname{tr}(M^kLN) = \operatorname{tr}(\hat{M}^k\hat{L}\hat{N}) = -b\lambda^k/\beta + c\beta/\lambda^k.$$

So it is simple to select i so $M_1 = M^iL$ and $M_2 = M^{i+1}L$ are both hyperbolic ($|\operatorname{tr}| > 2$) and also both satisfy $(M_1N)^2 \neq -I$ and $(M_2N)^2 \neq -I$. So M_1 and M_2 must satisfy (i) or (ii).

THEOREM 10. *Suppose v is a rational factor of automorphy with $(**)$. If Γ contains only matrices of infinite order, then Γ has a set of generators each of which has* (1) *infinite order and* (2) *factor of automorphy achieving full size n.*

Proof. Theorem 9 suffices for classes 1, 3, and 4. For class 2, all elliptics must be absent, and the proof is identical with the first part of the proof of Theorem 9 together with Lemma 1.

THEOREM 11. *Suppose v is a rational factor of automorphy with $(**)$. Suppose Γ contains no matrix of infinite order. If Γ contains a matrix of order at least $4n^4$, then Γ has a set of generators each of which has* (1) *order at least $4n^4$ and* (2) *factor of automorphy achieving full size n.*

Proof. If there is a bound on the order of matrices in Γ then Γ is cyclic [8], and its generator M has order $m \geq 4n^4$. For $n = 1$, $v(z, M)$ must have size 1; otherwise all $v(z, M^k) \sim 1$. For $n \geq 2$, M^k also generates Γ whenever k is relatively prime to m, and there are $\phi(m) > \sqrt{m} \geq 2n^2$ such possible k's. (The first inequality holds for all $m > 6$.) By the corollary to Theorem 8, $v(z, M^k)$ has full size if no $M^{-k}z_i$ equals any z_j and no $M^{-k}\zeta_i$ equals any ζ_j, and there are at most $2\alpha^2 + 2\beta^2 \leq 2n^2$ such equalities for $k = 1, 2, \ldots, m$. So there will be a k such that M^k generates and $v(z, M^k)$ is full.

If there is not a bound on the orders of matrices in Γ, then Γ is in class 4 and there is a set of generators all having finite orders as high as we wish [8]. Let M be a generator of order j and let N have factor of automorphy of full size. In the notation of Lemma 2, $\theta = 2\pi i/j$ and $\phi = 2\pi l/m$, where j can be assumed as large as we wish. The argument of Lemma 1 shows that for all but a few values of k, $v(z, M^kN)$ achieves full size. The order of M^kN appears as the denominator of the fraction $ik/j + l/m = (ikm + jl)/jm$ reduced to lowest terms. Thus by choosing j large enough, k can be found so both M^kN and $M^{k+1}N$ have orders at least $4n^4$ and have factors of automorphy which achieve full size. In the list of generators, M can be replaced by $M^{k+1}N$ and M^kN. The process is repeated for each generator.

THEOREM 12 (FUNDAMENTAL THEOREM). *Suppose v is a rational factor of automorphy with $(**)$. If Γ contains either a matrix of infinite order, or a matrix of order $\geq 4n^4$, then there exist disjoint lists $\{z_i\}$ and $\{\zeta_i\}$, and a fixed* *integer α such that for every M in Γ,*

$$v(z, M) \sim \prod^{\alpha}(z - z_i)/(z - M^{-1}z_i) \prod^{n-\alpha}(z - M^{-1}\zeta_i)/(z - \zeta_i).$$

Proof. Suppose Γ contains M of infinite order, and Γ is not in class 2. Let M and N be two of the generators guaranteed by Theorem 9 having factors of automorphy of full size n. If we write $\{x_i/y_i\}$ as shorthand for $(z - x_i)/(z - y_i)$, then by Theorem 6(ii),

$$v(z, M) \sim \prod^{\alpha}\{z_i/M^{-1}z_i\} \prod^{n-\alpha}\{M^{-1}\zeta_i/\zeta_i\}$$

and

$$v(z, N) \sim \prod^{\beta}\{x_i/N^{-1}x_i\} \prod^{n-\beta}\{N^{-1}y_i/y_i\},$$

so that

$$\begin{aligned} v(z, M^jN^k) &= v(z, N^k)v(N^kz, M^j) \\ &\sim \left[\prod^{\beta}\{x_i/N^{-k}x_i\} \prod^{n-\beta}\{N^{-k}y_i/y_i\}\right] \\ &\quad\times\left[\prod^{\alpha}\{N^{-k}z_i/N^{-k}M^{-j}z_i\} \prod^{n-\alpha}\{N^{-k}M^{-j}\zeta_i/N^{-k}\zeta_i\}\right]. \end{aligned}$$

Choose j so no $N^{-k}M^{-j}z_i$ cancels with any $N^{-k}z_i$ or $N^{-k}y_i$, and no $N^{-k}M^{-j}\zeta_i$ cancels with any $N^{-k}\zeta_i$ or $N^{-k}x_i$. The value of k is clearly irrelevent in this choice. Then k is chosen so no x_i cancels with any $N^{-k}x_i$ or $N^{-k}M^{-j}z_i$, and no y_i cancels with any $N^{-k}y_i$ or $N^{-k}M^{-j}\zeta_i$. The number of j's to be avoided is at most $2n^2 - n$, and the number of k's to be avoided is likewise $2n^2 - n$, but since M has infinite order, there are infinitely many j's to choose from, and similarly for k. Therefore the only way for the size of $v(z, M^jN^k)$ to not exceed n, is for the list of x_i's to be identical with the list of z_i's, and the list of y_i's to match the list of ζ_i's. We repeat the process for every pair of the particular generators mentioned above, and then apply Theorem 8 repeatedly to conclude Theorem 12.

If Γ contains only matrices of finite order, including one of order at least $4n^4$, then Theorem 11 guarantees a set of generators for which the argument just given is valid, since there are at least $4n^4$ j's to choose from, and similarly for k, and only a few k's and j's to avoid.

There remains only the case that Γ is in class 2 containing an M with infinite order. If we delete from Γ all N such that $N^2 = -I$ then all matrices in the resulting group Γ' have infinite order, and Theorem 10 produces a set of generators for Γ' with infinite orders which have factors of automorphy achieving size n', the largest size occurring in Γ'. The argument above is applicable to Γ' and shows there are two lists $\{z_i\}$ and $\{\zeta_i\}$ and an integer α, such that for each M in Γ',

$$(*)\quad v(z, M) \sim \prod^{\alpha}(z - z_i)/(z - M^{-1}z_i) \prod^{n'-\alpha} (z - M^{-1}\zeta_i)/(z - \zeta_i).$$

If there is no N in Γ with $N^2 = -I$, then $n' = n$, and the theorem is proved. If there is an N in Γ with $N^2 = -I$, then there must be one with factor of automorphy which achieves full size n. The reason is that if not, there would be an infinite order L with this property, and Lemma 1 of Theorem 9 would imply $v(z, L^kN)$ has size n for large enough k, and further it is automatic in class 2 that $(L^kN)^2 = -I$.

Select M in Γ' and N in Γ, with factors of automorphy having maximal size for Γ' and Γ respectively (these sizes are denoted by n' and n respectively) and $N^2 = -I$. Theorem 7(ii) implies

$$v(z, N) \sim \prod^{n}(z - x_i)/(z - N^{-1}x_i)$$

because the size is n and each $(z - N^{-1}\zeta_i)/(z - \zeta_i)$ term can be expressed as

$$(z - x_i)/(z - N^{-1}x_i) \quad \text{where } x_i = N^{-1}\zeta_i,$$

so $N^{-1}x_i = N^{-2}\zeta_i = (-I)\zeta_i = \zeta_i$. By $(*)$ and the corollary to Theorem 8,

$$v(z, M^kN) \sim v(z, N)v(Nz, M^k)$$
$$\sim \prod^{n}(z - x_i)/(z - N^{-1}x_i)\prod^{\alpha}(z - N^{-1}z_i)/(z - N^{-1}M^{-k}z_i)$$
$$\times \prod^{n'-\alpha} (z - N^{-1}M^{-k}\zeta_i)/(z - N^{-1}\zeta_i)$$

for all k. We vary k to ascertain properties about the x_i, z_i, and ζ_i. Let k be selected so no term with a k cancels with any other term. Each of the α denominator terms $(z - N^{-1}M^{-k}z_i)$ must have a companion numerator term $(z - z_i)$ by Theorem 7 since $(M^kN)^2 = -I$. Since the size can be at most n, each $N^{-1}x_i$ must cancel with some $N^{-1}z_i$. Thus

$$v(z, M^kN) \sim \prod^{\alpha}(z - z_i)/(z - N^{-1}M^{-k}z_i) \prod^{n-\alpha} (z - x_i)/(z - N^{-1}x_i)$$
$$\times \prod^{n'-\alpha} (z - N^{-1}M^{-k}\zeta_i)/(z - N^{-1}\zeta_i).$$

Similarly each of the $(n' - \alpha)$ numerators $(z - N^{-1}M^{-k}\zeta_i)$ requires a companion $(z - \zeta_i)$ denominator. Again the size can be at most n, so each $N^{-1}\zeta_i$ must cancel with an x_i. If $N^{-1}\zeta_i = x_j$, then $N^{-1}x_j = N^{-2}\zeta_i = \zeta_i$ since $N^2 = -I$. Thus for all k,

$$v(x, M^kN) \sim \prod^{\alpha}(z - z_i)/(z - N^{-1}M^{-k}z_i) \prod^{n'-\alpha}(z - N^{-1}M^{-k}\zeta_i)/(z - \zeta_i)$$
$$\times \prod^{n-n'}(z - x_i)/(z - N^{-1}x_i).$$

On the other hand, since $(M^kN)^2 = -I$, the last product must be (by Theorem 7)

$$\prod^{n-n'}(z - x_i)/(z - N^{-1}M^{-k}x_i)$$

for all k. If any x_i were not a fixed point of M, then k could be chosen so $N^{-1}M^{-K}x_i$ would be different from all the $N^{-1}x_i$, a contradiction. So all the x_i's in the last product satisfy $M^{-1}x_i = x_i$. We have therefore shown (*) is equivalent to

$$(**)\quad v(z, M) \sim \prod^{\alpha}(z - z_i)/(z - M^{-1}z_i) \prod^{n'-\alpha}(z - M^{-1}\zeta_i)/(z - \zeta_i)$$
$$\times \prod^{n-n'}(z - x_i)/(z - M^{-1}x_i)$$

for all matrices M of infinite order. Taking $k = 0$ in the previous equation, we see $(**)$ holds for $M = N$ with the same x_i, z_i, and ζ_i. If L is any other matrix in Γ, (so $L^2 = -I$), then LN^{-1} has infinite order, and $L = (LN^{-1})N$, so Theorem 8 can be applied to show $(**)$ also holds for $M = L$.

COROLLARY. *If Γ is any infinite subgroup of $SL(2, R)$ then the conclusion of Theorem* 12 *remains valid.*

Proof. Either Γ contains an infinite order matrix, or not. In the latter case, Γ must contain matrices of arbitrarity high order, since if not, there would be a uniform bound A on the orders, so Γ can be considered a period group of period $(2A)!$, thus a special case of a theorem of Burnside [7, p. 251] would imply Γ must be finite. In either case Theorem 12 applies.

VI. Conclusions

Theorem 12 and its corollary apply to any factor of automorphy which satisfies $(*)$ and $(**)$ of Section III. Condition $(**)$ was adopted purely for computational convenience, and shall be done away with, resulting in Theorem 13.

DEFINITION.

$$[z-w]^* = \begin{cases} z-w & \text{if} \quad w \neq \infty \\ 1 & \text{if} \quad w = \infty \end{cases} \quad \text{and} \quad \left[\frac{z-w}{z-v}\right]^* = \frac{[z-w]^*}{[z-v]^*}.$$

Suppose $v(z, M)$ satisfies (*) of Section III. Given

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

let $\zeta = T^{-1}z$. The conjugate group for Γ is $\hat{\Gamma} = \{\hat{M} : \hat{M} = T^{-1}MT, M \in \Gamma\}$.

$$\text{Define } \hat{v}(\zeta, \hat{M}) = v(T\zeta, T\hat{M}T^{-1}) = v(z, M).$$

The consistency condition $v(z, MN) = v(z, N)v(Nz, M)$ becomes

$$\hat{v}(\zeta, \hat{M}\hat{N}) = \hat{v}(\zeta, \hat{N})\hat{v}(\hat{N}\zeta, \hat{M}),$$

and $v(z, M) \sim \prod^{\mu}(z - x_i)/\prod^{\nu}(z - y_i)$ becomes

$$\hat{v}(\zeta, \hat{M}) \sim \left[\prod^{\mu}(\zeta - T^{-1}x_i)\Big/\prod^{\nu}(\zeta - T^{-1}y_i)\right](c\zeta + d)^{\nu-\mu}.$$

T can be chosen so the factor of automorphy $\hat{v}(\zeta, \hat{M})$ satisfies both (*) and (**) of Section III.

THEOREM 13. *Suppose v is a rational factor of automorphy. If Γ is infinite or contains an L of order at least $4n^4$, then there exist lists $\{z_i\}$ and $\{\zeta_i\}$, and a number α such that for each M in Γ,*

$$v(z, M) \sim \prod^{\alpha}[(z - z_i)/(z - M^{-1}z_i)]^* \prod^{n-\alpha}[(z - M^{-1}\zeta_i)/(z - \zeta_i)]^*.$$

Proof. With $\hat{T}$, $\hat{M}$, $\hat{v}$, and ζ as above, $v(z, M) = \hat{v}(\zeta, \hat{M})$. The order of $\hat{M}$ equals the order of M, so by Theorem 12 and its corollary there exist lists $\{\hat{z}_i\}$ and $\{\hat{\zeta}_i\}$, and a number α such that

$$\begin{aligned} v(z, M) &= \hat{v}(\zeta, \hat{M}) \\ &\sim \prod^{\alpha}(\zeta - \hat{z}_i)/(\zeta - \hat{M}^{-1}\hat{z}_i) \prod^{n-\alpha}(\zeta - \hat{M}^{-1}\hat{\zeta}_i)/(\zeta - \hat{\zeta}_i). \end{aligned}$$

We have

$$\zeta = T^{-1}z = \frac{dz - b}{-cz + a},$$

so for w arbitrary,

$$(\zeta - w) = \begin{cases} (z - Tw)/(-cz + a) & \text{if} \quad w \neq \infty \\ 1/(-cz + a) & \text{if} \quad w = \infty. \end{cases}$$

Therefore

$$\begin{aligned} v(z, M) &\sim \prod^{\alpha} [(z - T\hat{z}_i)/(z - T\hat{M}^{-1}\hat{z}_i)]^* \prod^{n-\alpha} [(z - T\hat{M}^{-1}\hat{\zeta}_i)/(z - T\hat{\zeta}_i)]^* \\ &\sim \prod^{\alpha} [(z - T\hat{z}_i)/(z - M^{-1}T\hat{z}_i)]^* \prod^{n-\alpha} [(z - M^{-1}T\hat{\zeta}_i)/(z - T\hat{\zeta}_i)]^*. \end{aligned}$$

Define $z_i = T\hat{z}_i$ and $\zeta_i = T\hat{\zeta}_i$, and the theorem is proved.

In the case z_i (or ζ_i) is infinite, $[(z - z_i)/(z - M^{-1}z_i)]^*$ becomes

$$1/[z - M^{-1}z_i]^*.$$

The question arises whether the few groups not covered by the proofs in this paper might also satisfy the conclusions of these theorems. Unfortunately not, as the following example shows. Let $N^7 = I$, $\Gamma = \langle N \rangle$, x be a nonfixed point of N, and

$$v(z, N) = \frac{(z - x)(z - N^{-2}x)}{(z - N^{-1}x)(z - N^{-4}x)}.$$

Then $v(z, N^m)$ always has size two or less; however, v cannot be written in the form which Theorem 12 (with $n = 2$) guarantees. The theorem cannot be applied in this case because the order 7 does not exceed $4n^4$.

THEOREM 14. *Let Γ be an infinite subgroup of $SL(2, R)$ or contain a matrix of order $\geq 4n^4$, and let D be a subset of the complex plane which has at least one finite limit point. If $f(Mz) = v(z, M)f(z)$ for $z \in D$, and $v(z, M)$ is rational in the sense of $(*)$ of Section* III, *then there is a rational function $R(z)$ such that $R(z)f(z)$ is an unrestricted automorphic form of integral degree.*

Proof. Define $R(z) = \prod^{\alpha}(z - z_i)^*/\prod^{n-\alpha}(z - \zeta_i)^*$ where the z_i and ζ_i are those which arise in the previous theorem. Define $g(z) = R(z)f(z)$. Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

belong to Γ. Then

$$\begin{aligned} g(Mz) &= R(Mz)f(Mz) = R(Mz)v(z, M)f(z) \\ &= [R(Mz)/R(z)]v(z, M)g(z). \end{aligned}$$

If $\zeta_i \neq \infty$, then $Mz - \zeta_i \sim (z - M^{-1}\zeta_i)^*(cz + d)^{-1}$, and if $\zeta_i = \infty$, then $(z - \zeta_i)$ fails to appear in $R(z)$ so $Mz - \zeta_i$ fails to appear in $R(Mz)$. Analogous results hold for z_i in place of ζ_i. Thus

$$\begin{aligned} &[R(Mz)/R(z)]v(z, M) \\ &\quad \sim 1 \cdot \left[\prod^{\varepsilon} (z - M^{-1}\zeta_i)^* \Big/ \prod^{\lambda} (z - M^{-1}z_i)^* \right] (cz + d)^{n - 2\alpha + \lambda - \varepsilon}, \end{aligned}$$

where $\prod^{\lambda}$ contains the terms for which z_i is infinite, and $\prod^{\varepsilon}$ contains the terms for which ζ_i is infinite. If $z_i = \infty$, then $M^{-1}z_i = -d/c$ so that

$$(z - M^{-1}z_i)^* \sim (cz + d).$$

Likewise if $\zeta_i = \infty$, then $(z - M^{-1}\zeta_i)^* \sim (cz + d)$. So

$$[R(Mz)/R(z)]v(z, M) \sim (cz + d)^{n - 2\alpha}.$$

We have shown $g(Mz) \sim (cz + d)^{n-2\alpha}g(z)$; that is, there is a function $u(M)$ such that

$$g(Mz) = u(M)(cz + d)^{n - 2\alpha}g(z),$$

and the theorem is proved.

The requirement in Theorem 14 on the set D is a minimal one. A stronger hypothesis yields the following.

COROLLARY. *Suppose Γ is infinite and D is the complex plane. Then $f(z)$ has a rational factor of automorphy if and only if $f(z)$ is itself rational.*

Proof. If $f(z)$ is rational, then $f(Mz)/f(z)$ is an obvious factor of automorphy and is rational. Conversely, if $f(Mz) = v(z, M)f(z)$ for v rational, then by Theorem 14, $R(z)f(z)$ is an unrestricted automorphic form for some rational $R(z)$. Any unrestricted automorphic form for an infinite group which is meromophic on the whole complex plane is known to be $u(M)(z - z_1)^a(z - z_2)^b$ for integers, a, b, (see [13]), so $R(z)f(z)$ must be rational, hence $f(z)$ is rational.

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