ILLINOIS JOURNAL OF MATHEMATICS
Volume 31, Number 1, Spring 1987

SURFACES WITH MINIMAL RANDOM WEIGHTS AND MAXIMAL FLOWS: A HIGHER DIMENSIONAL VERSION OF FIRST-PASSAGE PERCOLATION

BY

HARRY KESTEN[1]

1. Introduction

Random surfaces with independent plaquettes were studied by Aizenman et al. [2] in connection with bond percolation in $\mathbf{Z}^3$. Such surfaces arise naturally as dual objects to percolation clusters, and they exhibit critical phenomena similar to other percolation models (cf. [2]). Random surfaces with dependent plaquettes have recently been considered in several other physical contexts [1], [6] but here we stick to situations where the plaquettes have independent characteristics. Our original motivation was to generalize the estimates on the resistances of random two-dimensional subnetworks of $\mathbf{Z}^2$ of [8] and [12, Ch. 11], to three-dimensional subnetworks of $\mathbf{Z}^3$. Since our results in this direction are still unsatisfactory (as indicated in (2.23) below) we concentrate here on the relation with first-passage percolation and maximal flows.

We begin with a brief description of the fundamental results of first-passage percolation on $\mathbf{Z}^d$ (d not necessarily restricted to 2 or 3). A good introduction to the subject is the monograph [15] of Smythe and Wierman. For later results see also [13]. To each edge e of $\mathbf{Z}^d$ between two neighboring vertices[2] of $\mathbf{Z}^d$ one assigns a random nonnegative value $t(e)$. It is assumed that all $t(e)$, $e \in \mathbf{Z}^d$, are independent and have the same distribution function F with

$$F(0-) = 0; \tag{1.1}$$

$t(e)$ was interpreted by Hammersley and Welsh in [10]—the article which started the subject—as the passage time of e. A *path* on $\mathbf{Z}^d$ (from v_0 to v_n) is a sequence $(v_0, e_1, v_1, \dots, e_n, v_n)$ of vertices $v_0, \dots, v_n$ alternating with edges $e_1, \dots, e_n$, such that v_{i-1} and v_i are neighbors on $\mathbf{Z}^d$ with e_i the edge of $\mathbf{Z}^d$

Received February 15, 1985.

[1]Research supported by the National Science Foundation through a grant to Cornell University.

[2]The vertices $v = (v(1), \dots, v(d))$ and w in $\mathbf{Z}^d$ are neighbors (or adjacent) if and only if $\Sigma_1^d |v(i) - w(i)| = 1$.

between them, $1 \le i \le n$. If r is the path $(v_0, e_1, \dots, e_n, v_n)$ we set

$$T(r) = \sum_{i=1}^{n} t(e_i).$$

$T(r)$ can be interpreted as the passage time of r, and it is natural to introduce the following quantities which can be interpreted as travel times between the origin (denoted by $\mathbf{0}$) and the point $(k, 0, \dots, 0)$, and between $\mathbf{0}$ and the hyperplane

$$H_k := \{x : x(1) = k\},$$

respectively.

$$a_{0,k} := \inf\{T(r) : r \text{ a path from } \mathbf{0} \text{ to } (k, 0, \dots, 0)\},$$
$$b_{0,k} := \inf\{T(r) : r \text{ a path from } \mathbf{0} \text{ to a point in } H_k\}.$$

In many respects it is easier to work with the more restricted "cylinder passage times"

$$t_{0,k} := \{\inf T(r) : r \text{ a path from } \mathbf{0} \text{ to } (k, 0, \dots, 0) \text{ which, with the exception of its endpoints, lies strictly between the hyperplanes } H_0 \text{ and } H_k\},$$
$$s_{0,k} := \inf\{T(r) : r \text{ a path from } \mathbf{0} \text{ to } H_k \text{ which, with the exception of its endpoints lies strictly between the hyperplanes } H_0 \text{ and } H_k\}.$$

Finally, we shall need the passage times between hyperplanes:

$$l_{k,m} = \{\inf T(r) : r \text{ a path from } H_0 \text{ to } H_k \text{ which is contained in the box } [0, k] \times [0, m]^{d-1}\}.$$

Since we took the $t(e)$ non-negative we may restrict the inf in the definitions of a, b, t, s and l to *selfavoiding paths* r, i.e., to paths r all of whose vertices are distinct.

One of the fundamental results of first-passage percolation is the following theorem[3] (see [15, Sections 5.1–5.3], [8], [13]).

THEOREM A. *If*

$$Et(e) = \int_{[0,\infty)} x \, dF(x) < \infty, \tag{1.2}$$

[3] We do not state minimal conditions here; for refinements see [5] and [13].

then there exists a constant $\mu = \mu(F, d) < \infty$ *such that*

$$\lim_{k \to \infty} \frac{1}{k} \theta_{0,k} = \mu \quad \text{w.p.1} \quad \textit{and in } L^1 \tag{1.3}$$

for $\theta = a, b, t$ *or* s. *If* $m \to \infty$ *with* k *in such a way that*

$$\frac{1}{k} \log m \to 0,$$

then also

$$\lim \frac{1}{k} l_{k,m} = \mu \quad \text{w.p.1.} \quad \textit{and in } L^1 \tag{1.4}$$

Our main objective is to prove a version of this theorem when the role of the edges e is taken over by "plaquettes" in $\mathbf{Z}^3$. To explain why this is desirable we interpret $t(e)$ as a capacity of the edge e. In other words $t(e)$ is the maximal amount of fluid which can flow through e per unit time. A good introduction to this interpretation is given by Fulkerson in [7]. (See also [8] for an interpretation of $t(e)$ as a limitation on electrical current which can flow through e, i.e., as a conductivity.) For given $t(e)$ we denote by $\Phi(\mathbf{k}, m)$ the *maximal flow* through the restriction of $\mathbf{Z}^d$ to the box

$$B(\mathbf{k}, m) := [0, k_1] \times \cdots \times [0, k_{d-1}] \times [0, m]$$

from its bottom

$$F_0 := [0, k_1] \times \cdots \times [0, k_{d-1}] \times \{0\}$$

to its top

$$F_m := [0, k_1] \times \cdots \times [0, k_{d-1}] \times \{m\}.$$

(Of course $\mathbf{k}$ is short for $(k_1, \dots, k_{d-1})$ here.) By definition, such a *flow* is an assignment of nonnegative numbers $f(e)$ and a direction to all the edges e in $B(\mathbf{k}, m)$ such that $0 \le f(e) \le t(e)$ for all e, and such that for each vertex v outside $F_0 \cup F_m$ the total inflow equals the total outflow, that is, $\Sigma_v^+ f(e) = \Sigma_v^- f(e)$, where Σ_v^+ (Σ_v^-) is the sum over all edges incident to v and directed towards v (away from v). For any such assignment, the flow from F_0 to F_m is defined as $\Sigma^+ f(e) - \Sigma^- f(e)$ where Σ^+ (Σ^-) is the sum over all edges e with exactly one endpoint in F_m, and e directed towards this endpoint (away from this endpoint). The maximum of this expression over all possible choices of $f(\cdot)$ is $\Phi(\mathbf{k}, m)$. See [7], [4] for more details.[4]

[4] To make the connection with these references, which deal with directed graphs, we should replace in our graphs each edge e by a pair of edges between the endpoints of e, and assign opposite orientation to the two edges replacing e.

The max-flow min-cut theorem allows us to express $\Phi(\mathbf{k}, m)$ in a different way (which for $d = 2$ immediately expresses $\Phi(k, m)$ in terms of $l_{k+1, m-1}$). A set of edges E is said to *separate F_0 from F_m* in $B(\mathbf{k}, m)$ if there is no path in $B(\mathbf{k}, m) \setminus E$ from F_0 to F_m. We call E an *(F_0, F_m)-cut* if E separates F_0 from F_m in $B(\mathbf{k}, m)$ and if E is *minimal*, in the sense that no proper subset of E separates F_0 from F_m. Note the minimality requirement in this definition; some authors do not include this in the definition of a cut (cf. [4] and [7]). Note also that we shall usually not explicitly refer to $B(\mathbf{k}, m)$ when talking about an (F_0, F_m)-cut. However, we shall occasionally use the alternative phrase that E is a *cut which separates the bottom from the top in* $B(\mathbf{k}, m)$ to indicate that E is an (F_0, F_m)-cut in $B(\mathbf{k}, m)$.

To each set of edges E we assign the value

$$V(E) = \sum_{e \in E} t(e). \tag{1.5}$$

The max-flow min-cut theorem [4], [7] states that

$$\Phi(\mathbf{k}, m) = \min\{V(E) : E \text{ an } (F_0, F_m)\text{-cut}\}. \tag{1.6}$$

This much still holds for any d. However, it seems that only for $d = 2$ a simple description of all (F_0, F_m)-cuts is available. When $d = 2$, the *dual graph* of $\mathbf{Z}^2$ is $\mathscr{L}^* = \mathbf{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ (cf. [15, Section 2.1], [12, Section 2.6]). Its vertices are the points $(j_1 + \frac{1}{2}, j_2 + \frac{1}{2})$, $j_1, j_2 \in \mathbf{Z}$ and there is an edge of $\mathscr{L}^*$ between $(j_1 + \frac{1}{2}, j_2 + \frac{1}{2})$ and $(i_1 + \frac{1}{2}, i_2 + \frac{1}{2})$ if and only if $|j_1 - i_1| + |j_2 - i_2| = 1$; see Fig. 1. Each edge e of $\mathbf{Z}^2$ intersects exactly one edge e^* of $\mathscr{L}^*$ and vice versa. We call the intersecting e and e^* *associated to each other*. Through this association we have a one to one correspondence between the edges of $\mathbf{Z}^2$ and the edges of $\mathscr{L}^*$. It is therefore unambiguous to define $t(e^*)$ for an edge e^* of $\mathscr{L}^*$ as $t(e)$, where e is the edge of $\mathbf{Z}^2$ associated to e^*. It is a special case of Whitney's theorem (see [17, Theorem 4], [15, Section 2.1], [12,

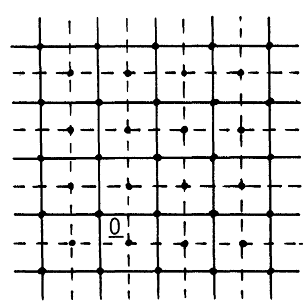

FIG. 1 The graph $\mathbf{Z}^2$ (solid edges) and its dual $\mathscr{L}^*$ (dashed edges).

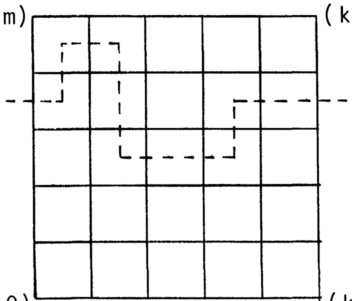

FIG. 2 The dashed path is an (F_0, F_m)-cut in $B(k, m)$.

Proposition 2.2]) that a set of edges E in $B(k, m) = [0, k] \times [0, m]$ is an (F_0, F_m)-cut if and only if E consists of the edges associated to the edges $e_1^*, \dots, e_\nu^*$ of a self-avoiding path

$$r^* = (v_1^*, e_1^*, \dots, e_\nu^*, v_\nu^*)$$

on $\mathscr{L}^*$ from a point on $x(1) = -\frac{1}{2}$ to a point on $x(1) = n + \frac{1}{2}$, and contained in $(-\frac{1}{2}, k + \frac{1}{2}) \times [\frac{1}{2}, m - \frac{1}{2}]$ (except for its endpoints). This has the intuitive meaning that the minimal sets which separate the top from the bottom of $[0, k] \times [0, m]$ correspond to paths on the dual graph through this rectangle from left to right. (See Fig. 2.)

From the above we see that for $d = 2$, if E consists of the edges associated to the edges e_i^*, $1 \le i \le \nu$, of the path

$$r^* = (v_0^*, e_1^*, \dots, e_\nu^*, v_\nu^*)$$

on $\mathscr{L}^*$, then

$$V(E) = T(r^*) := \sum_{i=1}^{\nu} t(e_i^*)$$

(cf. (1.5)), and by virtue of (1.6),

$$\begin{aligned}\Phi(k, m) = \{\min T(r^*) : r^* &\text{ a self-avoiding path on } \mathscr{L}^* \text{ from} \\ x(1) &= -\tfrac{1}{2} \text{ to } x(1) = n + \tfrac{1}{2} \text{ contained in} \\ [-\tfrac{1}{2}, k &+ \tfrac{1}{2}] \times [\tfrac{1}{2}, m + \tfrac{1}{2}]\}.\end{aligned}$$

Thus, for $d = 2$, $\Phi(k, m)$ is the analogue of $l_{k+1, m-1}$ on $\mathscr{L}^*$. Consequently, by (1.4),

$$\text{(1.7)} \qquad \lim \frac{1}{k} \Phi(k, m) = \mu \quad \text{w.p.1}$$

when $k \to \infty$, $m \to \infty$ such that

$$\frac{1}{k}\log m \to 0.$$

It is the result (1.7) which we want to generalize to higher dimensions, and in particular to $d = 3$. In view of (1.7), the most obvious guess is that under some restrictions on $k_1, \ldots, k_{d-1}$ and m,

$$(k_1 k_2, \ldots, k_{d-1})^{-1}\Phi(\mathbf{k}, m)$$

will converge to a constant. If one attempts to imitate for $d = 3$ the two-dimensional proof indicated above, then one probably will replace the edge e^* associated to e by the unit square π^* perpendicular to e and bisecting e (compare also [2]). We shall call these unit squares of the form

$$\left[j_1 - \tfrac{1}{2}, j_1 + \tfrac{1}{2}\right] \times \left[j_2 - \tfrac{1}{2}, j_2 + \tfrac{1}{2}\right] \times \left\{j_3 + \tfrac{1}{2}\right\}, \quad j_i \in \mathbf{Z},$$

or the similar forms obtainable by interchanging the roles of the coordinates, *plaquettes*. Thus the plaquettes are faces of the unit cubes with centers in $\mathbf{Z}^3$; the "corners" of the plaquettes are on $\mathscr{L}^* := \mathbf{Z}^3 + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Again each plaquette intersects a unique edge of $\mathbf{Z}^3$ and vice versa, so that plaquettes are associated in a one to one way to edges of $\mathbf{Z}^3$, and we can set $t(\pi^*) = t(e)$ if e is the edge associated to the plaquette π^*. However, we do not know how to characterize in a simple manner a collection of plaquettes E^* for which the associated edges form an (F_0, F_m)-cut in the box

$$B(k, l, m) = [0, k] \times [0, l] \times [0, m].$$

Loosely speaking, we expect such an E^* to be a "surface" which cuts $B(k, l, m)$ into a lower and an upper component. We define

$$\partial E^* = \text{collection of edges of } \mathscr{L}^* \text{ which belong to an odd number of plaquettes of } E^*. \tag{1.8}$$

We expect that if E^* corresponds to a cut, then ∂E^* lies outside $B(k, l, m)$. In fact, as we shall see in Lemma 3.6a, ∂E^* lies in

$$\begin{aligned}
&\left[-\tfrac{1}{2}, k + \tfrac{1}{2}\right] \times \left\{-\tfrac{1}{2}\right\} \times \left[\tfrac{1}{2}, m - \tfrac{1}{2}\right] \\
&\cup \left[-\tfrac{1}{2}, k + \tfrac{1}{2}\right] \times \left\{l + \tfrac{1}{2}\right\} \times \left[\tfrac{1}{2}, m - \tfrac{1}{2}\right] \\
&\cup \left\{-\tfrac{1}{2}\right\} \times \left[-\tfrac{1}{2}, l + \tfrac{1}{2}\right] \times \left[\tfrac{1}{2}, m - \tfrac{1}{2}\right] \\
&\cup \left\{k + \tfrac{1}{2}\right\} \times \left[-\tfrac{1}{2}, l + \tfrac{1}{2}\right] \times \left[\tfrac{1}{2}, m - \tfrac{1}{2}\right],
\end{aligned} \tag{1.9}$$

the "*vertical surface surrounding* $B(k, l, m)$". However, E^* can be quite complicated; the union of the plaquettes in E^* is not necessarily a two-manifold. Rather than getting involved in complicated topological classifications we simply decide to call a collection of plaquettes E^* an *(F_0, F_m)-cut* or a *cut which separates the bottom from the top in $B(k, l, m)$* if and only if the set E of associated edges of $\mathbf{Z}^3$ is an (F_0, F_m)-cut. We then study cut sets on $\mathscr{L}^*$, show how to patch several of them together, and finally obtain an analogue of (1.4) which also shows that[5] $(kl)^{-1}\Phi(k, l, m)$ converges under suitable conditions to a constant ν.

It is more ambiguous to decide what the proper analogues of $a_{0,n}$, $b_{0,n}$, $t_{0,n}$ and $s_{0,n}$ are for plaquettes. For any collection E^* of plaquettes define

$$V(E^*) = \sum_{\pi^* \in E^*} t(\pi^*). \tag{1.10}$$

We also must extend our definition of separating set and a cut. Let S be a rectangle of the form $[k_1, k_2] \times [l_1, l_2]$, $k_i, l_i \in \mathbf{Z}$, $k_1 \le k_2$, $l_1 \le l_2$. We say that a set E of edges of $\mathbf{Z}^3$, or the set E^* of associated plaquettes, *separates $-\infty$ from $+\infty$ over S* if there is no path on $\mathbf{Z}^3$ in $(S \times \mathbf{Z}) \setminus E$ from $S \times \{-N\}$ to $S \times \{+N\}$ for some (and hence all sufficiently large) $N > 0$. Similarly we call E, or the set E^* of associated plaquettes, a *cut over S* if E separates $-\infty$ from $+\infty$ over S, but no proper subset of E separates $-\infty$ from $+\infty$ over S. As analogues of $t_{0,n}$ and $a_{0,n}$ we now propose

$$\begin{aligned}\tau(k, l) = \{&\inf V(E^*) : E^* \text{ a cut over } [0, k] \times [0, l] \text{ whose} \\ &\text{boundary } \partial E^* \text{ consists of the edges of } \mathscr{L}^* \\ &\text{on the perimeter of } [-\tfrac{1}{2}, k + \tfrac{1}{2}] \times [-\tfrac{1}{2}, l + \tfrac{1}{2}] \times \{\tfrac{1}{2}\}\}\end{aligned} \tag{1.11}$$

and

$$\begin{aligned}\alpha(k, l) = \{&\inf V(E^*) : E^* \text{ separates } -\infty \text{ from } +\infty \text{ over} \\ &[0, k] \times [0, l], \text{ and } \partial E^* \text{ consists of the edges} \\ &\text{of } \mathscr{L}^* \text{ on the perimeter of} \\ &[-\tfrac{1}{2}, k + \tfrac{1}{2}] \times [-\tfrac{1}{2}, l + \tfrac{1}{2}] \times \{\tfrac{1}{2}\}\},\end{aligned} \tag{1.12}$$

respectively.

Remark. The greek names have been chosen to bring out the analogy with ordinary first passage percolation. The quantity α is the analogue of a, and τ of t. σ and β in (2.5) and (2.6) correspond to s and b, respectively.

---

[5] Of course $\Phi(k, l, m)$ is the flow which we denoted above by $\Phi(\mathbf{k}, m)$ for the case $\mathbf{k} = (k, l)$.

It will be seen in (3.1a) that the requirement that E^* separate $-\infty$ from $+\infty$ over $[0, k] \times [0, l]$ is superfluous in (1.12). It is already implied by the requirement on the form of ∂E^*.

As a partial analogue of (1.3) we show that under extra conditions there exists a constant $\nu < \infty$ such that

$$(1.13) \qquad \lim_{k, l \to \infty} \frac{1}{kl} \tau(k, l) = \lim_{k, l \to \infty} \frac{1}{kl} \alpha(k, l) = \nu \quad \text{w.p.1};$$

this ν equals $\lim(kl)^{-1} \Phi(k, l, m)$. Analogues of $b_{0,k}$ and $s_{0,k}$ and precise results are given in Section 2, to which the reader can skip now.

It is instructive, though, to discuss further the relationship with bond-percolation in $\mathbf{Z}^3$, and the random surfaces of Aizenman et al. [2]. Bond-percolation corresponds to the case where $t(e)$ can take only the values 0 or 1, with probabilities $q = 1 - p$ and p, respectively. This means that F is taken to be the Bernoulli distribution

$$B_p(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 - p & \text{if } 0 \le x < 1, \\ 1 & \text{if } x \ge 1. \end{cases}$$

We shall call an edge e with $t(e) = 1$ *open* and one with $t(e) = 0$ *closed*. The space of all configurations of open and closed edges can then be identified with $\Omega = \{0, 1\}^{\mathscr{E}}$, where $\mathscr{E}$ denotes the set of all edges of $\mathbf{Z}^3$. We write P_p for the joint distribution of the $t(e)$ in this case; it can be identified with the product measure on Ω according to which each coordinate is 0(1) with probability $q(p)$.

Call a path on $\mathbf{Z}^3$ *open* if all its edges are open. It follows from Menger's theorem [4, Theorem III.5.ii] that in the present case—with 0 and 1 the only possible values for $t(e)$—

$$(1.14) \qquad \Phi(k, l, m) = \text{maximal number of edge disjoint open paths on } \mathbf{Z}^3 \cap B(k, l, m) \text{ from } F_0 \text{ to } F_m.$$

Our Theorem 2.12 implies that for $d = 3$ and sufficiently large p and $k, l, m \to \infty$ such that $k \ge l$ and $k^{-1+\delta} \log m \to 0$ for some $\delta > 0$,

$$(1.15) \qquad (kl)^{-1}\{\text{maximal number of edge disjoint open paths on } \mathbf{Z}^3 \cap B(k, l, m) \text{ from } F_0 \text{ to } F_m\} \to \nu(B_p, 3) \quad \text{w.p.1}.$$

It is also known that $\Phi(k, k, k) = 0$ eventually, w.p.1 when p is sufficiently small. In fact, for bond-percolation on $\mathbf{Z}^d$ for any d, let

$$W = \text{open cluster of } \mathbf{0} = \text{collection of all edges belonging to an open path starting at } \mathbf{0},$$

and denote by $\#W$ the number of edges in W. One of the critical probabilities considered in percolation theory is

$$p_T = p_T(d) = \sup\{p : E_p\{\#W\} < \infty\}. \tag{1.16}$$

Then it is known ([12, Theorem 5.1] or [9, Theorem 2]) that for $p < p_T(d)$,

$$\begin{aligned} &P_p\{\text{there is any open path from } [0,k]^{d-1} \times \{0\} \text{ to} \\ &\quad [0,k]^{d-1} \times \{k\} \text{ in } [0,k]^d\} \to 0 \\ &\text{exponentially fast as } k \to \infty. \end{aligned} \tag{1.17}$$

One would hope that for $d = 3$, (1.15) holds as soon as $p > p_T(3)$, but we have not been able to prove this. (We note here that for $d = 2$ we do know the analogous result; $p_T(2) = \frac{1}{2}$ and for $p > \frac{1}{2}$,

$$\begin{aligned} k^{-1}\{&\text{maximal number of edge disjoint open paths on} \\ &\mathbf{Z}^2 \cap [0,k] \times [0,m] \text{ from } [0,k] \times \{0\} \text{ to } [0,k] \times \{m\}\} \\ &\to \mu(B_p, 2) > 0 \quad \text{w.p.1}, \end{aligned}$$

where $k, m \to \infty$ such that $k^{-1}\log m \to 0$; (see [12, p. 54] for $p_T(2) = \frac{1}{2}$, and [8, Cor. 4.1]).

Aizenman et al. [2] investigate among other things for $d = 3$ the behavior for large k, l of

$$\begin{aligned} P_p\{&\text{there exists a set of plaquettes } E^* \text{ such that } \partial E^* \text{ consists} \\ &\text{of the edges of the perimeter of } [-\tfrac{1}{2}, k + \tfrac{1}{2}] \\ &\times [-\tfrac{1}{2}, l + \tfrac{1}{2}] \times \{\tfrac{1}{2}\} \text{ and such that all edges } e \\ &\text{associated to } E^* \text{ are closed}\}. \end{aligned} \tag{1.18}$$

The above probability can be seen to equal $P_p\{\alpha(k, l) = 0\}$ when $F = B_p$ (see Remark 3.3 below). Thus [2] studies the probability of the existence of a separating set of zero value (in the case of Bernoulli distributions), whereas our interest here is more in the center of the distribution of the smallest value assigned to any separating set. Results on (1.18) are a special type of large deviation results for this distribution. In [2] it is shown that for p sufficiently large, $F = B_p$, and $d = 3$,

$$c(p) := \lim_{k,l\to\infty} \frac{1}{kl}\log P_p\{\alpha(k,l) = 0\} \text{ exists and is strictly positive.} \tag{1.19}$$

For $a_{0,k}$, $s_{0,k}$ and $t_{0,k}$ even more precise results are known (even for general F), namely

$$B(\varepsilon, F, d) = \lim_{k\to\infty} \frac{1}{k}\log P\{\theta_{0,k} \le k(\mu - \varepsilon)\} \tag{1.20}$$

exists and is strictly positive for $\varepsilon > 0$ and $\theta = a$, s or t (cf. [8], [13]). In the course of our proofs we also obtain some large deviation estimates for $P\{\alpha(k,l) \le kl(\nu - \varepsilon)\}$ but they are by no means the full analogue of (1.19), (1.20) (cf. Theorem 2.10).

Acknowledgement. The author is indebted to M. Cohen and M. Steinberger for help with some of the topological lemmas, and to R. Durrett for several helpful discussions about this paper in general.

2. Statement of results and open problems

We need a preliminary lemma to define the domain of validity of our results. This first lemma relies on a Peierls argument, and the restriction $F(0) < p_0$ in the sequel is akin to the restriction $F(0) < 1/\lambda$ which originally appeared in several theorems of [15]. Since Peierls arguments are rather crude one may venture that a much better understanding of percolation in dimension ≥ 3 will be needed to do away with the restriction on $F(0)$ (cf. Problem 2.23 below).

From now on we take $d = 3$.

(2.1) LEMMA. *There exists a $p_0 \ge 1/27$ with the following property: For every distribution function F with*

$$F(0-) = 0, \quad F(0) < p_0 \tag{2.2}$$

there exist constants $\Theta = \Theta(F) > 0$, $0 < C_i = C_i(F) < \infty$ such that

$$P\Big\{\textit{there exists a connected set } E^* \textit{ of } n \textit{ plaquettes of } \mathscr{L}^* \textit{ which contains the point } \left(-\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}\right) \textit{ and with } V(E^*) \le \Theta n\Big\} \le C_1 e^{-C_2 n}, \ n \ge 0, \tag{2.3}$$

and

$$P\{\text{there exists a cut } E^* \text{ over } [0,k]\times[0,l] \text{ which contains the point } (-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2}) \text{ and consists of at least } n \text{ plaquettes but has } V(E^*)\le \Theta n\} \le C_1 e^{-C_2 n},\ n\ge 0. \tag{2.4}$$

Next we define some analogues of $b_{0,k}$ and $s_{0,k}$. Recall that $\tau(k,l)$ and $\alpha(k,l)$ were defined in (1.11) and (1.12).

$$\sigma(k,l) := \{\inf V(E^*) : E^* \text{ a cut over } [0,k]\times[0,l] \text{ containing the point } (-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2}) \text{ and contained in } [-\tfrac{1}{2},k+\tfrac{1}{2}]\times[-\tfrac{1}{2},l+\tfrac{1}{2}]\times\mathbf{R}\}, \tag{2.5}$$

$$\beta(k,l) := \{\inf V(E^*) : E^* \text{ a connected set which separates } -\infty \text{ from } +\infty \text{ over } [0,k]\times[0,l] \text{ and which contains the point } (-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2})\}. \tag{2.6}$$

We point out that there is a certain arbitrariness in choosing $\sigma(k,l)$ and $\beta(k,l)$ as the analogues of $s_{0,k}$ and $b_{0,k}$. There is much less arbitrariness in the definitions of $\tau(k,l)$ and $\alpha(k,l)$ since, as pointed out in [2, Section 1(ii)], the analogues of $t_{0,k}$ and $a_{0,k}$ should be infima over cuts whose boundaries are completely described. However, $s_{0,k}$ and $b_{0,k}$ are defined as infima over paths of which only part of the boundary is fixed. Thus, for $\sigma(k,l)$ and $\beta(k,l)$ we should also fix only part of the boundary. In fact we chose the minimal restriction on E^*, namely that it contain the point $(-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$. One could, for instance, also define infima over all cuts E^* such that ∂E^* contains the edges on $\{-\frac{1}{2}\}\times[-\frac{1}{2},l+\frac{1}{2}]\times\{\frac{1}{2}\}$. But such infima would lie between $\alpha(k,l)$ and $\beta(k,l)$, or $\tau(k,l)$ and $\sigma(k,l)$, respectively. Thus, in a way, our theorem below is as extensive as possible, since other reasonable infima would be sandwiched between the ones dealt with here.

(2.7) THEOREM. *Assume F satisfies* (2.2) *or more generally*, $F(0-)=0$ *and* (2.3). *If in addition*,

$$Ee^{\gamma t(\pi^*)} = \int_{[0,\infty)} e^{\gamma x}\,dF(x) < \infty \tag{2.8}$$

for some $\gamma>0$, *then there exists a number* $\nu=\nu(F)\le\int_{[0,\infty)} x\,dF(x)<\infty$ *such that*

$$\lim_{k,l\to\infty}\frac{1}{kl}\theta(k,l)=\nu \quad \text{w.p.1} \quad \textit{and in } L^1 \tag{2.9}$$

for $\theta = \alpha, \beta, \sigma$ or τ. This ν is strictly positive.

(2.10) THEOREM. *Under the assumptions of Theorem 2.7 there exist for each $\delta > 0$, $\varepsilon > 0$ constants $0 < D_1(\varepsilon, \delta, F)$, $L = L(\varepsilon, \delta, F) < \infty$ such that for $\theta = \alpha, \beta, \sigma$ or τ, and ν as in* (2.9),

$$P\{\theta(k, l) \le kl(\nu - \varepsilon)\} \le D_1 \exp(-k^{1-\delta}),\ k \ge l \ge L. \tag{2.11}$$

(2.12) THEOREM. *If $m(k, l) \to \infty$ as $k \ge l \to \infty$ in such a way that for some $\delta > 0$,*

$$k^{-1+\delta} \log m(k, l) \to 0, \tag{2.13}$$

and if the hypotheses of Theorem 2.7 hold, then

$$\lim_{k, l \to \infty} \frac{1}{kl} \Phi(k, l, m) = \nu \quad \text{w.p.1}, \quad \textit{and in } L^1 \tag{2.14}$$

where Φ is $\Phi((k, l), m)$ in the notation of (1.6) *and ν is as in* (2.9).

(2.15) *Remark.* (2.9) and (2.11) for $\theta = \sigma$ or τ and (2.14) are valid for all F which satisfy $F(0-) = 0$, (2.4) and (2.8). The proof does not use (2.3) until (4.63), and there (2.4) would suffice if we restricted ourselves to $\theta = \sigma$ or τ. The replacement of (2.3) by (2.4) may be useful because (2.4) probably holds for a larger interval of $F(0)$-values than (2.3). (Compare also problem 2.23 below.)

(2.16) *Remark.* The estimates in Theorems 2.10 and 2.12 can be improved if one restricts oneself to k and l with k/l bounded away from 0 and ∞. For example in the course of the proof of (2.10) we show that

$$P\{\theta(k, k) \le k^2(\nu - \varepsilon)\} \le D_1 \exp(-k^{2-2\delta})$$

(use (4.55) and (4.63)). This implies that (2.14) with $l = k$ will hold as long as $k^{-2+2\delta} \log m(k, k) \to 0$ for any $\delta > 0$ (which is less restrictive than (2.13)).

(2.17) *Remark.* Condition (2.13) may appear somewhat strange. We remind the reader, though, that the two-dimensional analogue of (2.14), namely (1.7), was only proved under the condition $k^{-1} \log m(k) \to 0$, and that the latter condition is sharp by [8, Theorem 5.2]. Thus, we would hope that the condition (2.13) could be weakened to $(kl)^{-1} \log m(k, l) \to 0$. We could not prove this, but in any case (2.13) allows arbitrary polynomial growth in k for m.

(2.18) THEOREM. *Assume that*

$$(2.19)\quad F(0-)=0,\quad F(0)>1-p_T(3)\quad \textit{and}\quad \int_{[0,\infty)} x^6\,dF(x)<\infty.$$

Then

$$(2.20)\qquad \limsup_{k,l\to\infty}\frac{1}{k+l}\theta(k,l)<\infty \quad \text{w.p.1}$$

for $\theta=\alpha$, β, σ *and* τ. *A fortiori*, (2.9) *holds with* $\nu=0$.

Furthermore, there exists a constant $C_3=C_3(F)<\infty$ *such that* w.p.1,

$$(2.21)\qquad \Phi(k,l,m)=0 \quad \textit{for all sufficiently large } k,l,$$

whenever $m(k,l)\to\infty$ *as* $k,l\to\infty$ *in such a way that*

$$(2.22)\qquad \liminf_{k,l\to\infty}\frac{m(k,l)}{\log(kl)}>C_3.$$

Almost immediately, the above results raise a number of *problems*. We mention the most obvious ones.

(2.23) Prove analogous results for $p_0\le F(0)\le 1-p_T(3)$. The boldest conjecture would be that (2.9) holds, irrespective of the value of $F(0)$, but that $\nu>0$ if and only if $F(0)<1-p_T(3)$. This is suggested by two-dimensional results, if one believes that "there is only one critical probability" for various phenomena in bond percolation (compare also [2, Section 5]). Probably the situation with $F(0)=1-p_T(3)$ will be most difficult.

The fact that our approach does not seem to work for all $F(0)$ is disappointing for resistance estimations. Let R_n be the resistance between two opposite faces of the cube $[0,n]^3$ when the resistances of the edges of $\mathbf{Z}^3$ are chosen independently equal to 1 or ∞ with probability $1-F(0)$ and $F(0)$, respectively. As one can check from the proofs in [8, Section 5] or [12, pp. 372, 373], (2.14) with $\nu>0$ for this situation would imply $\limsup nR_n<\infty$ w.p.1. This is here obtained for $F(0)<p_0$. In this respect the results here are poor in comparison to [12], Theorem 11.3, which already shows that $\limsup nR_n<\infty$ for $F(0)<\frac{1}{2}$. J. Chayes and L. Chayes (Comm. Math. Physics, vol. 105 (1986), pp. 133–152) have even proved this for a still larger interval of $F(0)$-values.

(2.24) Prove results under weaker moment conditions than (2.8). Most first-passage percolation results require only very weak moment conditions, if any (see [5], [8], [13]). $Et^2(e)<\infty$ should suffice for most results.

(2.25) Does $\lim_{k,l\to\infty}\frac{1}{kl}\log P\{\theta(k,l) < kl(\nu - \varepsilon)\}$ exist and is it strictly negative for $\varepsilon > 0$?

(2.26) Obtain similar results for $d > 3$ and/or when the plaquettes are replaced by cells of dimension greater than 2.

3. Topological preliminaries

The reader is advised to skip this section at first reading and to refer to it only when the need arises. We discuss here various properties of cuts E^* and their boundaries ∂E^*. These properties are all of a topological nature and no probability is involved.

(3.1) LEMMA. (a) *If E^* is a collection of plaquettes such that ∂E^* consists of the edges on the perimeter of $[-\frac{1}{2}, k+\frac{1}{2}] \times [-\frac{1}{2}, l+\frac{1}{2}] \times \{\frac{1}{2}\}$, then E^* separates $-\infty$ from $+\infty$ over $[0,k] \times [0,l]$.*
(b) *If E^* is a cut over $[0,k] \times [0,l]$, then the interiors of all plaquettes in E^* are contained in $(-\frac{1}{2}, k+\frac{1}{2}) \times (-\frac{1}{2}, l+\frac{1}{2}) \times \mathbf{R}$.*

Proof. (a) Assume that E^* does not separate $-\infty$ from $+\infty$ over $[0,k] \times [0,l]$. Then there exists a path on $\mathbf{Z}^3$ from $-\infty$ to $+\infty$ in $[0,k] \times [0,l] \times \mathbf{R}$ which does not intersect any plaquette of E^*. Let D^* be the collection of plaquettes

$$\left[j_1 - \tfrac{1}{2}, j_1 + \tfrac{1}{2}\right] \times \left[j_2 - \tfrac{1}{2}, j_2 + \tfrac{1}{2}\right] \times \left\{\tfrac{1}{2}\right\} \quad \text{with } 0 \le j_1 \le k, 0 \le j_2 \le l.$$

The union of these plaquettes is the intersection of the hyperplane $x(3) = \frac{1}{2}$ with $[-\frac{1}{2}, k+\frac{1}{2}] \times [-\frac{1}{2}, l+\frac{1}{2}] \times \mathbf{R}$ and ∂D^* consists of the edges on the perimeter of

$$\left[-\tfrac{1}{2}, k+\tfrac{1}{2}\right] \times \left[-\tfrac{1}{2}, l+\tfrac{1}{2}\right] \times \left\{\tfrac{1}{2}\right\}.$$

Clearly ϕ, which starts below D^* and ends above D^*, must intersect D^* an odd number of times. However, it follows from $\partial E^* = \partial D^*$ that

$$\begin{aligned}(3.2)\quad &(\text{number of intersections of } \phi \text{ with } D^*)\\ &\quad - (\text{number of intersections of } \phi \text{ with } E^*) \text{ is an even integer.}\end{aligned}$$

This follows from general topological considerations (see also [2, Prop. 2.1]); for completeness we give a simple standard proof. Let ϕ run along the perimeter of some unit square π (with "corners" on $\mathbf{Z}^3$) from the vertex v to

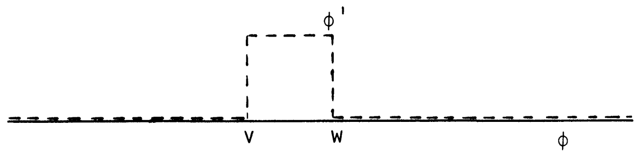

FIG. 3 The curve ϕ and a perturbation ϕ' (dashed).

the vertex w of $\mathbf{Z}^3$, and let ϕ' be obtained from ϕ by replacing the arc from v to w by the other arc of the perimeter of π between π and w (see Fig. 3). Let e^* be the edge of $\mathscr{L}^*$ which intersects π in its midpoint. Since $\partial D^* = \partial E^*$, the number of plaquettes in D^* containing e^* differs by an even integer from the number of plaquettes in E^* containing e^*. One easily obtains from this that

$$\begin{aligned}&(\text{number of intersections of } \phi \text{ with } D^*)\\ &\qquad -(\text{number of intersections, of } \phi \text{ with } E^*)\\ &\qquad = \text{even integer} + (\text{number of intersections of } \phi' \text{ with } D^*)\\ &\qquad -(\text{number of intersections of } \phi' \text{ with } E^*).\end{aligned}$$

Thus, perturbing ϕ to ϕ' only changes the number of intersections with D^* minus the number of intersections with E^* by an even integer. Since we can change ϕ by a number of such perturbations to a curve which intersects neither D^* nor E^*, (3.2) follows.

On the other hand (3.2) cannot hold in our situation since ϕ intersects D^* an odd number of times, and by construction, ϕ is disjoint from E^*. It follows from this contradiction that E^* must separate $-\infty$ from $+\infty$ over $[0, k] \times [0, l]$.

(b) Let E be the collection of edges of $\mathbf{Z}^3$ associated with the plaquettes of E^*. If E^* is a cut over $[0, k] \times [0, l]$, then by definition E is a minimal set of edges separating $-\infty$ from $+\infty$ in $[0, k] \times [0, l] \times \mathbf{R}$. Clearly such a minimal set contains no edges outside $[0, k] \times [0, l] \times \mathbf{R}$. From this (b) follows immediately.

(3.3) *Remark.* 3.1(a) also shows that (1.18) equals $P\{\alpha(k, l) = 0\}$ when $F = B_p$. Indeed the sets E^* appearing in (1.18) must separate $-\infty$ from $+\infty$ over $[0, k] \times [0, l]$. It also shows that the requirement that E^* separates $-\infty$ from $+\infty$ over $[0, k] \times [0, l]$ can be dropped in the definition (1.12) of $\alpha(k, l)$. ■

In the next four lemmas, $R = [0, k] \times [0, l]$ and E is an (F_0, F_m)-cut in $B = R \times [0, m]$ ($F_0 = R \times \{0\}$, $F_m = R \times \{m\}$) and E^* is the set of plaquettes associated with the edges of E. Furthermore we define

$$\Delta B = \text{“vertical part of the boundary” of } \left[-\tfrac{1}{2}, k+\tfrac{1}{2}\right] \times \left[-\tfrac{1}{2}, l+\tfrac{1}{2}\right] \times [0, m].$$

ΔB consists of the four rectangles parallel to the third coordinate axis, obtained by replacing $[\frac{1}{2}, m - \frac{1}{2}]$ by $[0, m]$ in (1.9).

(3.4) LEMMA. *Let B, ΔB, F_0, F_m and E be as above.*
(a) *E contains no edges in F_0 or F_m so that E^* is contained in*

$$\left[-\tfrac{1}{2}, k+\tfrac{1}{2}\right] \times \left[-\tfrac{1}{2}, l+\tfrac{1}{2}\right] \times \left[\tfrac{1}{2}, m-\tfrac{1}{2}\right].$$

E^* contains no plaquettes in ΔB.
(b) *Let K_- (K_+) be the set of edges and vertices of $\mathbf{Z}^3$ which lie on a path on $\mathbf{Z}^3$ in B, which contains no edges in E and which starts at some point of F_0 (F_m). Then, when viewed as subsets of $\mathbf{R}^3$, K_- and K_+ are closed, connected and disjoint. Moreover each edge of $\mathbf{Z}^3$ in B which does not belong to E lies in K_- or K_+. Finally $F_0 \subset K_-$, $F_m \subset K_+$.*
(c) *E consists precisely of those edges of $\mathbf{Z}^3$ in B which have one endpoint in K_- and the other endpoint in K_+.*

Proof. Part (a) again follows from the minimality of E; E contains no edges of F_0 or F_m or outside B (compare also the proof of (3.1b)).

As for (b) and (c), K_- and K_+ are closed and connected, and $F_0 \subset K_-$, $F_m \subset K_+$, all by definition. Moreover, K_- and K_+ must be disjoint, since otherwise F_0 and F_m can be connected by a path in $K_- \cup K_+$. Such a path would contain no edge in E, contradicting the assumption that E separates F_0 from F_m in B. Thus $K_- \cap K_+ = \varnothing$. The remainder of (b) and (c) follows easily from Theorem 5 of [17]. ■

We maintain the notation of the last lemma. In addition we introduce the *lower edge* and *upper edge* of ΔB. These are defined as

$$\Delta_- := \text{perimeter of } \left[-\tfrac{1}{2}, k+\tfrac{1}{2}\right] \times \left[-\tfrac{1}{2}, l+\tfrac{1}{2}\right] \times \{0\}$$

and

$$\Delta_+ := \text{perimeter of } \left[-\tfrac{1}{2}, k+\tfrac{1}{2}\right] \times \left[-\tfrac{1}{2}, l+\tfrac{1}{2}\right] \times \{m\},$$

respectively.

Our aim is to decompose ∂E^* into a number of circuits on $\Delta B \setminus (\Delta_- \cup \Delta_+)$. To do this we must also discuss separation properties of such paths. First, more definitions. A *circuit* on $\mathscr{L}^*$ is a sequence

$$(v_0^*, e_1^*, \dots, e_n^*, v_n^*)$$

with v_i^* and e_i^* vertices and edges, respectively, of $\mathscr{L}^*$ such that e_i^* is the edge between v_{i-1}^* and v_i^*, $1 \le i \le n$, and $v_i^* \ne v_j^*$, for $i \ne j$ with the single exception $v_0^* = v_n^*$. To discuss the separation properties of such circuits on ΔB we introduce a number of graphs. $\mathscr{G}^*$ is the restriction of $\mathscr{L}^*$ to ΔB. $\mathscr{G}_0$ is the graph whose vertices are the points of the form

$$\left(z_1 + \tfrac{1}{2}, z_2, z_3\right) \quad \text{or} \quad \left(z_1, z_2 + \tfrac{1}{2}, z_3\right)$$

with $z_i \in \mathbf{Z}$, $i = 1, 2, 3$, which lie in ΔB. Two points

$$\left(z_1' + \tfrac{1}{2}, z_2', z_3'\right) \quad \text{and} \quad \left(z_1'' + \tfrac{1}{2}, z_i'', z_3''\right)$$

are adjacent on $\mathscr{G}_0$ if and only if

$$|z_1' - z_1''| + |z_2' - z_2''| + |z_3' - z_3''| = 1,$$

and similarly for two points $(z_1', z_2' + \frac{1}{2}, z_3')$ and $(z_1'', z_2'' + \frac{1}{2}, z_3'')$. Also the following pairs are adjacent on $\mathscr{G}_0$:

$$\begin{array}{llll} (3.5) \qquad & (-\tfrac{1}{2}, 0, z_3) & \text{and} & (0, -\tfrac{1}{2}, z_3), \\ & (k + \tfrac{1}{2}, 0, z_3) & \text{and} & (k, -\tfrac{1}{2}, z_3), \\ & (-\tfrac{1}{2}, l, z_3) & \text{and} & (0, l + \tfrac{1}{2}, z_3), \\ & (k + \tfrac{1}{2}, l, z_3) & \text{and} & (k, l + \tfrac{1}{2}, z_3). \end{array}$$

Part of $\mathscr{G}_0$ is drawn in Fig. 4. Finally $\mathscr{G}$ is obtained from $\mathscr{G}_0$ by identifying all vertices of $\mathscr{G}_0$ on Δ_- as one vertex v_-, and identifying all vertices of $\mathscr{G}_0$ on Δ_+ as another vertex Δ_+, see Fig. 5.

$\mathscr{G}^*$ and $\mathscr{G}$ are planar graphs, since ΔB is homeomorphic to part of sphere. Moreover $\mathscr{G}$ has one vertex to each face of $\mathscr{G}^*$, and each edge of $\mathscr{G}$ intersects exactly one edge of $\mathscr{G}^*$ and vice versa (cf. Fig. 5). Therefore $\mathscr{G}$ and $\mathscr{G}^*$ are dual to each other in the sense of [16] (see proof of Theorem 29). Therefore, by Theorems 4 and 5 of [17], the minimal sets $\mathscr{C}$ of edges of $\mathscr{G}$ which separate $\mathscr{G}$ into two components, are precisely the sets for which the set $\mathscr{C}^*$ of associated edges of $\mathscr{G}^*$ forms a circuit on $\mathscr{G}^*$. Notice that there is also a 1-1 correspondence between the edges of $\mathscr{G}^*$ and the edges of $\mathscr{G}_0$ which do not lie in $\Delta_+ \cup \Delta_-$. Again one edge of the former type intersects a unique one of the latter type, and vice versa. It follows from this that if $\mathscr{C}^*$ is a circuit on $\mathscr{G}^*$, then the collection $\mathscr{C}_0$ of edges of $\mathscr{G}_0$ which intersect some edge in $\mathscr{C}^*$ is a minimal set separating $\mathscr{G}_0$ into two components. $\mathscr{C}_0$ will not contain any

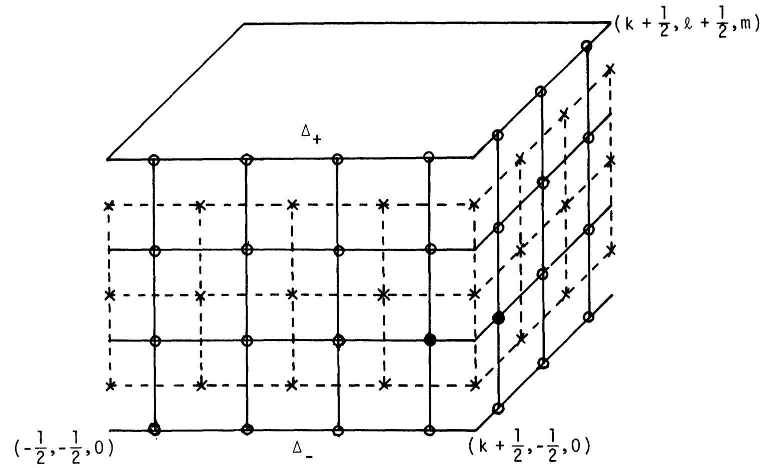

FIG. 4 The parts of $\mathscr{G}^*$ and $\mathscr{G}_0$ on the "front and right face of ΔB". The edges of $\mathscr{G}^*$ are dashed, the edges of $\mathscr{G}_0$ are drawn as solid curves. the vertices of $\mathscr{G}^*$ and $\mathscr{G}_0$ are indicated by $\times$ and $\circ$ respectively. The two vertices marked with a solid dot are adjacent on $\mathscr{G}_0$.

edge in $\Delta_+ \cup \Delta_-$, so that Δ_+ and Δ_- each belong entirely to one component of $\mathscr{G}_0 \setminus \mathscr{C}_0$. Thus each circuit $\mathscr{C}^*$ of $\mathscr{G}^*$ must belong to one of the two classes which we now define. We say that a circuit $\mathscr{C}^*$ of $\mathscr{G}^*$ is of *class I* (*class II*) if Δ_+ and Δ_- lie in the same component (in different components) of $\mathscr{G}_0 \setminus \mathscr{C}_0$.

The above discussion also shows that any set F^* of edges of $\mathscr{G}^*$ which separates Δ_- from Δ_+ in ΔB must contain a circuit of class II. Indeed, if F is the set of edges of $\mathscr{G}$ which intersect some edge of F^*, then F does not contain any edge in $\Delta_+ \cup \Delta_-$, and therefore also separates v_- from v_+ in $\mathscr{G}$.

A circuit $\mathscr{C}^*$ of $\mathscr{G}^*$ can also be viewed in an obvious way as a Jordan curve (i.e., a simple closed curve) on ΔB. In fact $\mathscr{C}^*$ must lie in $\Delta B \setminus (\Delta_+ \cup \Delta_-)$, because the restriction of $\mathscr{L}^*$ to ΔB lies in $\mathbf{R}^2 \times [\frac{1}{2}, m - \frac{1}{2}]$. Thus $\mathscr{C}^*$ divides ΔB into two path components. If $\mathscr{C}_0$ corresponds to $\mathscr{C}^*$ as above, and v, w belong to the same component of $\mathscr{G}_0 \setminus \mathscr{C}_0$, then v and w can be connected by a path on $\mathscr{G}_0$ which contains no edge from $\mathscr{C}_0$, and hence does not intersect $\mathscr{C}^*$. Therefore the components of $\mathscr{G}_0 \setminus \mathscr{C}_0$ each lie in one path component of $\Delta B \setminus \mathscr{C}^*$. Conversely, with a bit more work (using an argument similar to [12], pp. 410, 411) one can see that if v and w are two vertices of $\mathscr{G}_0$ which are connected by a path ϕ in $\Delta B \setminus \mathscr{C}^*$, then one can deform ϕ to a path on $\mathscr{G}_0$ from v to w which does not intersect $\mathscr{C}^*$. Consequently all vertices of $\mathscr{G}_0$ in one path component of $\Delta B \setminus \mathscr{C}^*$ belong to the same component of $\mathscr{G}_0 \setminus \mathscr{C}_0$, and each path component of $\Delta B \setminus \mathscr{C}^*$ contains exactly one component of $\mathscr{G}_0 \setminus \mathscr{C}_0$. We introduce the following notation for the path components of $\Delta B \setminus \mathscr{C}^*$. If $\mathscr{C}^*$ is of class I, then the *exterior of* $\mathscr{C}^*$ (*denoted* $\mathscr{C}^*$(ext)) is the path component of $\Delta B \setminus \mathscr{C}^*$ which contains $\Delta_+ \cup \Delta_-$, and the other path

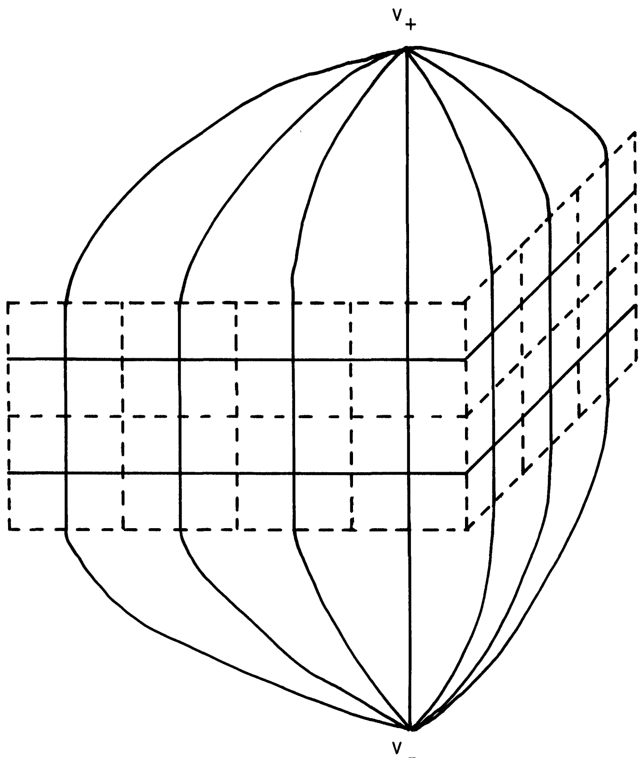

FIG. 5 Part of $\mathscr{G}$, obtained by collapsing the vertices of $\mathscr{G}_0$ on Δ_+ (Δ_-) to one vertex v_+ (v_-).

component of $\Delta B \setminus \mathscr{C}^*$ is called the *interior of* $\mathscr{C}^*$ (*denoted* $\mathscr{C}^*$(int)). If $\mathscr{C}^*$ is of class II, then we denote by $\mathscr{C}_+^*$ ($\mathscr{C}_-^*$) the path component of $\Delta B \setminus \mathscr{C}^*$ which contains Δ_+ (Δ_-). (See Fig. 6 for some examples.) Finally, $\overline{\mathscr{C}^*\text{(int)}} = \mathscr{C}^*\text{(int)} \cup \mathscr{C}^*$, and similarly for $\overline{\mathscr{C}^*\text{(ext)}}, \overline{\mathscr{C}_+^*}, \overline{\mathscr{C}_-^*}$.

(3.6) LEMMA. (a) *$\partial E^* \subset \Delta B \cap (\mathbf{R}^2 \times [\frac{1}{2}, m - \frac{1}{2}])$. An edge e^* of $\mathscr{L}^*$ in ΔB belongs to ∂E^* if and only if e^* is an edge of a "boundary plaquette", i.e., a plaquette of the form*

$$[i - \tfrac{1}{2}, i + \tfrac{1}{2}] \times [j - \tfrac{1}{2}, j + \tfrac{1}{2}] \times \{p + \tfrac{1}{2}\}$$

with $i = 0$ or k or $j = 0$ or l, $0 \le p \le m - 1$, or

$$[i - \tfrac{1}{2}, i + \tfrac{1}{2}] \times \{j + \tfrac{1}{2}\} \times [p - \tfrac{1}{2}, p + \tfrac{1}{2}]$$

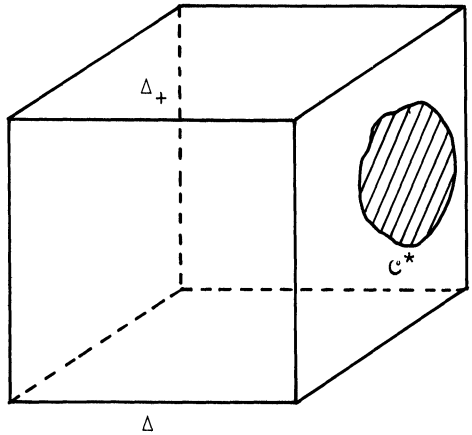

FIG. 6I A circuit $\mathscr{C}^*$ of class I (in the "right" face) of ΔB. The hatched region is $\mathscr{C}^*(\text{int})$.

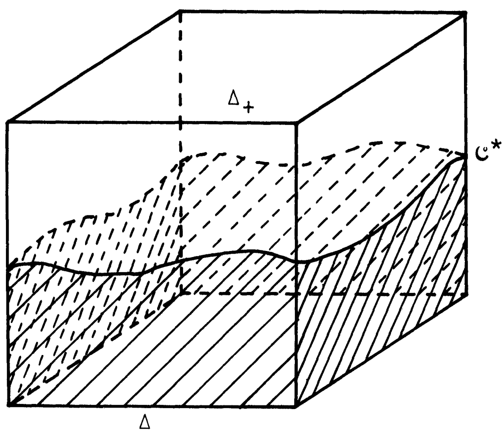

FIG. 6II A circuit $\mathscr{C}^*$ of class II. The hatched part of the vertical boundary is $\mathscr{C}^*_-$.

with $i = 0$ *or* k, $0 \le j \le l - 1$, $1 \le p \le m - 1$ *or*

$$\{i + \tfrac{1}{2}\} \times [j - \tfrac{1}{2}, j + \tfrac{1}{2}] \times [p - \tfrac{1}{2}, p + \tfrac{1}{2}]$$

with $j = 0$ *or* l, $0 \le i \le k - 1$, $j = 0$ *or* l, $1 \le p \le m - 1$.

(b) *If* v^* *is a vertex of* $\mathscr{L}^*$ *in* ΔB *then there is always an even number of edges of* ∂E^* *incident to* v^*.

(c) $(\partial E^*)_0$, *defined as the collection of edges of* $\mathscr{G}_0$ *which intersect* ∂E^*, *separates* Δ_+ *from* Δ_- *in* $\mathscr{G}_0$. *Consequently* ∂E^* *contains at least one circuit of class II.*

Proof. (a) Since $E^* \subset [-\frac{1}{2}, k+\frac{1}{2}] \times [-\frac{1}{2}, l+\frac{1}{2}] \times [\frac{1}{2}, m-\frac{1}{2}]$ by (3.4a), also

$$\partial E^* \subset \mathbf{R}^2 \times \left[\tfrac{1}{2}, m - \tfrac{1}{2}\right].$$

Note also that once we prove the first statement, the second statement in (a) follows immediately, because E^* contains no plaquettes in ΔB. Therefore we only have to prove that $\partial E^* \subset \Delta B$, or equivalently, that ∂E^* cannot contain any edge e^* whose interior lies in

$$\left(-\tfrac{1}{2}, k+\tfrac{1}{2}\right) \times \left(-\tfrac{1}{2}, l+\tfrac{1}{2}\right) \times \left[\tfrac{1}{2}, m-\tfrac{1}{2}\right].$$

We prove this by giving an alternative description of E^*. Let U be the closed unit cube with center at the origin:

$$U = \left\{x \in \mathbf{R}^3 : -\tfrac{1}{2} \le x(i) \le \tfrac{1}{2}, i = 1,2,3\right\}. \tag{3.7}$$

Note that the faces of the cubes of the form $v + U$, $v \in \mathbf{Z}^3$, are the plaquettes. We call $v + U$ a $+$ cube ($-$ cube) if $v \in K_+$ (K_-) (see (3.4b) for $K_\pm$), and set

$$\hat{K}_+ = \bigcup_{v \in K_+} (v + U), \quad \hat{K}_- = \bigcup_{v \in K_-} (v + U). \tag{3.8}$$

We claim that

(3.9) E^* is the collection of plaquettes π^* which are a face of a $-$ cube and of a $+$ cube.

(3.9) is immediate from (3.4c). Indeed, if e is the edge of $\mathbf{Z}^3$ associated to a plaquette π^*, then π^* belongs to E^* if and only if $e \in E$, and this holds if and only if there exist vertices $v \in K_-$, $w \in K_+$ such that e is the edge between v and w (by (3.4c)).

We return to the proof of (a), by means of (3.9). Assume e^* is an edge of $\mathscr{L}^*$ whose interior lies in

$$\left(-\tfrac{1}{2}, k+\tfrac{1}{2}\right) \times \left(-\tfrac{1}{2}, l+\tfrac{1}{2}\right) \times \left[\tfrac{1}{2}, m-\tfrac{1}{2}\right].$$

Then e^* is an edge of four cubes $v_i + U$ with v_i a vertex of $\mathbf{Z}^3$ in B. We can number these in such a way that $v_i + U$ and $v_{i+1} + U$ have as common face a plaquette, π_i^* say, which contains e^*, $1 \le i \le 4$ ($v_5 = v_1$). Now $e^* \in \partial E^*$ if and only if the number of π_i^*, $1 \le i \le 4$, which belong to E^* is odd. However, a simple examination of cases shows that this number is always even

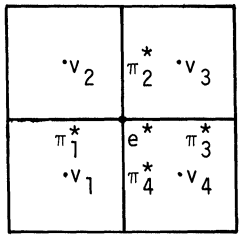

FIG. 7 Projection of the $(v_i + U)$ and the π_i^* in the direction of e^*; e^* itself projects onto the point at the center. No matter how we partition the v_i among K_- and K_+, an even number of the π_i^* will lie between a $-$ cube and a $+$ cube.

(see Fig. 7), so that the edge e^* under consideration cannot belong to ∂E^*. This proves (a).

(b) This also follows from the argument which we just completed.

(c) This statement is a reflection of the fact that E^* separates F_0 from F_m. To prove (c) let ϕ be a path on $\mathscr{G}_0$ from Δ_- to Δ_+. We shall map ϕ to a path ψ on $\mathbf{Z}^3$ in B from F_0 to F_m, by "pushing ϕ inwards". ψ must intersect E^* and this will imply that ϕ intersects ∂E^*, thereby showing that ∂E^* has the required separation property.

To describe how ϕ is "pushed inwards" we merely have to say where an edge f of $\mathscr{G}_0$ is mapped to. Roughly speaking, when f lies entirely in one face of ΔB, then it is mapped to the nearest edge of $\mathbf{Z}^3$ on the boundary of B which is parallel to f. Thus, if f runs from $(z_1, -\frac{1}{2}, z_3)$ to $(z_1 + 1, -\frac{1}{2}, z_3)$, then its image is the edge from $(z_1, 0, z_3)$ to $(z_1 + 1, 0, z_3)$. Similarly the edge from $(z_1, l + \frac{1}{2}, z_3)$ to $(z_1 + 1, l + \frac{1}{2}, z_3)$ is mapped to the edge from (z_1, l, z_3) to $(z_1 + 1, l, z_3)$. Similarly for vertical edges or edges in the other faces of ΔB. Only the edges between any of the pairs in (3.5) are special. In fact the edge f_1 from $(-\frac{1}{2}, 0, z_3)$ to $(-\frac{1}{2}, 0, z_3 + 1)$ is mapped to the edge e from $(0, 0, z_3)$ to $(0, 0, z_3 + 1)$. The edge f_2 from $(0, -\frac{1}{2}, z_3)$ to $(0, -\frac{1}{2}, z_3 + 1)$ has the same image. Accordingly, the whole edge between the first pair in (3.5) should be mapped to the single point $(0, 0, z_3)$. The edges between the other pairs in (3.5) are similarly mapped to one point in the boundary of B. It is not hard to see that this "pushing in transformation" does indeed take ϕ to a path ψ on $\mathbf{Z}^3$ from F_0 to F_m. ψ must intersect some plaquette π^* of E^*. Since ψ lies on the boundary of B, π^* must be a "boundary plaquette" as described in part (a). π^* has either one edge e_1^*, or two edges e_1^* and e_2^* in ΔB. By (a) this edge (or these edges, respectively) belong to ∂E^*, and one easily checks that ϕ must intersect e_1^* (or $e_1^* \cup e_2^*$, respectively). This proves that ∂E^* separates Δ_+ from Δ_-. The rest follows from Whitney's theorem as discussed before Lemma 3.6, because $(\partial E^*)_0$ must contain a minimal set which separates Δ_- from Δ_+. ∎

The next lemma describes the structure of ∂E^* in greater detail. Fig. 8 illustrates this rather lengthy description.

(3.10) LEMMA. *∂E^* can be decomposed into a finite number of circuits $\mathscr{C}_1^*, \ldots, \mathscr{C}_\rho^*$ of $\mathscr{L}^*$ on $\Delta B \setminus (\Delta_+ \cup \Delta_-)$. These circuits can be chosen in such a way that the following properties hold*:

(3.11) *Any two circuits $\mathscr{C}_i^*, \mathscr{C}_j^*$ with $i \neq j$ have no edges in common.*

(3.12) *There is an odd number of circuits of class II; these can be numbered as $\mathscr{C}_1^*, \ldots, \mathscr{C}_\tau^*$ (τ odd) such that $\mathscr{C}_j^* \subset \overline{\mathscr{C}_{i-}^*}$ and $\mathscr{C}_i^* \subset \overline{\mathscr{C}_{j+}}$ if $i < j$. In words $\mathscr{C}_i^*$ lies "above" $\mathscr{C}_j^*$ if $i < j$.*

(3.13) *If $\mathscr{C}_k^*$ is a circuit of class I, then it "lies between" two successive circuits of class II, i.e., for some $0 \le i \le \tau$,*

$$\mathscr{C}_k^* \subset \overline{\mathscr{C}_{i-}^*} \cap \overline{\mathscr{C}_{(i+1)+}^*}.$$

(*Here we interpret $\mathscr{C}_0^*$ as Δ_+ and $\mathscr{C}_{\tau+1}^*$ as Δ_-.*)

(3.14) *If $\mathscr{C}_i^*$ and $\mathscr{C}_j^*$ are both of class I, $i \neq j$, then either*

$$\mathscr{C}_j^* \subset \overline{\mathscr{C}_i^* (\text{int})} \quad or \quad \mathscr{C}_i^* \subset \overline{\mathscr{C}_j^*(\text{int})}$$

or both

$$\mathscr{C}_j^* \subset \overline{\mathscr{C}_i^*(\text{ext})} \quad and \quad \mathscr{C}_i^* \subset \overline{\mathscr{C}_j^*(\text{ext})}\,.$$

(3.15) *If $1 \le i \le \tau$, i odd, then all plaquettes π^* in ΔB which contain an edge of $\mathscr{C}_i^*$ and with interior in $\mathscr{C}_{i-}^*$ ($\mathscr{C}_{i+}^*$) are faces of a $-$ cube ($+$ cube). For i even the plaquettes adjacent to $\mathscr{C}_i^*$ in $\mathscr{C}_{i-}^*$ ($\mathscr{C}_{i+}^*$) are faces of $+$ cubes ($-$ cubes).*

(Note that despite the notation $\mathscr{C}_{i-}^*$ may contain faces of $+$ cubes.)

Proof. By (3.6b) and Euler's theorem (cf. [4], Theorem 1.10) one can decompose ∂E^* into a number of edge disjoint circuits. However, we have to choose these circuits with some care to guarantee (3.11)–(3.15). By (3.6c) ∂E^* contains at least one circuit of class II. Exactly as in [11], Lemma 1, we can

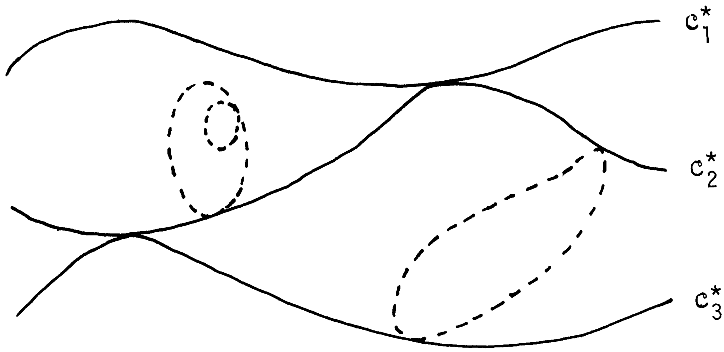

FIG. 8 A typical configuration of circuits in ∂E^*. The circuits of class I are dashed and those of class II are solidly drawn.

find a "highest" circuit $\mathscr{C}_1^*$ among all such circuits. That is we can choose $\mathscr{C}_1^*$ such that for any other circuit $\mathscr{C}^*$ of class II in ∂E^* $\mathscr{C}^* \subset \overline{\mathscr{C}_{1-}^*}$ and $\mathscr{C}_1^* \subset \overline{\mathscr{C}_+^*}$. We claim that once $\mathscr{C}_1^*$ has been chosen in this way, any other circuit $\mathscr{C}^*$ in ∂E^*, which is edge disjoint from $\mathscr{C}_1^*$ must lie entirely in $\overline{\mathscr{C}_{1-}^*}$ or entirely in $\overline{\mathscr{C}_{1+}^*}$. This is true for $\mathscr{C}^*$ of class I as well as of class II, and follows from the proof of Lemma 1 in [11]. Indeed if $\mathscr{C}^*$ contains a point x in $\mathscr{C}_{1-}^*$ and a point y in $\mathscr{C}_{1+}^*$, then $\mathscr{C}^*$ must contain a whole arc $\mathscr{A}$ in $\mathscr{C}_{1+}^*$ from some vertex v^* on $\mathscr{C}_1^*$ to some vertex $w^* \neq v^*$, also on $\mathscr{C}_1^*$. By replacing the arc from v to w of $\mathscr{C}_1^*$ by $\mathscr{A}$ we would obtain a circuit of class II in ∂E^*, which lies above $\mathscr{C}_1^*$, contradicting the choice of $\mathscr{C}_1^*$ as the highest circuit of class II in ∂E^*. This proves our claim.

Now remove $\mathscr{C}_1^*$ and consider any vertex v^* of $\mathscr{L}^*$ in $\overline{\mathscr{C}_{1+}^*} = \mathscr{C}_{1+}^* \cup \mathscr{C}_1^*$. If $v^* \in \mathscr{C}_{1+}^*$, then there are still an even number of edges of $\partial E^* \setminus \mathscr{C}_1^*$ incident to v^*, since no edges incident to v^* were removed. If $v^* \in \mathscr{C}_1^*$ then two edges of $\mathscr{C}_1^*$ incident to v^* were removed, so that there is still an even number of edges of $\partial E^* \setminus \mathscr{C}_1^*$ incident to v^*. Thus the edges of $\partial E^* \setminus \mathscr{C}_1^*$ in $\overline{\mathscr{C}_{1+}^*}$ can be decomposed into a number of edge disjoint circuits. All of these must belong to class I, since $\mathscr{C}_1^*$ was the highest of class II. We want to choose these circuits such that they satisfy (3.14). In the rest of this paragraph we only use edges in $\partial E^* \setminus \mathscr{C}_1^*$ which lie in $\overline{\mathscr{C}_{1+}^*}$. If there are such edges, we first choose a circuit $\mathscr{C}_\kappa^*$ say, such that there is no other circuit $\mathscr{C}^*$ with $\mathscr{C}_\kappa^* \subset \overline{\mathscr{C}^*}(\text{int})$, nor another circuit $\mathscr{C}^*$ which is edge disjoint from $\mathscr{C}_\kappa^*$ and contains points in $\mathscr{C}_\kappa^*(\text{int})$ as well as points in $\mathscr{C}_\kappa^*(\text{ext})$. Such a "maximal circuit" can be constructed by imitating the proof of [11], Lemma 1. If $\mathscr{C}_\kappa^*$ and $\mathscr{C}^*$ are located as in Fig. 9, then we can construct a larger circuit. Of course if $\mathscr{C}_\kappa^* \subset \overline{\mathscr{C}^*(\text{int})}$, then we simply replace $\mathscr{C}_\kappa^*$ by $\mathscr{C}^*$. After a finite number of such steps we end up with a $\mathscr{C}_\kappa^*$ with the above property. We can then remove the edges of $\mathscr{C}_\kappa^*$, and repeat the argument separately with the

FIG. 9 In (a) the circuit of short dashes is a candidate for $\mathscr{C}_\kappa^*$ and the circuit of long dashes represents $\mathscr{C}^*$. Then $\mathscr{C}_\kappa^*$ should be replaced by the larger, solidly drawn circuit of (b).

edges of $\partial E^* \setminus \mathscr{C}_1^* \cup \mathscr{C}_\kappa^*$ in $\overline{\mathscr{C}_\kappa^*(\text{ext})}$ and the edges in $\overline{\mathscr{C}_\kappa^*(\text{int})}$. We continue in this way until we exhausted all edges in $\overline{\mathscr{C}_{1+}^*}$. The resulting set of circuits, $\mathscr{C}_1^*$ and the circuits in $\overline{\mathscr{C}_{1+}^*}$ then satisfy (3.13) and (3.14).

We now proceed by induction. We remove all edges of $\mathscr{C}_1^*$ and of the circuits of class I in $\overline{\mathscr{C}_{1+}^*}$ which we just constructed. From the remaining edges of ∂E^* we construct the next highest circuit of class II, if one exists. Assume it exists. Denote it by $\mathscr{C}_2^*$. Automatically $\mathscr{C}_2^* \subset \overline{\mathscr{C}_{1-}^*}$, since we had removed all edges of $\overline{\mathscr{C}_{1+}^*}$. Remove $\mathscr{C}_2^*$ as well and decompose all edges between $\mathscr{C}_2^*$ and $\mathscr{C}_1^*$ (i.e., the edges in $\overline{\mathscr{C}_{2+}^*} \cap \overline{\mathscr{C}_{1-}^*}$ which do not belong to $\mathscr{C}_1^*$ or $\mathscr{C}_2^*$) into circuits of class I which satisfy (3.14), in the same way as with the edges of $\overline{\mathscr{C}_{1+}^*}$. These circuits then automatically satisfy (3.13).

We continue in this way until ∂E^* is exhausted. This yields a system of circuits which satisfies (3.13) and (3.14) and the circuits of class II will be ordered, i.e.,

$$\mathscr{C}_j^* \subset \overline{\mathscr{C}_{i-}^*} \quad \text{and} \quad \mathscr{C}_i^* \subset \overline{\mathscr{C}_{j+}^*} \quad \text{for } 1 \le i < j \le \tau.$$

We must show that τ is odd, and that (3.15) holds. As we shall see, τ odd will follow from (3.15), so we prove (3.15) first. Let ϕ be a path on $\mathscr{G}_0$ from a vertex v of $\mathscr{G}_0$ to a vertex w of $\mathscr{G}_0$. Whenever ϕ crosses an edge e^* of ∂E^* it goes from a face of a + cube to a face of a − cube or vice versa, since e^* is the edge of a boundary plaquette π^* which has a + cube and a − cube at its two sides (see (3.6a) and (3.9)). Thus,

(3.16) if v and w belong to the same component of $\Delta B \setminus \mathscr{C}_i^*$ for each of the circuits $\mathscr{C}_i^*$, then v and w both lie in the face of a + cube, or both lie in the face of a − cube.

(Since ϕ must cross each circuit $\mathscr{C}_i^*$ an even number of times in this situation.)

Next observe that if π^* is a plaquette in ΔB which has an edge e^* in common with the circuit $\mathscr{C}_i^*$ of class II (i.e., $1 \le i \le \tau$), then $\mathring{\pi}^*$, the interior of π^*, must lie in $\mathscr{C}_j^*(\text{ext})$ for each $\mathscr{C}_j^*$ of class I. Indeed if $\mathscr{C}_j^* \subset \overline{\mathscr{C}_{i+}^*}$, then we can connect e^* to Δ_- by a path in $\mathscr{C}_{i-}^*$ (except for its initial point on e^*), and this path avoids $\mathscr{C}_j^*$ (note that e^* is not an edge of $\mathscr{C}_j^*$ since $\mathscr{C}_i^*$ and $\mathscr{C}_j^*$ are edge disjoint). Thus the interior of e^* and the interior of π^* lie in $\mathscr{C}_j^*(\text{ext})$. A similar argument works if $\mathscr{C}_j^* \subset \overline{\mathscr{C}_{i-}^*}$.

Now let the interiors of π_1^* and π_2^* both lie in $\mathscr{C}_{i+}^*$ for some $1 \le i \le \tau$ and both with an edge in common with $\mathscr{C}_i^*$. Denote the edge of π_l^* in $\mathscr{C}_i^*$ by e_l^*, $l = 1, 2$. Then by the last paragraph, $\mathring{\pi}_1^*$ and $\mathring{\pi}_2^*$ belong to the same component of $\Delta B \setminus \mathscr{C}_j^*$ for $j > \tau$. Also, $\mathring{\pi}_1^*$ and $\mathring{\pi}_2^*$ belong to $\mathscr{C}_{j+}^*$ for $i < j \le \tau$ since $\mathscr{C}_{i+}^* \subset \mathscr{C}_{j+}^*$, for $\mathscr{C}_i^*$ above $\mathscr{C}_j^*$. Finally e_1^* and e_2^* belong to $\mathscr{C}_i^* \subset \overline{\mathscr{C}_{j-}^*}$ for $1 \le j < i$. From this one sees that π_1^* and π_2^* also belong to $\overline{\mathscr{C}_{j-}^*}$ for $1 \le j < i$. Thus, $\mathring{\pi}_1^*$ and $\mathring{\pi}_2^*$ belong to the same component of $\Delta B \setminus \mathscr{C}_j^*$ for all j. Therefore, by (3.16) they both are faces of + cubes or both faces of − cubes.

The faces in $\overline{\mathscr{C}_i^*}$ which contain e_1^* and e_2^* belong both to cubes of the opposite parity of the cubes corresponding to π_1^* and π_2^*, since we reach these faces by crossing the single edge e_1^* from π_1^* or e_2^* from π_2^*, respectively. To complete the proof of (3.15) we therefore only have to decide whether π_1^* and π_2^* are both faces of + cubes or both faces of − cubes. Assume first that π_1^* and π_2^* are in $\mathscr{C}_{1+}^*$ with e_1^* and e_2^* edges of $\mathscr{C}_1^*$ (i.e., take $i = 1$ in the last paragraph). Then the center v of π_1^* can be connected to Δ_+ by a path ϕ on $\mathscr{G}_0$ which stays in $\mathscr{C}_{1+}^*$ and hence does not intersect any $\mathscr{C}_i^*$ of class II. Moreover, as we saw above the initial point v of ϕ lies in $\mathscr{C}_j^*(\text{ext})$ for any $\mathscr{C}_j^*$ of class I. The same is true for the endpoint, w say, of ϕ, since this endpoint lies on Δ_+. The endpoint w on Δ_+ belongs to the face of a + cube, by definition. Thus, by (3.16) π_1^* (and π_2^*) will also be the face of a + cube. This proves (3.15) for $i = 1$. For general i it follows by induction on i. For example, let π_3^* (π_4^*) have an edge in common with $\mathscr{C}_1^*$ ($\mathscr{C}_2^*$) and let π_3^* lie in $\overline{\mathscr{C}_{1-}^*}$ (π_4^* lie in $\overline{\mathscr{C}_{2+}^*}$). Then π_3^* is the face of a − cube, as we just proved. Moreover, $\mathring{\pi}_3^*$ and $\mathring{\pi}_4^*$ lie in the same component of $\Delta B \setminus \mathscr{C}_j^*$ for every j. We already saw this for $\mathscr{C}_j^*$ of class I, and for $\mathscr{C}_j^*$ of class II it follows from the fact (see the part of (3.12) which we already proved) that

$$\pi_3^*, \pi_4^* \subset \overline{\mathscr{C}_{1-}^*} \cap \overline{\mathscr{C}_{2+}^*} \subset \overline{\mathscr{C}_{j+}^*}, \quad 2 \le j \le \tau.$$

Thus, π_4^* is the face of a − cube. In this way (3.15) follows by induction.

Finally, it follows that τ is odd. Indeed the same argument which showed that a π^* in $\overline{C_{1+}^*}$ with an edge in common with $\mathscr{C}_1^*$ belongs to a + cube, shows that a π^* in $\overline{\mathscr{C}_{\tau-}^*}$ with an edge in common with $\mathscr{C}_\tau^*$ belongs to a − cube. By (3.15) this means that τ is odd. ∎

We only give an outline of the proof of our last purely topological lemma, which is very intuitive in any case. Here we identify E^* with the union of the plaquettes in E^*, i.e., we view E^* as a closed subset of $\mathbf{R}^3$.

(3.17) LEMMA. *E^*, when viewed as a subset of $\mathbf{R}^3$, is connected.*

Indication of proof. Fix some $0 < \varepsilon < \frac{1}{8}$ and let

$$U^\varepsilon = \left\{x \in \mathbf{R}^3 : -\tfrac{1}{2} - \varepsilon \le x(i) \le \tfrac{1}{2} + \varepsilon,\ i = 1,2,3\right\}$$

and

$$\hat{K}_-(\varepsilon) = \bigcup_{v \in K_-} \{v + U^\varepsilon\}.$$

Compare with (3.8); $\hat{K}_- = \hat{K}_-(0)$ so that for $\varepsilon > 0$, $\hat{K}_-(\varepsilon)$ is a fattened $\hat{K}_-$. We also set

$$\begin{aligned} B(\varepsilon) &= \bigcup_{v \in B} \{v + U^\varepsilon\} \\ &= \left[-\tfrac{1}{2} + \varepsilon, k + \tfrac{1}{2} + \varepsilon\right] \times \left[-\tfrac{1}{2} - \varepsilon, l + \tfrac{1}{2} + \varepsilon\right] \\ &\quad \times \left[-\tfrac{1}{2} - \varepsilon, m + \tfrac{1}{2} + \varepsilon\right] \end{aligned}$$

and

$$\begin{aligned} \Delta B(\varepsilon) = {} & \text{vertical part of the boundary of} \\ & \left[-\tfrac{1}{2} - \varepsilon, k + \tfrac{1}{2} + \varepsilon\right] \times \left[-\tfrac{1}{2} - \varepsilon, l + \tfrac{1}{2} + \varepsilon\right] \times [0, m]. \end{aligned}$$

With each edge e of E we now associate an "ε-plaquette" defined as follows. If $e \in E$, then it has exactly one endpoint, v say, in a $-$ cube (see 3.4c). Then the ε-plaquette, $\pi^*(\varepsilon)$ say, associated with e will be the unique face of $v + U^\varepsilon$ which intersects e. Thus $\pi^*(\varepsilon)$ is a $(1 + 2\varepsilon) \times (1 + 2\varepsilon)$ square parallel and close to the plaquette π^* associated to e. Let $E^*(\varepsilon)$ be the intersection of the boundary of $\hat{K}_-(\varepsilon)$ with the union of all ε-plaquettes associated to edges e of E. One easily checks that two ε-plaquettes $\pi_1^*(\varepsilon)$ and $\pi_2^*(\varepsilon)$ intersect if and only if the corresponding plaquettes π_1^* and π_2^* of E have at least one point in common. Moreover, for each $e \in E$, $E^*(\varepsilon)$ will contain a rectangle from the ε-plaquette associated to e. Therefore, it suffices to prove that $E^*(\varepsilon)$ is connected. The advantage of $E^*(\varepsilon)$ over E^* is that $E^*(\varepsilon)$ is smoother. In fact one can show that $E^*(\varepsilon)$ is a topological two-manifold with boundary, and this boundary lies in $\Delta B(\varepsilon)$ (see [13], proof of Step (i) in Lemma 2.23 for a similar argument). Note that there exists a homeomorphism h from $B(\varepsilon)$ onto the unit ball

$$C := \left\{x \in \mathbf{R}^3 : x^2(1) + x^2(2) + x^2(3) \le 1\right\}.$$

If M is the image under h of $E^*(\varepsilon)$, then M is a two-manifold in C with boundary, and M is contained in ∂C, the two-sphere in $\mathbf{R}^3$. Moreover $C \setminus M$

consists of two components, since $B(\varepsilon) \setminus E^*(\varepsilon)$ consists of two components. (One of these is $\hat{K}_-(\varepsilon)$ and the other consists of a union of boxes around vertices v in K_+. This second piece is still connected, since any pair of vertices v and w in K_+ can be connected by a path ϕ on $\mathbf{Z}^3$ in K_+. The path ϕ will lie entirely in the complement of $\hat{K}_-(\varepsilon)$.) Thus we have reduced the lemma to the purely topological problem of showing that a manifold M with the above properties is connected.

One can show that M is connected by mapping C to the upper half of S^3 in $\mathbf{R}^4$ by means of the map which takes

$$(x(1), x(2), x(3)) \in C$$

to

$$\left(x(1), x(2), x(3), +\left(1 - x^2(1) - x^2(2) - x^3(2)\right)^{1/2}\right).$$

Let M_+ be the image of M under this mapping. Then ∂M_+ lies on the equator, i.e., on

$$\left\{x \in \mathbf{R}^4 : x(4) = 0, x^2(1) + x^2(2) + x^2(3) = 1\right\}.$$

Let M_- be the reflection of M_+ in this equator. Then $M_+ \cup M_-$ is a compact manifold *without boundary* on S^3. One can also check that $S^3 \setminus (M_+ \cup M_-)$ still has exactly two components and that $M_+ \cup M_-$ has exactly as many components as M_+ (or M). But by Alexander duality, a compact two-manifold without boundary on S^3, whose complement has two components, must be connected (compare proof of Step (i) in Lemma 2.23 of [13] again). Thus M_+ and $E^*(\varepsilon)$ and E^* are connected. ∎

4. Probabilistic part of the proofs

Throughout, C_i and D_i will denote constants strictly between 0 and ∞. The precise values of these constants will have no importance and may vary from one occurrence to another. We use C_i for constants which depend on F only, while the D_i may depend on additional parameters such as ε, δ.

In broad outline the proof of Theorem 2.7 consists of the following steps:

(i) The constant ν is defined as

$$\lim_{k \to \infty} \frac{1}{k^2} \tau(k, k) = \limsup_{k, l \to \infty} \frac{1}{kl} \tau(k, l). \tag{4.1}$$

These limits exist and are equal w.p.1; we obtain this as an easy consequence of the multiparameter subadditive ergodic theorem. Since $\alpha(k, l)$, $\beta(k, l)$,

$\sigma(k,l) \le \tau(k,l)$ (see proof of (2.10) below) it follows from (4.1) that

$$\limsup_{k,l\to\infty} \frac{1}{kl}\theta(k,l) \le \nu \tag{4.2}$$

for $\theta = \alpha, \beta, \sigma, \tau$ and the difficult part is to handle $\liminf (kl)^{-1}\theta(k,l)$.

(ii) We next derive a kind of recurrence relation for

$$f(k,l,m,\varepsilon) := P\{\exists (F_0, F_m)\text{-cut } E^* \text{ of } [0,k]\times[0,l]\times[0,m] \text{ with } V(E^*) \le (\nu - 5\varepsilon)kl\}. \tag{4.3}$$

Roughly speaking this relation (see (4.12)) says that this f grows at most linearly in m. We prove this relation by constructing from a cut in $[0,k]\times[0,l]\times[0,m]$ another cut of height at most a constant times $\max(k,l)$. This recurrence relation is combined with an estimate which says that $f(k,l,m,\varepsilon+\delta)$ is smaller than $\exp(-D_2pq)$ if $f(k/p, l/q, m, \varepsilon)$ is smaller than some multiple of δ. The latter estimate is based on a simple large deviation argument and the observation that any (F_0, F_m)-cut in $[0,k]\times[0,l]\times[0,m]$ contains $\frac{1}{4}pq$ cuts which separate the bottom from the top in

$$\left[i(p^{-1}k+1), i(p^{-1}k+1)+p^{-1}k\right]\times\left[j(q^{-1}l+1), j(q^{-1}l+1)+q^{-1}l\right] \times[0,m], \quad 0\le i<p/2, \quad 0\le j<q/2.$$

These two estimates are then combined in Prop. 4.34 to prove a rapid decrease of $f(k,l,m,\varepsilon)$ in kl, provided we have a moderately good estimate for $f(k/p, l/q, m, \varepsilon)$. More specifically, Prop. 4.34 says that (4.35) implies (4.36), where

$$f(s_1, s_2, D_3(4\varepsilon)k, \varepsilon) \le \frac{\varepsilon}{\nu}, \tag{4.35}$$

and

$$f(k,l,m,4\varepsilon) \le \frac{9m}{k}\exp\left(-C_8\varepsilon\frac{kl}{s_1s_2}\right) + 6mD_6(4\varepsilon)\exp(-D_7(4\varepsilon)kl). \tag{4.36}$$

(p and q above correspond to k/s_1 and l/s_2, respectively).

(iii) To provide the "moderately good" estimate of the type (4.35) which we need to exploit Prop. 4.34 we first estimate $f(k,k,m,\varepsilon)$ in terms of

$$P\{\tau((r,r)) \le r^2(\nu-\varepsilon)\} \tag{4.4}$$

when $r \sim pk$, for a large p. This is done by "patching together" p^2 cuts in

$$[i(k+1), i(k+1)+k]\times[j(k+1), j(k+1)+k]\times[0, D_3k],$$

$0 \le i < p$, $0 \le j < p$, each with value at most $(\nu - 5\varepsilon)k^2$. After some modifications at the boundary this provides a cut E^* in

$$[0, r]^2 \times [0, D_3 k]$$

with boundary equal to the edges in the perimeter of $[-\frac{1}{2}, r + \frac{1}{2}]^2 \times \{\frac{1}{2}\}$. In this way we find in Prop. 4.45 the following inequality for $f(k, k, m, \varepsilon)$ in terms of (4.4): if $(p+1)k \le r \le (p+2)k$ and $m \ge k$ then

$$\text{(4.46)} \quad P\{\tau((r,r)) \le r^2(\nu - \varepsilon)\} \ge \frac{1}{4}\left[\frac{k}{9D_{11}m}\{f(k, k, m, \varepsilon) - 6mD_6\exp(-D_7k^2)\}\right]^{p^2}$$

(iv) Finally, we show by an easy subadditivity argument that

$$\text{(4.5)} \qquad P\{\tau((r,r)) \le r^2(\nu - \varepsilon)\} = O\left(\frac{1}{\log r}\right).$$

along some subsequence of r's. This estimate, together with step (iii) yields the required moderate estimate for $f(k, k, m, \varepsilon)$, which is turned into a good estimate by repeated application of step (ii). Here are some more details. (4.5) gives us the moderate estimate

$$f(k_0, k_0, m, \varepsilon) \le \frac{9D_{11}m}{k_0}\left\{\frac{4D_{13}}{\log r}\right\}^{p^{-2}} + 6mD_6\exp(-D_7k_0^2),$$

for some large starting value k_0 and an r, with $\log k_0$ and $\log r$ of the same order. We take

$$m = m_0 := D_3(4\varepsilon)k_1 \quad \text{with } k_1 = k_0(\log k_0)^{1/2p^2}.$$

Substitution of this m_0 easily yields

$$f(k_0, k_0, D_3(4\varepsilon)k_1, \varepsilon) \le \varepsilon/\nu,$$

which is precisely (4.35) for $s_1, s_2 = k_0$, $k = k_1$. As outlined under step (ii) we can then conclude from (4.36) that

$$f(k_1, k_1, m, 4\varepsilon) \le \frac{9m}{k_1}\exp\left(-C_8\varepsilon\left(\frac{k_1}{k_0}\right)^2\right) + 6mD_6(4\varepsilon)\exp(-D_7(4\varepsilon)k_1^2).$$

This sets us up for an iteration of Prop. 4.34. By choosing

$$m = m_1 := D_3(16\varepsilon)k_1\log k_1$$

we find

$$f(k_1, k_1, D_3(16\varepsilon)k_1 \log k_1, 4\varepsilon) \le 4\varepsilon/\nu,$$

which is (4.35) for $s_1 = s_2 = k_1$, ε replaced by 4ε and $k = k_2 := k_1 \log k_1$. Then (4.36) gives an estimate for $f(k_2, k_2, m, 16\varepsilon)$. We iterate once more by choosing

$$m = m_2 := D_3(64\varepsilon)k_2^{\lambda+1} \quad \text{for some} \quad \lambda \ge 1.$$

We get (4.35) for $s_1 = s_2 = k_2$, ε replaced by 16ε and k by $k_2^{\lambda+1}$. The final result read off from (4.36) with $n = k_2^{\lambda+1}$ is (see (4.55))

$$f(n, n, m, \varepsilon) \le \frac{9m}{n}\exp\left(-D_{15}(\varepsilon)n^{2-2\delta}\right), \; m \ge n,$$

for $n \ge n_0(\varepsilon, \delta)$. This is our required good estimate of $f(k, l, m, \varepsilon)$ when $k = l$. Analogous arguments work for $k \ge l$, and Theorems 2.7, 2.10, 2.12 follow easily after this. As a prologue we give:

Proof of Lemma 2.1. We first give an equivalent formulation of (2.3) in first-passage percolation terms. Let $\mathscr{H}$ be a graph whose vertices are in one-to-one correspondence with the plaquettes of $\mathscr{L}^*$, and two of whose vertices are adjacent if and only if the corresponding plaquettes intersect. We can think of these vertices as being located at the centers of the corresponding plaquettes, or—what is the same—at the midpoints of the edges of $\mathbf{Z}^3$. Then the vertex at $(0, 0, \frac{1}{2})$ has neighbors in $\mathscr{H}$ at $(\pm\frac{1}{2}, 0, a)$, $(0, \pm\frac{1}{2}, a)$, $(\pm 1, \pm\frac{1}{2}, a)$, $(\pm\frac{1}{2}, \pm 1, a)$ for $a = 0, 1$ (all combinations of $+$ and $-$ are permitted). Also, there are four neighbors at $(\pm 1, 0, \frac{1}{2})$ and $(0, \pm 1, \frac{1}{2})$. In total each vertex has 28 neighbors. Now attach to each vertex v of $\mathscr{H}$ a random variable $t(v)$ with distribution F and take all the $t(v)$, $v \in \mathscr{H}$, independent. To a set V of vertices attach the value

$$T(V) = \sum_{v \in V} t(v).$$

Then the probability in (2.3) equals

(4.6) $P\{$there exists a connected set V of n vertices of $\mathscr{H}$ which contains v_0 and with $T(V) \le \Theta n\}$,

where v_0 is any fixed vertex of $\mathscr{H}$, and a set V of vertices is connected if for any pair $v', v'' \in V$ there is a sequence $v' = v_0, v_1, \dots, v_k = v''$ of vertices in V such that v_i and v_{i+1} are neighbors on $\mathscr{H}$ for $0 \le i < k$.

If we are content with a poor estimate of p_0 we can now argue that there exist at most C_4^n connected sets V containing a fixed v_0 for some $C_4 < \infty$

($C_4 = 29^{29}28^{-28}$ will do; cf. [12], (5.22)). Therefore, for each $\lambda \geq 0$, (4.6) is bounded by

$$C_4^n P\{t_1 + \cdots + t_n \leq \Theta n\} \leq \left(C_4 e^{\lambda\Theta} E\{e^{-\lambda t(v)}\}\right)^n$$

(for $t_1, \ldots, t_n$ i.i.d. with the distribution F). If $P\{t(v) = 0\} < C_4^{-1}$ we can choose λ and $\Theta > 0$ so that this expression decreases exponentially in n.

Not much is gained by improving the lower bound for p_0 to $1/27$, so we merely list the main ingredients of the proof. First we define the critical probability corresponding to p_T for site percolation on $\mathscr{H}$. Specifically, denote by P_p the distribution of $\{t(v) : v \in \mathscr{H}\}$ when $P\{t(v) = 0\} = p$, $P\{t(v) = 1\} = 1 - p$. For a fixed v_0 let N be the number of vertices v for which there exists a sequence $v_0, v_1, \ldots, v_n = v$ of distinct vertices with v_{i+1} adjacent to v_i, $0 \leq i < n$ and $t(v_j) = 0$, $0 \leq j \leq n$ (N is the number of points which can be reached from v_0 "along a path of zero passage time"). The critical probability we want is

$$p_0 := \sup\{p : E_p N < \infty\}.$$

Since there are at most $28(27)^{n-1}$ sequences $v_0, \ldots, v_n$ with distinct v_i, and v_{i+1} adjacent to v_i, a simple Peierls argument shows that this $p_0 \geq 1/27$.

Next one shows that (2.3) holds for $F(0)$ less than the above p_0. This is done by combining a block argument almost identical to Lemmas 5.2 and 5.3 of [12] with the first part of this proof. Going over to large blocks is useful, because for $F(0) < p_0$ one has for sufficiently small $\Theta' > 0$,

$$\begin{aligned} P\{&\text{there exists a connected set of plaquettes } E^* \text{ which intersects} \\ &\text{the boundaries of } [0, l]^3 \text{ and of } [-l, 2l]^3 \text{ and} \\ &V(E^*) \leq \Theta' l\} \to 0,\ l \to \infty. \end{aligned}$$

This follows from the (proof of) Prop. 5.8 in [13] applied to $\mathscr{H}$. In fact this proposition shows that the above probability converges to 0 exponentially fast in l, which already indicates that (2.3) holds. We omit further details.

(2.4) follows from (2.3) since a cut E^* over $[0, k] \times [0, l]$ has to be connected by Lemma 3.17. ■

We now start on *Step* (*i*). *In all the succeeding lemmas the hypotheses of Theorem* 2.7 *are tacitly assumed*. Recall that Θ is the constant of Lemma 2.1.

(4.7) LEMMA. *There exists a constant $\nu \in [\Theta, Et(e)]$ such that both limits in* (4.1) *equal ν* w.p.1.

Proof. The existence of the limit ν will be immediate from the multi-parameter subadditive ergodic theorems of Akcoglu and Krengel (Theorem 2.4 of [3]) and Smythe (Theorem 1.1 in [14]) once we show that τ is subadditive in the sense of [3]. More specifically, for $k_1 \le k_2$, $l_1 \le l_2$, $k_i, l_i \in \mathbf{Z}$, define

$$\begin{aligned}\tau([k_1, k_2 + 1), [l_1, l_2 + 1)) \\ = \inf\{ V(E^*) : E^* \text{ a cut over} [k_1, k_2] \times [l_1, l_2] \\ \text{whose boundary } \partial E^* \text{ consists of the edges} \\ \text{of } \mathscr{L}^* \text{ on the perimeter of} \\ [k_1 - \tfrac{1}{2}, k_2 + \tfrac{1}{2}] \times [l_1 - \tfrac{1}{2}, l_2 + \tfrac{1}{2}] \times \{\tfrac{1}{2}\}\}.\end{aligned}$$

We need only show that

$$\tau(S) \le \sum_1^{\rho} \tau(S_i) \tag{4.8}$$

whenever

$$S = [k_1, k_2 + 1) \times [l_1, l_2 + 1) \quad \text{and} \quad S_i = [k_1^i, k_2^i + 1) \times [l_1^i, l_2^i + 1)$$

are disjoint rectangles of the above form such that $S = \cup_1^{\rho} S_i$ (ρ arbitrary, but finite; see Fig. 10).

To prove (4.8) we choose cuts E_i^* over

$$[k_1^i, k_2^i] \times [l_1^i, l_2^i], 1 \le i \le \rho,$$

such that ∂E_i^* consists of the edges on the perimeter P_i of

$$[k_i - \tfrac{1}{2}, k_2 + \tfrac{1}{2}] \times [l_1^i - \tfrac{1}{2}, l_2^i + \tfrac{1}{2}] \times \{\tfrac{1}{2}\}.$$

Then $E^* := \cup_1^{\rho} E_i^*$ is a collection of plaquettes whose boundary consists of the edges on the perimeter P of

$$[k_1 - \tfrac{1}{2}, k_2 + \tfrac{1}{2}] \times [l_1 - \tfrac{1}{2}, l_2 + \tfrac{1}{2}] \times \{\tfrac{1}{2}\}.$$

This is easily seen from the fact that $\partial(\cup E_i^*) \subset \cup(\partial E_i^*)$, and that each edge on some P_i which does not belong to P, must belong to exactly one other P_j with $j \neq i$ (of course P_i is the perimeter of S_i; see Fig. 10). The latter edges therefore do not belong to ∂E^* (see (1.8)). It follows from Lemma 3.1(a) that E^* separates $-\infty$ from $+\infty$ over S. Finally, it follows from Lemma 3.1(b) that the different E_i^* have no plaquettes in common (recall that the S_i are disjoint). Therefore E^* must be a *minimal* separating set, because if we remove a plaquette from E^*, say we remove π^* from E_i^*, then there exists a

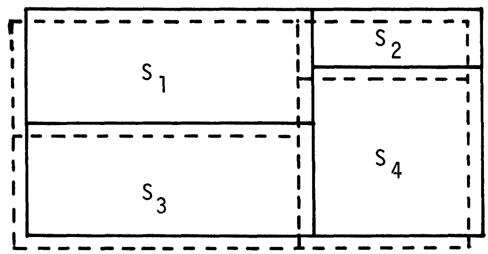

FIG. 10 $S = \cup_1^4 S_i$. The solid segments consist of edges of $\mathbf{Z}^2$ which form the boundaries of the S_j. The dashed segments consist of edges of $\mathbf{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ and the ∂E_i^* lie above then.

path ϕ from $-\infty$ to $+\infty$ in

$$[k_1^i, k_2^i] \times [l_1^i, l_2^i] \times \mathbf{R}$$

which does not intersect $E_i^* \setminus \pi^*$ (since $E_i^* \setminus \pi^*$ no longer separates $-\infty$ from $+\infty$ over $[k_1^i, k_2^i] \times [l_1^i, l_2^i]$); ϕ does not intersect $E^* \setminus \pi^*$ either (again by Lemma 3.1(b)).

The above proves that E^* is a cut over $[k_1, k_2] \times [l_1, l_2]$ with ∂E^* consisting of the edges on the perimeter of

$$[k_1 - \tfrac{1}{2}, k_2 + \tfrac{1}{2}] \times [l_1 - \tfrac{1}{2}, l_2 + \tfrac{1}{2}].$$

Thus

$$\tau(S) \le V(E^*) \le \sum_1^\rho V(E_i^*). \tag{4.9}$$

Taking the infimum over $E_1^*, \dots, E_\rho^*$ now yields (4.8).

Theorem 2.4 in [3] now states that the first limit in (4.1) exists w.p.1, while Theorem 1.1 of [14] shows that the lim sup in the second member of (4.1) exists and equals the first limit w.p.1. If we denote the common value by ν, then ν is a function of the $\{t(e) : e \in \mathbf{Z}^3\}$ which is invariant under the shifts θ_1 and θ_2 which take $t(e)$ to $t(e + (1, 0, 0))$ and $t(e + (0, 1, 0))$, respectively (as remarked in the proof of Theorem 1.1 of [14]). By Kolmogorov's zero-one law all sets in the σ-field generated by $\{t(e) : e \in \mathbf{Z}^3\}$ and invariant under θ_1 or θ_2 have probability zero or one. Thus ν is a constant w.p.1.

To prove (4.7) it remains to show that $\Theta \le \nu \le Et(e)$. The second inequality is immediate from the strong law of large numbers and the fact that $\tau((k, k))$ is at most equal to the value of the collection of plaquettes

$$\{[i - \tfrac{1}{2}, i + \tfrac{1}{2}] \times [j - \tfrac{1}{2}, j + \tfrac{1}{2}] \times \{\tfrac{1}{2}\} : 0 \le i \le k, 0 \le j \le k\}.$$

To prove $\nu \ge \Theta$ we observe that any cut E^* which separates $-\infty$ from $+\infty$

over $[0, k] \times [0, k]$ has $V(E^*) \geq \Theta(k+1)^2$, outside a set of probability at most $C_1 \exp(-C_2(k+1)^2)$ (by (2.4), since E^* must intersect each vertical line $\{i\} \times \{j\} \times \mathbf{R}$, $0 \leq i, j \leq k$). The Borel-Cantelli lemma and (4.1) now imply $\nu \geq \Theta$. ∎

This takes care of Step (i) and we turn to *Step (ii)*. We recall that $B(k, l, m) = [0, k] \times [0, l] \times [0, m]$ and that $f(k, l, m, \varepsilon)$ is defined in (4.3).

(4.10) Lemma. *There exists an $\varepsilon_0 > 0$ and for $0 < \varepsilon \leq \varepsilon_0$ constants $0 < D_i = D_i(F, \varepsilon) < \infty$ such that for $m \geq k \geq l \geq D_4$ there exists $K = K(k, l, m, \varepsilon)$ and $L = L(k, l, m, \varepsilon)$ for which*

$$(4.11) \qquad (1-\varepsilon^2)k \leq K \leq k, \quad (1-\varepsilon^2)l \leq L \leq l$$

and

$$(4.12) \qquad \begin{aligned} P\{&\text{there exists an } (F_0, F(D_3K))\text{-cut } E_1^* \text{ of } B(K, L, D_3K) \\ &\text{such that } V(E_1^*) \leq (\nu - 3\varepsilon)KL \text{ and such that } \partial E_1^* \\ &\text{contains fewer than } D_5K \text{ edges}\} \\ &\geq \frac{k}{9m}\{f(k, l, m, \varepsilon) - 6mD_6\exp(-D_7kl)\}. \end{aligned}$$

Note that we assumed $k \geq l$, and that the dimensions of B in (4.12) are different from those in (4.3). We have written $F(D_3K)$ instead of F_{D_3K} for typographical reasons. The lemma is of interest only if m is much larger than k; we shall see that (4.12) is easy when $m \leq Dk$ with $D_3 = 2D$.

Proof. Let E^* be an (F_0, F_m)-cut in $B = B(k, l, m)$ with $\nu(E^*) \leq (\nu - 5\varepsilon)kl$. Then E^* is a cut over $[0, k] \times [0, l]$ and must contain one of the plaquettes $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \times \{p + \frac{1}{2}\}$, and hence the point $(-\frac{1}{2}, -\frac{1}{2}, p + \frac{1}{2})$, $0 \leq p < m$. (Otherwise, we can connect F_0 and F_m along the coordinate axis $\{0\} \times \{0\} \times \mathbf{R}$ without hitting E^*.) For each fixed p, the probability that there exists a cut over $[0, k] \times [0, l]$ through $(-\frac{1}{2}, -\frac{1}{2}, p + \frac{1}{2})$ with value $\leq \nu kl$ and containing more than $\nu\Theta^{-1}kl$ plaquettes is at most

$$C_1\exp(-C_2\nu\Theta^{-1}kl) \leq C_1\exp(-C_2kl),$$

by virtue of (2.4) and $\nu \geq \Theta$. We shall write $|E^*|$ for the number of plaquettes in E^* and $|\partial E^*|$ for the number of edges in ∂E^*. Also we write B for $B(k, l, m)$ and C_5 for $\Theta^{-1}\nu$. With this notation the above argument shows that

$$(4.13) \qquad \begin{aligned} P\{&\text{there exists an } (F_0, F_m)\text{-cut } E^* \text{ of } B \text{ with } V(E^*) \leq \\ &\leq (\nu - 5\varepsilon)kl \text{and } |E^*| \leq C_5kl\} \geq f - mC_1\exp(-C_2kl). \end{aligned}$$

Now we take an E^* with the properties in (4.13) and denote by Z_j the number of plaquettes of E^* which intersect the vertical part of the boundary of

(4.14) $$\left[j-\tfrac{1}{2}, k-j+\tfrac{1}{2}\right] \times \left[j-\tfrac{1}{2}, l-j+\tfrac{1}{2}\right] \times [0, m].$$

Since each plaquette intersects at most two such vertical boundaries,

$$\sum_{0 \le j < \varepsilon^2 l/2} Z_j \le 2|E^*| \le 2C_5 kl.$$

Consequently, the number of $j < \varepsilon^2 l/2$ with $Z_j \le 8C_5\varepsilon^{-2}k$ is at least $\varepsilon^2 l/4$. In view of (4.13), with $D_8 = 8C_5\varepsilon^{-2}$, this implies

$$\begin{aligned}
&\frac{\varepsilon^2 l}{4}(f - mC_1\exp(-C_2 kl)) \\
&\quad \le E\big\{\text{number of } j < \varepsilon^2 l/2 \text{ for which there exists an } (F_0, F_m)\text{-cut} \\
&\qquad\quad E^* \text{ in } B \text{ with } V(E^*) \le (\nu - 5\varepsilon)kl \text{ and } |E^*| \le C_5 kl \\
&\qquad\quad \text{and } Z_j \le D_8 k\big\} \\
&\qquad \le \sum_{j < \varepsilon^2 l/2} P\big\{\text{there exists an } (F_0, F_m)\text{-cut } E^* \text{ in } B \text{ with} \\
&\qquad\qquad\quad V(E^*) \le (\nu - 5\varepsilon)kl \text{ and } |E^*| \le C_5 kl \\
&\qquad\qquad\quad \text{and } Z_j \le D_8 k\big\}.
\end{aligned}$$

In particular, we can find a $j_1 < \varepsilon^2 l/2$ such that the corresponding probability in the right hand side above is at least

$$\tfrac{1}{3}(f - mC_1\exp(-C_2 kl)).$$

We choose $K = k - 2j_1$, $L = l - 2j_1$, and observe that any (F_0, F_m)-cut E^* in B contains a subset E_2^*, which is a minimal set separating the bottom from the top of

$$B' = [j_1, k - j_1] \times [j_1, l - j_1] \times [0, m].$$

Moreover, $V(E_2^*) \le V(E^*)$, $|E_2^*| \le |E^*|$, and the number of plaquettes of E_2^* which intersect the vertical part of the boundary of (4.14) for $j = j_1$ is at most Z_{j_1}. Finally observe that B' is just a translate of $B(K, L, m)$, and recall that $\Delta B(K, L, m)$ denotes the vertical part of the boundary of

$$\left[-\tfrac{1}{2}, K+\tfrac{1}{2}\right] \times \left[-\tfrac{1}{2}, L+\tfrac{1}{2}\right] \times [0, m].$$

We have therefore proven that

(4.15) $$P\{\text{there exists an } (F_0, F_m)\text{-cut } E_2^* \text{ in } B(K, L, m) \text{ with } V(E_2^*) \le (\nu - 5\varepsilon)kl,\ |E_2^*| \le C_5 kl \text{ and the number of plaquettes of } E_2^* \text{ which intersect } \Delta B(K, L, m) \text{ is at most } D_8 k\} \ge \tfrac{1}{3}\{f - mC_1\exp(-C_2 kl)\}.$$

By construction, our choice of K, L also satisfies (4.11).

For the remainder of this proof we denote $B(K, L, m)$ by B_1. Let E_2^* be a cut in B_1 with $V(E_2^*) \le (\nu - 5\varepsilon)kl$ and $|\partial E_2^*| \le$ number of plaquettes of E_2^* which intersect $\Delta B_1 \le D_8 k$. Such an E_2^* satisfies almost all requirements in (4.12). Our only job is to reduce the height, i.e., to replace the m in $B(K, L, m)$ by $D_3 K$. We shall do this by using the part of E_2^* between two horizontal planes of the form $H_p := \{x(3) = p + \frac{1}{2}\}$ plus certain plaquettes in those two H_p's. The succeeding steps serve to choose the H_p such that only few plaquettes in H_p have to be added to E_2^*. Let

$$U_p = \text{number of plaquettes of } E_2^* \text{ which intersect } H_p,$$
$$V_p = \text{number of plaquettes of } E_2^* \text{ which intersect } H_p \cap \Delta B_1.$$

Since

$$\sum_{p=0}^{m-1} U_p \le 2|E_2^*| \le 2C_5 kl$$

there are at most $2\varepsilon^{-1}C_5 k$ values of p for which $U_p \ge \varepsilon l$. Similarly,

$$\sum_{p=0}^{m-1} V_p \le 2(\text{number of plaquettes in } E_2^* \text{ which intersect } \Delta B_1) \le 2D_8 k.$$

so that there are at most $2D_8 k$ values of p with $V_p \neq 0$. Assume now that we have two $H_{p(i)}$, $i = 1, 2$ such that

(4.16) $$U_{p(i)} \le \varepsilon l, \quad V_{p(i)} = 0.$$

We claim that in this case for fixed i ($i = 1$ or 2) all plaquettes in $H_{p(i)}$ which have one edge in ΔB_1 are faces of cubes of only one parity, i.e., they are all only faces of + cubes or all only faces of − cubes. (Here and in the sequel + and − cubes are defined with respect to $B_1 = B(K, L, m)$ rather than B.) To

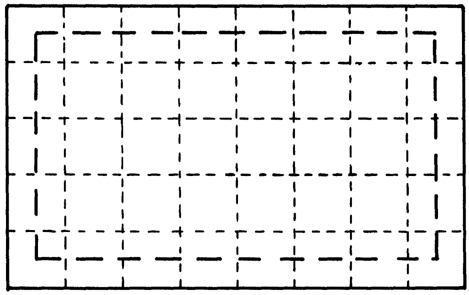

FIG. 11 A picture of $[-\frac{1}{2}, K+\frac{1}{2}] \times [-\frac{1}{2}, L+\frac{1}{2}] \times \{p(i)\}$. The vertices of $\mathbf{Z}^3$ in this square adjacent to its boundary are connected by the path of long dashes on $\mathbf{Z}^3$. This path does not cross any plaquette of E_2^*, because $V_{p(i)} = 0$.

see this, observe that the centers of all the cubes with such a face can be connected to each other by paths on $\mathbf{Z}^3$ in B_1 which avoid E_2^*, and use (3.9). The existence of such paths is illustrated in Fig. 11. Also the edge from $(a, b, p(i))$ to $(a, b, p(i)+1)$ from a cube below $H_{p(i)}$ to one above $H_{p(i)}$ does not intersect E_2^* if $a = 0$ or K or $b = 0$ or L, again because $V_{p(i)} = 0$ implies that the plaquettes in $H_{p(i)}$ adjacent to ΔB_1 do not belong to E_2^*.

We have shown that under (4.16) all cubes with a face in $H_{p(i)}$ adjacent to ΔB_1 are of one parity. We next assume that

(4.17) for $i = 1$ all the above cubes are $-$ cubes and for $i = 2$ all the above cubes are $+$ cubes,

and

(4.18) let $S^*(1)$ ($S^*(2)$) be the collection of plaquettes in $H_{p(1)}$ ($H_{p(2)}$) which are a face of at least one $+$ cube ($-$ cube).

The next observation—which allows us to construct a narrower cut—is that irrespective of (4.16),

(4.19) $S^*(1) \cup S^*(2) \cup$ (collection of all plaquettes in E_2^* between $H_{p(1)}$ and $H_{p(2)}$) separates the bottom from the top in

$$[0, K] \times [0, L] \times [\min(p(1), p(2)), \max(p(1), p(2)) + 1].$$

To simplify the notation in the proof of (4.19) we assume $p(1) < p(2)$: the case $p(2) < p(1)$ is similar. Write T^* for the collection of the plaquettes of E_2^* in

$$[-\tfrac{1}{2}, K+\tfrac{1}{2}] \times [-\tfrac{1}{2}, L+\tfrac{1}{2}] \times [p(1)+\tfrac{1}{2}, p(2)+\tfrac{1}{2}].$$

Now let ϕ be a path on $\mathbf{Z}^3$ from some point $(a, b, p(1))$ to $(c, d, p(2) + 1)$, such that ϕ, except for its endpoints, lies in

$$[0, K] \times [0, L] \times (p(1), p(2) + 1).$$

To prove (4.19) we must show that any such ϕ intersects a plaquette of

$$S^*(1) \cup S^*(2) \cup T^*.$$

If the plaquette

$$\left[a - \tfrac{1}{2}, a + \tfrac{1}{2}\right] \times \left[b - \tfrac{1}{2}, b + \tfrac{1}{2}\right] \times \left\{p(1) + \tfrac{1}{2}\right\}$$

lies in $S^*(1)$, then already the first edge of ϕ intersects this plaquette of $S^*(1)$. Thus, we may assume that

$$\left[a - \tfrac{1}{2}, a + \tfrac{1}{2}\right] \times \left[b - \tfrac{1}{2}, b + \tfrac{1}{2}\right] \times \left\{p(1) + \tfrac{1}{2}\right\}$$

is not in $S^*(1)$, so that by (4.18), $(a, b, p(1))$, the initial point of ϕ, is the center of a $-$ cube. For similar reasons we may assume that the final point of ϕ, $(c, d, p(2) + 1)$ is the center of a $+$ cube. But then ϕ runs from a $-$ cube to a $+$ cube, and since E_2^* separates the $+$ cubes from the $-$ cubes (cf. (3.9)) ϕ must intersect a plaquette in E_2^*. Since ϕ lies between the hyperplanes $\{x(3) = p(1)\}$ and $\{x(3) = p(2) + 1\}$ the only plaquettes of E_2^* which it can intersect belong to T^*. Thus (4.19) follows.

We remark that (4.19) remains valid if $p(1) = -1$ with $S^*(1) = \varnothing$ and/or $p(2) = m$ with $S^*(2) = \varnothing$. If $p(1) = -1$, then we do not have to consider cubes below $H_{p(1)} = H_{-1}$, but all the cubes directly above H_{-1}, i.e., the $v + U$ with $v = (a, b, 0)$, $0 \le a \le K$, $0 \le b \le L$, are $-$ cubes by definition, and E_2^* does not intersect H_{-1}. The above argument needs no change therefore, if $p(1) = -1$, and the same holds if $p(2) = m$.

Our final cut set will be (a translate of)

$$\begin{aligned} E_3^* := {} & \text{a minimal subset of } S^*(1) \cup S^*(2) \cup T^* \text{ which} \\ & \text{separates } [0, K] \times [0, L] \times \{p(1)\} \text{ from} \\ & [0, K] \times [0, L] \times \{p(2) + 1\} \text{ in } [0, K] \times [0, L] \times \mathbf{R}. \end{aligned}$$

To show that this is a good choice we first observe that

$$\partial E_3^* \subset \partial E_2^*, \tag{4.20}$$

by virtue of (3.6a) and the fact that $S^*(1)$ and $S^*(2)$ contain no edges in ΔB_1

because of (4.17), (4.18). Next we note that

$$V(E_3^*) \le V(S^*(1)) + V(S^*(2)) + V^*(T) \le V(S^*(1)) + V(S^*(2)) + V(E_2^*). \tag{4.21}$$

Clearly an estimate for $V(S^*(i))$ is needed now. First we estimate $|S^*(i)|$, the number of plaquettes in $S^*(i)$. Note that each plaquette of $S^*(1)$ is a face of at least one + cube. On the other hand all plaquettes in $H_{p(1)}$ adjacent to ΔB_1 are faces only of $-$ cubes, by (4.17). Thus, any path ψ in $H_{p(1)}$ from a plaquette in $S^*(1)$ to ΔB_1 must at some time cross from a face of a + cube to a face of a $-$ cube. At this place ψ must intersect an edge in $H_{p(1)}$ of a plaquette in E_2^*. In other words, $S^*(1)$ in $H_{p(1)}$ is separated from $H_{p(1)} \cap \Delta B_1$ by edges of plaquettes of E_2^*. This means that $S^*(1)$ is surrounded by curves made up of edges of plaquettes of E_2^* in $H_{p(1)}$. There are at most $U_{p(1)} \le \varepsilon l$ such edges. Now any planar area surrounded by a curve of length λ has diameter $\le \lambda$ and hence area $\le \lambda^2$. Consequently

$$|S^*(1)| = \text{area of } S^*(1) \le \varepsilon^2 l^2$$

and similarly

$$|S^*(2)| \le \varepsilon^2 l^2.$$

It follows that

$$V(S^*(1)) + V(S^*(2)) \le 2 \max_{0 \le p < m} \max_{S^*} V(S^*),$$

where the second max runs over all sets S^* of plaquettes in the rectangle $[-\frac{1}{2}, K + \frac{1}{2}] \times [-\frac{1}{2}, L + \frac{1}{2}] \times \{p + \frac{1}{2}\}$ which are surrounded by curves which contain in total at most εl edges. The number of choices for p is at most m, and for fixed p one can choose a number of curves which together have no more than εl edges in at most $(8KL)^{\varepsilon l}$ ways—since there are at most $2(K + 2)(L + 2)$ edges of $\mathscr{L}^*$ in $[-\frac{1}{2}, K + \frac{1}{2}] \times [-\frac{1}{2}, L + \frac{1}{2}] \times \{p + \frac{1}{2}\}$. Once the boundary curves are chosen, the value of any set surrounded by these curves is at most the value of the set of all plaquettes inside at least one of these curves. As argued above, there are at most $\varepsilon^2 l^2$ such plaquettes. Therefore, if $t_1, t_2, \ldots$ are independent random variables each with the distribution function F, then for ε_0 small enough and γ as in (2.8)

$$\begin{aligned} P\{V(S^*(1)) + V(S^*(2)) \ge \varepsilon k l\} &\le P\Big\{\max_{p, S^*} V(S^*) \ge \tfrac{1}{2}\varepsilon k l\Big\} \\ &\le m(8KL)^{\varepsilon l} P\{t_1 + \cdots + t_{\varepsilon^2 l^2} \ge \tfrac{1}{2}\varepsilon k l\} \\ &\le m(8KL)^{\varepsilon l} \exp(-\tfrac{1}{2}\gamma \varepsilon k l)(E \exp \gamma t_1)^{\varepsilon^2 l^2} \\ &\le m(8KL)^{\varepsilon l} \exp(-\tfrac{1}{4}\gamma \varepsilon k l) \\ &\le m D_6 \exp(-D_7 k l). \end{aligned} \tag{4.22}$$

(4.22) guarantees us that with overwhelming probability

$$V(S^*(1)) + V(S^*(2))$$

will be negligible, if we can find $H_{p(i)}$, $i = 1, 2$, which satisfy (4.16) and (4.17). Recall that we want a cut set of height at most D_3K for (4.12), and we therefore want $|p(1) - p(2)| \le D_3K$ for some D_3. To guarantee this we must now look somewhat closer at ∂E_2^*. Assume that E_2^* has the properties described in (4.15). Then ∂E_2^* in ΔB_1 can be decomposed into circuits $\mathscr{C}_1^*, \mathscr{C}_2^*, \dots, \mathscr{C}_\rho^*$ with the properties (3.11)–(3.15). The circuits $\mathscr{C}_1^*, \dots, \mathscr{C}_\tau^*$ are the ones of class II, and they are ordered such that $\mathscr{C}_j^*$ lies "below" $\mathscr{C}_i^*$ if $j > i$. Now set $D_9 = 2\varepsilon^{-1}C_5 + 2D_8 + 1$ and call $\mathscr{C}_i^*$ and $\mathscr{C}_{i+1}^*$ *linked* if

$$d(\mathscr{C}_i^*, \mathscr{C}_{i+1}^*) := \min\{|x - y| : x \in \mathscr{C}_i^*, \ y \in \mathscr{C}_{i+1}^*\} \le D_9 k.$$

For $i < j$, call $\mathscr{C}_i^*$ and $\mathscr{C}_j^*$ linked if $\mathscr{C}_t^*$ and $\mathscr{C}_{t+1}^*$ are linked for $i \le t < j$. Then $\mathscr{C}_1^*, \dots, \mathscr{C}_\tau^*$ break up into blocks, such that all circuits in one block are linked, but circuits in distinct blocks are not linked. Since τ is odd (cf. (3.12)), at least one of these groups contains an odd number of circuits, that is there exist $1 \le \lambda \le \xi \le \tau$ such that

$$d(\mathscr{C}_t^*, \mathscr{C}_{t+1}^*) \le D_9 k \quad \text{for} \quad \lambda \le t < \xi, \tag{4.23}$$

$$d(\mathscr{C}_{\lambda-1}^*, \mathscr{C}_\lambda^*) > D_9 k \quad \text{if} \quad \lambda > 1, \tag{4.24}$$

$$d(\mathscr{C}_\xi^*, \mathscr{C}_{\xi+1}^*) > D_9 k \quad \text{if} \quad \xi < \tau, \tag{4.25}$$

and

$$\xi - \lambda \text{ is even.} \tag{4.26}$$

First consider the case $1 < \lambda \le \xi < \tau$. Since there are at most

$$2\varepsilon^{-1}C_5 k + 2D_8 k = (D_9 - 1)k$$

values of p for which $U_p > \varepsilon l$ or $V_p \ne 0$ we can choose $p(i)$ such that (4.16) holds and such that $H_{p(1)}$ ($H_{p(2)}$) lies between $\mathscr{C}_{\xi+1}^*$ and $\mathscr{C}_\xi^*$ ($\mathscr{C}_\lambda^*$ and $\mathscr{C}_{\lambda-1}^*$) and at a distance at most $D_9 k$ from $\mathscr{C}_\xi^*$ ($\mathscr{C}_\lambda^*$). More precisely,

$$\begin{aligned} &\mathscr{C}_j^* \subset \mathbf{R}^2 \times \left(-\infty, p(1) + \tfrac{1}{2}\right) \quad \text{for} \quad \xi + 1 \le j \le \tau, \\ &\mathscr{C}_j^* \subset \mathbf{R}^2 \times \left(p(1) + \tfrac{1}{2}, p(2) + \tfrac{1}{2}\right) \quad \text{for} \quad \lambda \le j \le \xi, \\ &\mathscr{C}_j^* \subset \mathbf{R}^2 \times \left(p(2) + \tfrac{1}{2}, \infty\right) \quad \text{for} \quad j < \lambda, \end{aligned} \tag{4.27}$$

(see Fig. 12) and

$$d(\mathscr{C}_\lambda^*, H_{p(1)}), d(\mathscr{C}_\xi^*, H_{p(2)}) \le D_9 k. \tag{4.28}$$

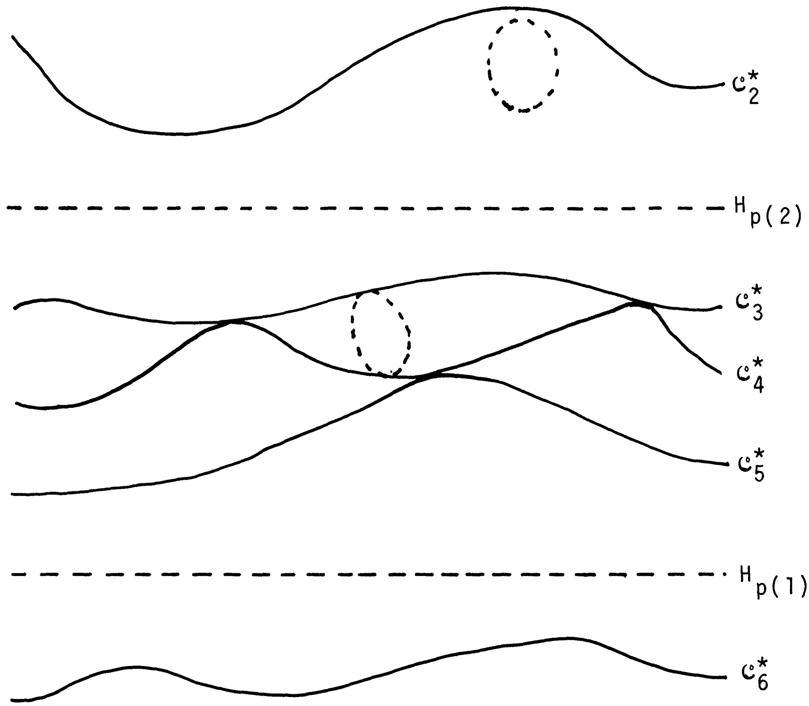

FIG. 12 $\lambda = 3, \xi = 5$ in this illustration. The dashed circuits are of class I.

If $\xi = \tau$, then (4.25) does not hold, but if $d(\mathscr{C}_\tau^*, \Delta_-) > D_9 k$ we can still choose $p(1)$ such that $H_{p(1)}$ lies between $\mathscr{C}_\tau^*$ and Δ_- and such that $H_{p(1)}$ is at distance at most $D_9 k$ from $\mathscr{C}_\tau^*$. If $d(\mathscr{C}_\tau^*, \Delta_-) \le D_9 k$, then we take $p(1) = -1$. As remarked before, (4.19) will still hold for this choice of $p(1)$ and $S^*(1) = \varnothing$. Similarly, if $\lambda = 1$ we can still choose $p(2)$ such that $H_{p(2)}$ is at distance at most $D_9 k$ from $\mathscr{C}_1^*$ and such that (4.19) holds. For simplicity we restrict ourselves for the remainder of the proof to the case $1 < \lambda \le \xi < \tau$ and show that (4.17), and hence (4.19), holds.

To prove (4.17) let ϕ be a path on $\mathscr{G}_0$ in B_1 (with $\mathscr{G}_0$ as defined after Lemma (3.4) with B_1 for B) from the center v of a plaquette π^* in ΔB_1 to $\Delta_- = [-\frac{1}{2}, K + \frac{1}{2}] \times [-\frac{1}{2}, L + \frac{1}{2}] \times \{0\}$, where π^* contains an edge of $\mathscr{C}_\xi^*$ and $\mathring{\pi}^* \subset \mathscr{C}_{\xi-}^*$. By definition of $\mathscr{C}_{\xi-}^*$ we can choose ϕ so that it does not intersect $\mathscr{C}_\xi^*$. However, by (4.27) the initial point v of ϕ—which lies in the plaquette π^* adjacent to $\mathscr{C}_\xi^*$—lies above $H_{p(1)}$ while the final point on Δ_- lies below $H_{p(1)}$. Thus ϕ intersects $H_{p(1)}$, for the first time in x, say. Denote the piece of ϕ from v to x by ψ. ψ lies in $\mathscr{C}_{\xi-}^*$ but above $H_{p(1)}$ and, again by (4.27), does not intersect any $\mathscr{C}_j^*$ with $1 \le j \le \tau$. ψ may intersect a circuit $\mathscr{C}_j^*$ of class I (with $j > \tau$). However, none of these circuits intersect $H_{p(1)}$

(because $V_{p(1)} = 0$) and hence each $\mathscr{C}_j^*$ of class I lies entirely above or entirely below $H_{p(1)}$. Thus $x \in \mathscr{C}_j^*(\text{ext})$ for each $\mathscr{C}_j^*$ of class I. Also, the initial point of ψ, the center of π^*, lies in $\mathscr{C}_j^*(\text{ext})$ for each $\mathscr{C}_j^*$ of class I; this was shown for any plaquette $\pi^* \subset \Delta B_1$ which has an edge in common with some circuit of class II right after (3.16). Now let w be the last vertex of $\mathscr{G}_0$ on ψ; w is the center of some plaquette $\pi_1^* \subset \Delta B_1$ which has an edge f^* in common with $H_{p(1)}$. $x \in f^*$ and by the above, v, x and w lie in the same component of $\Delta B_1 \setminus \mathscr{C}_j^*$ for each j. By (3.16), π^* and π_1^* are therefore both faces of a $+$ cube or both faces of a $-$ cube. Since $\mathring{\pi}^* \subset \mathscr{C}_{\xi-}^*$ we see from (3.15) that they are both faces of a $-$ cube ($+$ cube) if ξ is odd (even). Thus, if ξ is odd, then (4.17) holds for $i = 1$. By (4.26), λ is then also odd and the same argument with $p(1)$ and Δ_- replaced by $p(2)$ and Δ_+ will establish (4.17) also for $i = 2$. If both λ and ξ are even we merely have to interchange the role of $p(1)$ and $p(2)$. Thus, we can always choose $p(i)$ such that (4.27), (4.28) and (4.19) hold.

It remains to show that $|p(2) - p(1)|$ is not too large if the $p(i)$ are chosen as above. This, however, is easy, since by (4.27), (4.28) and the definition of linked circuits,

$$\begin{aligned}|p(2) - p(1)| &\leq 2D_9k + \sum_{j=\lambda}^{\xi-1} d(\mathscr{C}_j^*, \mathscr{C}_{j+1}^*) + \sum_{j=\lambda}^{\xi} (\text{length of } \mathscr{C}_j^*)\\ &\leq (\tau + 2)D_9k + |\partial E_2^*|.\end{aligned}$$

Moreover, any circuit of class II must contain at least $2(K + L) \geq 2K$ edges of ∂E_2^* (since it must go all the way "around ΔB_1") so that

$$2K\tau \leq |\partial E_2^*| \leq D_8k. \tag{4.29}$$

Thus, for $\varepsilon < \frac{1}{2}$ and $D_{10} = (D_8 + 3)D_9$

$$|p(2) - p(1)| \leq (\tau + 3)D_9k \leq (D_8 + 3)D_9k = D_{10}k. \tag{4.30}$$

Thus, we have shown that if E_2^* with the properties in (4.15) exists, then we can find a cut E_3^* which separates the bottom from the top of

$$[0, K] \times [0, L] \times [\min(p(1), p(2)), \max(p(1), p(2)) + 1]$$

for some $p(1)$, $p(2)$ satisfying (4.30). In addition E_3^* satisfies (4.20) and (4.21). Thus, if we discard a set of probability at most $mD_6 epx - D_7kl$ (cf. (4.22)), then outside this set

$$\begin{aligned}&V(E_3^*) \leq V(E_2^*) + \varepsilon kl \leq (\nu - 4\varepsilon)kl \leq (\nu - 3\varepsilon)KL,\\ &|\partial E_3^*| \leq |\partial E_2^*| \leq D_8k \leq 2D_8K.\end{aligned}$$

If $qk \le \min(p(1), p(2)) < (q+1)k$, and hence

$$\max(p(1), p(2)) \le (q + D_{10} + 1)k,$$

then any such cut E_3^* also separates bottom from top in

$$[0, K] \times [0, L] \times [qk, (q + D_{10} + 1)k].$$

Thus, if we choose $D_6 \ge C_1$, $D_7 \le C_2$ and $D_3 = 2D_{10} + 2$, then by virtue of (4.15) and (4.22), for large k and l we have

$$\begin{aligned}P\{&\text{there exist } -1 \le q \le m/k \text{ and a cut-set of plaquettes } E_3^* \text{ which} \\ &\text{separates bottom from top in } [0, K] \times [0, L] \times [qk, qk + D_3K] \\ &\text{with } V(E^*) \le (\nu - 3\varepsilon)KL \text{ and } |\partial E_3^*| \le 2D_8K\} \\ &\ge \tfrac{1}{3}\{f - 6mD_6\exp(-D_7kl)\}.\end{aligned}$$

(4.12) follows immediately if we take into account that there are at most $k^{-1}m + 2$ possible values for q, and use translation invariance to take a cut E_3^* in $[0, K] \times [0, L] \times [qk, qk + D_3K]$ to a cut E_1^* in $B(K, L, D_3K)$. ■

(4.31) Lemma. *Let $0 < \varepsilon, \delta < \nu/20$ and $m \ge k \ge l \ge 1$ be such that*

$$f(k, l, m, \varepsilon) \le \delta/\nu$$

(cf. (4.3) for f). There exist constants $2 \le C_6 < \infty$, $0 < C_7 < \infty$ (which depend only on F) such that for $k, l \ge C_6\delta^{-1}$, $k^{-1}r_1 \ge C_6\delta^{-1}$ and $l^{-1}r_2 \ge C_6\delta^{-1}$ one has

$$f(r_1, r_2, m, (\varepsilon + \delta)) \le \exp\left(-C_7\frac{r_1r_2}{kl}\delta\right). \tag{4.32}$$

Also, for $k \ge C_6\delta^{-1}$, $k^{-1}r_1 \ge C_6\delta^{-1}$ one has

$$f(r_1, l, m, (\varepsilon + \delta)) \le \exp\left(-C_7\frac{r_1}{k}\delta\right). \tag{4.33}$$

Proof. We prove (4.32); the proof of (4.33) is similar. Let p_i, $i = 1, 2$, be such that

$$\begin{aligned}p_1(k+1) &\le r_1 < (p_1 + 1)(k+1), \\ p_2(l+1) &\le r_2 < (p_2 + 1)(l+1).\end{aligned}$$

Then $[0, r_1] \times [0, r_2]$ contains the disjoint squares

$$S(i, j) := [i(k+1), i(k+1) + k] \times [j(l+1), j(l+1) + l],$$

$0 \le i < p_1$, $0 \le j < p_2$. If E^* separates the bottom from the top in

$$[0, r_1] \times [0, r_2] \times [0, m],$$

then E^* contains cuts $E^*(i, j)$ which separate the bottom from the top in $S(i, j) \times [0, m]$, $0 \le i < p_1$, $0 \le j < p_2$. Thus, if we write

$$\Psi(i, j) = \min\{V(D^*) : D^* \text{ is a cut which separates the bottom from the top in } S(i, j) \times [0, m]\},$$

then for any (F_0, F_m)-cut E^* in $[0, r_1] \times [0, r_2] \times [0, m]$,

$$V(E^*) \ge \sum_{\substack{0 \le i < p_1 \\ 0 \le j < p_2}} \Psi(i, j),$$

and the $\Psi(i, j)$ are i.i.d. (since the $S(i, j)$ are disjoint). Consequently, for any $\lambda > 0$, the left hand side of (4.32) is bounded by

$$\begin{aligned}
&P\left\{\sum_{\substack{0 \le i < p_1 \\ 0 \le j < p_2}} \Psi(i, j) \le r_1 r_2 (\nu - 5\varepsilon - 5\delta)\right\} \\
&\quad \le \exp(\lambda r_1 r_2(\nu - 5\varepsilon - 5\delta))\left[Ee^{-\lambda\Psi(0,0)}\right]^{p_1 p_2} \\
&\quad \le \exp\{\lambda(p_1 + 1)(p_2 + 1)(k + 1)(l + 1)(\nu - 5\varepsilon - 5\varepsilon)\} \\
&\qquad \cdot \left[f(k, l, m, \varepsilon) + (1 - f(k, l, m, \varepsilon))e^{-\lambda kl(\nu - 5\varepsilon)}\right]^{p_1 p_2} \\
&\quad \le \left[e^{-4\delta\lambda kl} + \frac{\delta}{\nu}\left(e^{\nu\lambda kl} - e^{-4\delta\lambda kl}\right)\right]^{p_1 p_2} \\
&\quad \le \left[e^{-4\delta\lambda kl - \delta/\nu} + \frac{\delta}{\nu}e^{\nu\lambda kl}\right]^{p_1 p_2},
\end{aligned}$$

as soon as

$$(p_1 + 1)(p_2 + 1)(k + 1)(l + 1)(\nu - 5\varepsilon - 5\delta) \le p_1 p_2 kl(\nu - 5\varepsilon - 4\delta).$$

The lemma now follows by taking $\lambda > 0$ such that

$$e^{\nu\lambda kl} = 4e^{-4\delta\lambda kl - \delta/\nu}.$$

(4.34) PROPOSITION. *Let $0 < \varepsilon \le \varepsilon_0/4$ and $m \ge k \ge l \ge D_4(4\varepsilon)$. Assume further that there exist some*

$$C_6\varepsilon^{-1} \le s_1 \le \varepsilon k(2C_6)^{-1}, \quad C_6\varepsilon^{-1} \le s_2 \le \varepsilon l(2C_6)^{-1}$$

for which

$$f(s_1, s_2, D_3(4\varepsilon)k, \varepsilon) \le \varepsilon/\nu. \tag{4.35}$$

Then

$$f(k, l, m, 4\varepsilon) \le \frac{9m}{k}\exp\left(-C_8\varepsilon\frac{kl}{s_1 s_2}\right) + 6mD_6(4\varepsilon)\exp(-D_7(4\varepsilon)kl). \tag{4.36}$$

Proof. Obviously $f(k, l, m, \varepsilon)$ is decreasing in ε. Moreover, any (F_0, F_m)-cut in $[0, K] \times [0, L] \times [0, m]$ of value $\le (\nu - 3\varepsilon)KL$ contains a cut which separates the bottom from the top in

$$[0, K'] \times [0, L'] \times [0, m]$$

with value $\le (\nu - 3\varepsilon)KL \le (\nu - 3\varepsilon)KL(K'L')^{-1}K'L'$, whenever $K' \le K$, $L' \le L$. In particular, in view of (4.11), the left hand side of (4.12) is at most

$$f\left((1-\varepsilon^2)k, (1-\varepsilon^2)l, D_3K, \tfrac{1}{2}\varepsilon\right)$$

provided $(\nu - 3\varepsilon)(1-\varepsilon^2)^{-2} \le (\nu - \frac{5}{2}\varepsilon)$. The latter inequality may be assumed without loss of generality (if necessary, reduce ε_0). Finally it is easy to see that $f(k, l, m, \varepsilon)$ is increasing in m. Thus, the left hand side of (4.12) is bounded above by

$$f\left((1-\varepsilon^2)k, (1-\varepsilon^2)l, D_3k, \tfrac{1}{2}\varepsilon\right).$$

Now replace ε by 4ε and write r_1 for $(1 - 16\varepsilon^2)k$ and r_2 for $(1 - 16\varepsilon^2)l$. This yields

$$\begin{aligned} f(k, l, m, 4\varepsilon) \le {} & \frac{9m}{k} f(r_1, r_2, D_3(4\varepsilon)k, 2\varepsilon) \\ & + 6mD_6(4\varepsilon)\exp(-D_7(4\varepsilon)kl). \end{aligned} \tag{4.37}$$

Finally, estimate $f(r_1, r_2, D_3k, 2\varepsilon)$ by means of (4.32) with $\delta = \varepsilon$ and $m = D_3k$ to obtain (4.36). (Note that (4.35) is precisely the condition needed in Lemma 4.31 when $\delta = \varepsilon$, $m = D_3k$.) ∎

This completes Step (ii) and we next carry out *Step (iii)*. The aim is to patch together p^2 translates of cuts with the properties listed in (4.12) to produce a cut in

$$[0, p(K+1)] \times [0, p(L+1)] \times [0, D_3K]$$

with a value not larger than $(\nu - \varepsilon)p^2KL$. To do this consider the problem of combining two cuts[6] E_1^* and E_2^*, separating the bottom from the top in

$$[0, K] \times [0, L] \times [0, D_3K] \quad \text{and} \quad [K+1, 2K+1] \times [0, L] \times [0, D_3K],$$

respectively. If

$$\partial E_2^* \cap \{x(1) = K + \tfrac{1}{2}\}$$

agrees with

$$\partial E_1^* \cap \{x(1) = K + \tfrac{1}{2}\},$$

then it is easy to combine E_1^* and E_2^* since they match at their common boundary. However, this is usually not the case and one must add plaquettes to $E_1^* \cup E_2^*$ to obtain a cut in

$$[0, 2K+1] \times [0, L] \times [0, D_3K].$$

In order not to have to add too many plaquettes we want

$$\partial E_1^* \cap \{x(1) = K + \tfrac{1}{2}\} \quad \text{and} \quad \partial E_2^* \cap \{x(1) = K + \tfrac{1}{2}\}$$

to be "close together" with a reasonable probability. We do this by approximating the "long" circuits in ∂E_i^* by a bounded number of sets, and by covering up the interiors of the remaining "short" circuits. The details follow.

Let E_1^* be a cut which separates bottom from top in

$$B_1 := [0, K] \times [0, L] \times [0, D_3K]$$

with $V(E_1^*) \le (\nu - 3\varepsilon)KL$ and $|\partial E_1^*| \le D_5K$. Let $\eta = \eta(\varepsilon, F)$ be a small number which satisfies

$$0 < \eta < 1, \quad 21\eta^{1/2}D_5(\varepsilon, F) \le \varepsilon^2, \tag{4.38}$$

and subdivide the vertical boundary ΔB_1 of

$$[-\tfrac{1}{2}, K + \tfrac{1}{2}] \times [-\tfrac{1}{2}, L + \tfrac{1}{2}] \times [0, D_3K]$$

into rectangles approximately of size ηK by ηK. For simplicity assume that ηK is an integer which divides $(K+1)$, $(L+1)$ and D_3K, so that the above

---

[6] The subscripts of the B's and the cuts E^* here are unrelated to the previous subscripts.

rectangles can be taken as squares with perimeter on the family of lines

$$\begin{gathered}\{a\} \times \mathbf{R} \times \{i\eta K\}, \quad \{a\} \times \{j\eta K - \tfrac{1}{2}\} \times \mathbf{R},\\ \mathbf{R} \times \{b\} \times \{i\eta K\} \text{ and } \{h\eta K - \tfrac{1}{2}\} \times \{b\} \times \mathbf{R},\\ a = -\tfrac{1}{2} \text{ or } K + \tfrac{1}{2}, \quad b = -\tfrac{1}{2} \text{ or } L + \tfrac{1}{2}\end{gathered}$$

and

$$0 \le i \le \eta^{-1} D_3, \quad 0 \le j \le (\eta K)^{-1}(L+1), \quad 0 \le h \le (\eta K)^{-1}(K+1).$$

Let ∂E_1^* be decomposed into circuits $\mathscr{C}_1^*, \dots, \mathscr{C}_\rho^*$ as in Lemma 3.10. $\mathscr{C}_1^*, \dots, \mathscr{C}_\tau^*$ are the circuits of class II, as before. Number the remaining circuits in such a way that $\mathscr{C}_{\tau+1}^*, \dots, \mathscr{C}_\sigma^*$ are all the circuits of class I which contain at least $\eta^{3/2}K$ edges of ∂E_1^*. Exactly as in (4.29) one sees that

$$\sigma \le \frac{D_5 K}{\eta^{3/2} K} = \eta^{-3/2} D_5. \tag{4.39}$$

We next define the *type* of E_1^*. The type of E_1^* specifies τ, σ, and for each of the above $\eta K \times \eta K$ squares S, and $j \le \tau$, whether $\mathscr{C}_j^*$ intersects S, or if not, whether S belongs to $\mathscr{C}_{j-}^*$ or $\mathscr{C}_{j+}^*$. Furthermore the type specifies how many of the circuits $\mathscr{C}_j^*$, with $\tau < j \le \sigma$, S intersects, and for how many $\tau < j \le \sigma$, $S \subset \mathscr{C}_j^*(\text{int})$. Thus, we can think of the type as a large vector (with a variable number of components). The first two components are τ and σ, respectively; then there is a component corresponding to each $\eta K \times \eta K$ square S and each $j \le \tau$, and finally for each such S two integer components. The component corresponding to S and $j \le \tau$ can take the values $-1, 0, +1$; these values indicate that $S \subset \mathscr{C}_{j-}^*$, S intersects $\mathscr{C}_j^*$ or $S \subset \mathscr{C}_{j+}^*$, respectively. The last two integer components corresponding to S give the number of j, $\tau < j \le \sigma$, for which S intersects $\mathscr{C}_j^*$ and $S \subset \mathscr{C}_j^*(\text{int})$, respectively.

We need a few simple estimates. The first gives an upper bound for the number of possible different types. It is immediate from (4.39) and the fact that there are at most

$$(\eta K)^{-2}(2K + 2L + 4) D_3 K$$

squares, that for $L \le 2K$ the number of types is bounded by some $D_{11} = D_{11}(\eta, \varepsilon, F)$. The second estimate is for the number of plaquettes in $\overline{\mathscr{C}_j^*(\text{int})}$ for some $j > \sigma$. Denote by F_1^* the collection of plaquettes in ΔB_1 which lie in $\overline{\mathscr{C}_j^*}(\text{int})$ for some $\mathscr{C}_j^*$ which contains fewer than $\eta^{3/2}K$ edges. If $|F_1^*|$ denotes the number of plaquettes in F_1^*, and $|\mathscr{C}_j^*|$ denotes the number of edges in $\mathscr{C}_j^*$, then

$$|F_1^*| \le \sum_{j>\sigma} |\mathscr{C}_j^*|^2 \le \eta^{3/2} K \sum_{j>\sigma} |\mathscr{C}_j^*| \le \eta^{3/2} K |\partial E_1^*| \le \eta^{3/2} D_5 K^2. \tag{4.40}$$

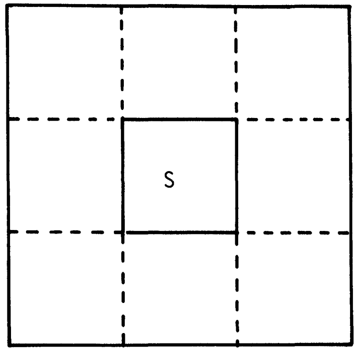

FIG. 13 S is the small square at the center, T the large square. The dashed lines delineate the squares surrounding S.

We need a similar estimate for $|G_1^*|$, the number of plaquettes in G_1^*, where G_1^* is the union of the squares S in ΔB_1 which intersect some $\mathscr{C}_j^*$ with $j \le \sigma$. We claim that

$$|G_1^*| \le 20\eta^{1/2} D_5 K^2. \tag{4.41}$$

To see this, let S be one of the $\eta K \times \eta K$ squares in ΔB_1, and assume that for a given j, $\mathscr{C}_j^*$ contains a point in S. S is surrounded by at most 8 of the $\eta K \times \eta K$ squares (see Fig. 13). Let T be the union of S and its 8 surrounding squares. It takes at least ηK edges of $\mathscr{C}_j^*$ to connect a point in S with the boundary or exterior of T. Thus once $\mathscr{C}_j^*$ enters S it cannot enter any but the nine squares of T with the next $\eta K - 1$ edges, so that $\mathscr{C}_j^*$ intersects at most $1 + 9(\eta K - 1)^{-1}|\mathscr{C}_j^*|$ of the $\eta K \times \eta K$ squares, each of which contains $\eta^2 K^2$ plaquettes. Thus, for large K,

$$\begin{aligned} |G_1^*| &\le \eta^2 K^2 \sum_{j \le \sigma} \left\{1 + 9(\eta K - 1)^{-1} |\mathscr{C}_j^*|\right\} \\ &\le \eta^2 K^2 \left(\sigma + \frac{18}{\eta K} |\partial E_1^*|\right) \\ &\le \eta^2 \left(\eta^{-3/2} + \frac{18}{\eta}\right) D_5 K^2 \end{aligned}$$

(see (4.39)). This proves (4.41).

Finally, let $H_1^* = F_1^* \cup G_1^*$. Note that H_1^* consists of plaquettes in ΔB_1, and that their values are independent of all the plaquettes with interior in $(-\frac{1}{2}, K + \frac{1}{2}) \times (-\frac{1}{2}, L + \frac{1}{2}) \times [\frac{1}{2}, D_3 K - \frac{1}{2}]$. In particular, given E_1^*, and hence H_1^*, the conditional distribution of $V(H_1^*)$ is just that of

$$v_1 + \cdots + v_{|H_1^*|}$$

where $v_1, v_2, \ldots$ are i.i.d. with distribution F. By (4.40), (4.41) and (4.38),

$$|H_1^*| \leq \left(\eta^{3/2} + 20\eta^{1/2}\right) D_5 K^2 \leq \varepsilon^2 K^2. \tag{4.42}$$

By the usual argument, for sufficiently small ε and any E_1^* with $|\partial E_1^*| \leq D_5 K$, we then have

$$\begin{aligned} P\{V(H_1^*) \geq \varepsilon K^2 | E_1^*\} &\leq e^{-\gamma \varepsilon K^2} \left\{ \int e^{\gamma x}\, dF(x) \right\}^{|H_1^*|} \\ &\leq \exp\left(-\tfrac{1}{2}\gamma\varepsilon K^2\right). \end{aligned} \tag{4.43}$$

(4.44) LEMMA. *Let E_1^* (E_2^*) be a cut which separates bottom from top in*

$$B_1 = [0, K] \times [0, L] \times [0, D_3 K]$$
$$\left(B_2 = [K+1, 2K+1] \times [0, L] \times [0, D_3 K]\right).$$

Let H_1^* be as above, and let H_2^* be the corresponding set of plaquettes for E_2^* in the vertical boundary of

$$\left[K + \tfrac{1}{2}, 2K + \tfrac{3}{2}\right] \times \left[-\tfrac{1}{2}, L + \tfrac{1}{2}\right] \times [0, D_3 K].$$

Finally, assume that the reflection of E_2^* in the plane $\{x(1) = K + \frac{1}{2}\}$ has the same type as E_1^*. Then $E_1^* \cup E_2^* \cup H_1^* \cup H_2^*$ separates the bottom from the top in $[0, 2K+1] \times [0, L] \times [0, D_3 K]$.

Proof. Let ϕ be any path on $\mathbf{Z}^3$ in $[0, 2K+1] \times [0, L] \times [0, D_3 K]$ connecting

$$F_0 := [0, 2K+1] \times [0, L] \times \{0\}$$

and

$$F(D_3 K) := [0, 2K+1] \times [0, L] \times \{D_3 K\}.$$

We call a cube $v + U$ with $v \in B_1 \cup B_2$ a + cube (− cube) if v can be connected by a path on $\mathbf{Z}^3$ from v to $F(D_3 K)(F_0)$ which lies *entirely in one B_i* but does not intersect the corresponding E_i^*, $i = 1, 2$. Then ϕ starts in a − cube and ends in a + cube, and therefore contains an edge e which connects the center v_1 of a − cube to the center v_2 of a + cube. If v_1 and v_2 lie in the same B_i, then e intersects E_i^*, since E_i^* separates the bottom from the top in B_i. Next consider the case in which v_1 and v_2 lie in different B_i. For the sake of argument, let $v_1 \in B_1$ and $v_2 \in B_2$. Then $v_1(1) = K$, $v_2(1) = K + 1$ and e is associated to a plaquette π^* in $\{x(1) = K + \frac{1}{2}\}$. π^* is the common face of

$v_1 + U$ and $v_2 + U$. $\pi^* \subset S$ for some $\eta K \times \eta K$ square S. It suffices to show that $S \subset H_1^* \cup H_2^*$, for then any path ϕ from F_0 to $F(D_3K)$ intersects $E_1^* \cup E_2^* \cup H_1^* \cup H_2^*$. Equivalently, it suffices to prove that $S \not\subset H_1^* \cup H_2^*$ implies that $v_1 + U$ and $v_2 + U$ are both $+$ cubes or both $-$ cubes. In turn we shall prove this by showing that if $S \not\subset H_1$, then we can read off from the type of E_1^* whether $v_1 + U$ is a $+$ cube or a $-$ cube. If also $S \not\subset H_2$, then the parity of $v_2 + U$ can be read off from the type of E_2^*, and as will be clear from the next paragraph, the fact that the reflection of E_2^* has the same type as E_1^* forces $v_1 + U$ and $v_2 + U$ to have the same parity.

Suppose then that $S \not\subset H_1^*$, and a fortiori $S \not\subset G_1^*$. Then S does not intersect any of the circuits $\mathscr{C}_j^*$ in the decomposition of ∂E_1^*, with $|\mathscr{C}_j^*| \geq \eta^{3/2}K$. The type of E_1^* determines for how many $j \leq \tau$ we have $S \subset \mathscr{C}_{j+}^*$, and for how many $\tau < j \leq \sigma$ we have $S \subset \mathscr{C}_j^*(\text{int})$. Say these numbers are ν_1 and ν_2. Let $\mathscr{G}_0$ be the graph defined after (3.4) (with $B = B_1$ now) and let ψ be a path on $\mathscr{G}_0$ from a point on $[-\frac{1}{2}, K + \frac{1}{2}] \times [-\frac{1}{2}, L + \frac{1}{2}] \times \{0\}$ to the center of π^*. As we saw just before (3.16), whenever ψ crosses one of the $\mathscr{C}_j^*$, then ψ goes from a face of a $+$ cube of B_1 to the face of a $-$ cube, or vice versa. ψ starts in $\mathscr{C}_{j-}^*$ for each $j \leq \tau$, but ends in $\mathscr{C}_{j+}^*$ for ν_1 values of $j \leq \tau$. Each $\mathscr{C}_j^*$ with $j \leq \tau$ for which the endpoint of ψ lies in $\mathscr{C}_{j-}^*$ ($\mathscr{C}_{j+}^*$) has been crossed an even (odd) number of times by ψ. Similarly ψ starts in $\mathscr{C}_j^*(\text{ext})$ for each $j > \tau$, and thus ψ crosses each such $\mathscr{C}_j^*$ an even (odd) number of times if the endpoint of ψ lies in $\mathscr{C}_j^*(\text{ext})$ ($\mathscr{C}_j^*(\text{int})$). But if $\pi^* \notin F_1^*$, then the endpoint of ψ lies in $\mathscr{C}_j^*(\text{ext})$ for each $j > \sigma$ and lies in $\mathscr{C}_j^*(\text{int})$ for exactly ν_2 values of $\tau < j \leq \sigma$. Thus, ψ crosses the $\mathscr{C}_j^*$'s ($\nu_1 + \nu_2 +$ even integer) many times and π^* is the face of a $(-1)^{\nu_1+\nu_2}$ cube of B_1. Thus $v_1 + U$ is a $(-1)^{\nu_1+\nu_2}$ cube. Exactly the same argument works for the mirror image of E_2^* in $\{x(1) = K + \frac{1}{2}\}$, which is also a cut in B_1. Since this mirror image has by assumption the same type as E_1^*, we find that if $S \not\subset H_2^*$, then π^* is also the face of a $(-1)^{\nu_1+\nu_2}$ cube (with the same ν_1, ν_2 as before) for

$$B_2 = [K + 1, 2K + 1] \times [0, K] \times [0, D_3K].$$

(Note that π^* lies in $\{x(1) = K + \frac{1}{2}\}$, and therefore does not change under the reflection.) ∎

(4.45) Proposition. *Let ε_0 and D_4 be as in* (4.10). *For $0 < \varepsilon \leq \varepsilon_0$ there exist constants $0 < D_i < \infty$, which depend on ε and F only, such that for $p \geq D_{12}$, $m \geq k \geq D_4$ and $(p + 1)k \leq r < (p + 2)k$*

$$(4.46) \quad P\{\tau((r, r)) \leq r^2(\nu - \varepsilon)\}$$
$$\geq \frac{1}{4}\left[\frac{k}{9D_{11}m}\{f(k, k, m, \varepsilon) - 6mD_6\exp(-D_7k^2)\}\right]^{p^2}.$$

Proof. Take $m \geq l = k$ in Lemma 4.10 and choose K, L such that (4.11) and (4.12) hold. If $\varepsilon_0 \leq \frac{1}{2}$ we then have $K/2 \leq L \leq 2K$, and as we saw before the number of possible types of cuts E_1^* which separate the bottom from the top in

$$B_1 = [0, K] \times [0, L] \times [0, D_3 K]$$

with $|\partial E_1^*| \leq D_5 K$ is then bounded by D_{11}. Thus, we can find a fixed type T such that

(4.47)

$$\begin{aligned}&P\{\text{there exists an } (F_0, F(D_3K))\text{-cut } E_1^* \text{ in } B_1 \text{ with } V(E_1^*) \leq (\nu - 3\varepsilon)KL,\\ &\quad |\partial E_1^*| \leq D_5 K \text{ and type of } E_1^* \text{ is } T\}\\ &\quad \geq \frac{k}{9D_{11}m}\{f(k, k, m, \varepsilon) - 6mD_6\exp(-D_7k^2)\}.\end{aligned}$$

Write $\Gamma(i, j)$ for the box

$$[i(K+1), i(K+1) + K] \times [j(L+1), j(L+1) + L] \times [0, D_3K], \qquad 0 \leq i, j < p.$$

Let $E^*(i, j)$ be a cut which separates the bottom from the top in $\Gamma(i, j)$ with $|\partial E^*(i, j)| \leq D_5 K$ and such that a "suitable" reflection and translate of $E_{i,j}^*$ has type T. How we should reflect and translate $E_{i,j}^*$ depends on the parity of i and j. If i is odd we reflect in the plane $\{x(1) = i(K+1) - \frac{1}{2}\}$ and if j is odd in the plane $\{x(2) = j(L+1) - \frac{1}{2}\}$; if i and j are odd we perform both reflections, but we do not carry out reflections for even i and/or j. The reflection $\tilde{E}^*(i, j)$ of $E^*(i, j)$ is a cut which separates the bottom from the top in the reflection $\tilde{\Gamma}(i, j)$ of $\Gamma(i, j)$. We translate $\tilde{\Gamma}(i, j)$ and $\tilde{E}^*(i, j)$ so that $\tilde{\Gamma}(i, j)$ coincides with $\Gamma(0, 0) = B_1$, and it is this translate of $\tilde{E}^*(i, j)$ which should have type T. Now let $H^*(i, j)$ correspond to $E^*(i, j)$ in exactly the same way as H_1^* to E_1^* (cf. lines preceding (4.42)). Then a repetition of the proof of Lemma 4.44 shows that

$$I^* := \bigcup_{0 \leq i, j < p} \{E^*(i, j) \cup H^*(i, j)\}$$

is a set of plaquettes in

$$[-\tfrac{1}{2}, p(K+1) - \tfrac{1}{2}] \times [-\tfrac{1}{2}, p(L+1) - \tfrac{1}{2}] \times [0, D_3 K]$$

which separates the bottom from the top in

$$B_0 := [0, p(K+1) - 1] \times [0, p(L+1) - 1] \times [0, D_3 K].$$

Now consider the set of plaquettes $I^* \cup J^*$, where J^* consists of all plaquettes in the vertical boundary Δ_0 of

$$[-\tfrac{1}{2}, p(K+1) - \tfrac{1}{2}] \times [-\tfrac{1}{2}, p(L+1) - \tfrac{1}{2}] \times [0, D_3K],$$

plus all plaquettes of the form

$$[a - \tfrac{1}{2}, a + \tfrac{1}{2}] \times [b - \tfrac{1}{2}, b + \tfrac{1}{2}] \times \{\tfrac{1}{2}\} \tag{4.48}$$

with

$$(a = -1 \text{ or } a \in [p(K+1), r-1]) \quad \text{and} \quad -1 \le b \le r-1 \tag{4.49}$$

or with

$$(b = -1 \text{ or } b \in [p(L+1), r-1]) \quad \text{and} \quad -1 \le a \le r-1. \tag{4.50}$$

These are the plaquettes between the square

$$[-\tfrac{3}{2}, r - \tfrac{1}{2}]^2 \times \{\tfrac{1}{2}\}$$

and the rectangle

$$[-\tfrac{1}{2}, p(K+1) - \tfrac{1}{2}] \times [-\tfrac{1}{2}, p(L+1) - \tfrac{1}{2}] \times \{\tfrac{1}{2}\}.$$

(See Fig. 14.) It is easy to see that $I^* \cup J^*$ separates the bottom from the top in

$$B := [-1, r-1]^2 \times [0, D_3K].$$

Indeed, any path ϕ on $\mathbf{Z}^3$ from the bottom to the top of B which goes upwards from a point $(a, b, 0)$ with a, b as in (4.49) or (4.50) must cross J^*

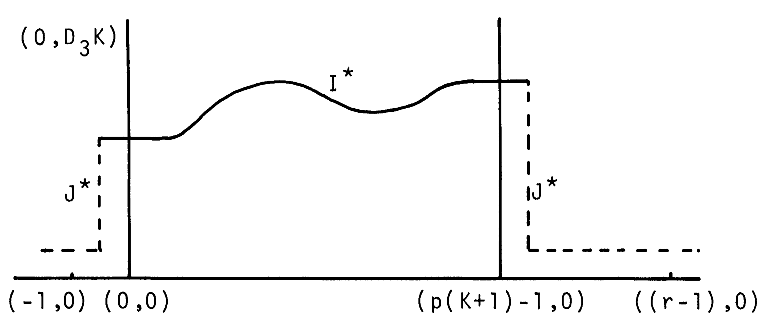

FIG. 14

immediately. Also, if ϕ contains a point in B_0, then ϕ cannot leave B_0 without hitting $\Delta_0 \subset J^*$. Finally ϕ cannot go from the bottom to the top of B_0, while staying in B_0, without hitting I^*. Thus $I^* \cup J^*$ indeed separates the bottom from the top of B.

To complete the proof we select a minimal cut E^* from $I^* \cup J^*$ which separates the bottom from the top in B. Since $I^* \cup J^*$ intersects the boundary of this box only in plaquettes of the form (4.48) with a or b equal to -1 or $r-1$, it follows that ∂E^* consists only of edges in the perimeter of $[-\frac{3}{2}, r-\frac{1}{2}]^2 \times \{\frac{1}{2}\}$ (cf. (3.6a)). Since E^* separates the bottom from the top of B one easily sees that E^* must contain *all* the plaquettes in (4.48)–(4.50) so that ∂E^* consists of all the edges in the perimeter of $[-\frac{3}{2}, r-\frac{1}{2}]^2 \times \{\frac{1}{2}\}$. Thus, apart from a translation by the vector $(-1,-1,0)$ E^* is one of the cuts in the inf in (1.11) (with $k = l = r$). Thus

$$P\{\tau(r,r) \le r^2(\nu - \varepsilon)\} \ge P\{V(E^*) \le r^2(\nu-\varepsilon)\}.$$

To estimate $V(E^*)$ observe that

$$\begin{aligned} V(E^*) &\le V(I^*) \cup V(J^*) \\ &\le \sum_{0 \le i,j<p} \{V(E^*(i,j)) + V(H^*(i,j))\} + V\Big(J^* \setminus \bigcup_{i,j} H^*(i,j)\Big). \end{aligned}$$

This will be at most $r^2(\nu-\varepsilon)$ provided

$$\begin{aligned} &V(E^*(i,j)) \le KL(\nu - 3\varepsilon) \quad \text{for } 0 \le i, j < p, \\ &V(\cup H^*(i,j)) \le \varepsilon p^2 K^2 \\ &V(J^* \setminus H^*(i,j)) \le \varepsilon r^2. \end{aligned}$$

Conditionally on all $E^*(i,j)$, $V(\cup H^*(i,j))$ has the distribution of $v_1 + \cdots + v_t$ with $t = |\cup H^*(i,j)| \le p^2\varepsilon^2K^2$ (by (4.42)). Exactly as in (4.43) this implies

$$\begin{aligned} P\{V(\cup H^*(i,j) > \varepsilon p^2K^2 | E^*(i,j) \quad &\text{with} \quad |\partial E^*(i,j)| \\ &\le D_5 K, 0 \le i,j<p\} \le \exp\left(-\tfrac{1}{2}\gamma\varepsilon p^2 K^2\right). \end{aligned}$$

Also, for $p \ge D_{12} := 8\varepsilon^{-2}(D_3+2)$, $J^* \setminus \cup H^*(i,j)$ contains at most

$$\begin{aligned} &2p(K+L+2)D_3K + 2(r+1)(r - p(K+1) + r - p(L+1) + 2) \\ &\quad \le \frac{8}{p} D_3 r^2 + 8r\{(p+2)k - p(1-\varepsilon^2)k\} \le 20\varepsilon^2 r^2 \end{aligned}$$

plaquettes, and all these plaquettes are outside I^*. Thus by an estimate similar to (4.43),

$$P\{V(J^*\setminus \cup H^*(i,j)) \le \varepsilon r^2 | E^*(i,j), H^*(i,j), 0 \le i,j < p\} \ge \tfrac{1}{2}.$$

Combining all these estimates we find

$$\begin{aligned}
P\{\tau((r,r)) \le r^2(\nu-\varepsilon)\} &\ge P\Big\{\text{there exists a cut } E^* \text{ which separates}\\
&\quad \text{bottom from top in } B \text{ with } \partial E^* \text{ consisting of the edges}\\
&\quad \text{in the perimeter of } \left[-\tfrac{3}{2}, r-\tfrac{1}{2}\right]^2 \times \left\{\tfrac{1}{2}\right\} \text{ and } V(E^*) \le r^2(\nu-\varepsilon)\Big\}\\
&\ge \tfrac{1}{2}P\Big\{\text{For } 0 \le i,j<p, \text{ there exists a cut } E^*(i,j) \text{ which separates the}\\
&\qquad \text{bottom from the top in } \Gamma(i,j) \text{ with } V(E^*(i,j)) \le (\nu-3\varepsilon)KL,\\
&\qquad \text{and such that the proper reflection and translate of}\\
&\qquad E^*(i,j) \text{ is of type } T, \text{ and } V(\cup H^*(i,j)) \le \varepsilon p^2K^2\Big\}\\
&\ge \tfrac{1}{2}\{g(k,m,\varepsilon)\}^{p^2}\left\{1-\exp\left(-\tfrac{1}{2}\gamma\varepsilon p^2K^2\right)\right\},
\end{aligned}$$

where $g(k, m, \varepsilon)$ stands for the probability in the left hand side of (4.47). The lemma now follows from (4.47). ∎

The reader will be relieved to know that this completes Step (iii) and that Step (iv) is short. To begin *Step (iv)* we establish (4.5) in the next lemma.

(4.51) LEMMA. *For $\varepsilon > 0$ there exists a constant $D_{13} = D_{13}(\varepsilon, F) < \infty$ such that for all $M \ge 4$ there exists an $r \in [M^{1/2}, M]$ with*

$$P\{\tau((r,r)) \le r^2(\nu-\varepsilon)\} \le \frac{D_{13}}{\log r}. \tag{4.52}$$

Proof. We already saw in the proof of Lemma 4.7 (cf. (4.8)) that

$$\begin{aligned}
\tau([0,2^{k+1})\times[0,2^{k+1})) \le{}& \tau([0,2^k)\times[0,2^k)) + \tau([2^k,2^{k+1})\times[0,2^k))\\
&+\tau([0,2^k)\times[2^k,2^{k+1}))\\
&+\tau([2^k,2^{k+1})\times[2^k,2^{k+1})).
\end{aligned}$$

Moreover, the four terms in the right hand side are independent, and each has the distribution of $\tau([0,2^k)\times[0,2^k)) = \tau(2^k-1, 2^k-1)$. Exactly as in [15], Section 2.4, it follows that

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{1}{2^{4k}}\sigma^2(\tau(2^k-1,2^k-1)) &= \sum_{k=0}^{\infty}\frac{1}{2^{4k}}\sigma^2(\tau([0,2^k)\times[0,2^k)))\\
&\le \tfrac{4}{3}E\{\tau^2([0,1)\times[0,1))\}
\end{aligned}$$

and

$$E\tau(2^k - 1, 2^k - 1) = E\tau([0, 2^k) \times [0, 2^k)) \geq \nu 2^{2k}.$$

Thus, for all M,

$$\sum_{M^{1/2} < 2^k \leq M} 2^{-4k}\sigma^2(\tau(2^k - 1, 2^k - 1)) \leq C_8,$$

and hence for some $M^{1/2} + 1 < 2^k \leq M$,

$$\sigma^2(\tau(2^k - 1, 2^k - 1)) \leq 2^{4k}\frac{C_9}{\log 2}(\log M - \log M^{1/2} - 1)^{-1}.$$

Take $r = 2^k - 1$ and apply Chebyshev's inequality to obtain the lemma.

Proof of Theorems 2.7 and 2.10. L^1 convergence is trivial once one has convergence w.p.1. Theorem 2.10 plus Lemma 4.7 and the fact that $\theta(n_1, n_2)$ has the same distribution as $\theta(n_2, n_1)$ will imply (2.9). Thus, it suffices to prove 2.11. In turn, in Theorem 2.10 we may restrict ourselves to $\theta = \beta$ because

$$\beta(k, l) \leq \sigma(k, l), \tau(k, l), \alpha(k, l).$$

The inequalities $\beta(k, l) \leq \sigma(k, l), \tau(k, l)$ are obvious because the cuts in the definitions of σ and τ must be connected (cf. (3.17)). To see that also $\beta(k, l) \leq \alpha(k, l)$ we show that in (1.12) we may also restrict the inf to connected sets E^*. Indeed, if ∂E^* consists of the edges on the perimeter of

$$[-\tfrac{1}{2}, k + \tfrac{1}{2}] \times [-\tfrac{1}{2}, l + \tfrac{1}{2}] \times \{\tfrac{1}{2}\},$$

then ∂E^* is connected and therefore belongs to a single component, E_0^* say, of E^*. One easily sees that $\partial E_0^* = \partial E^*$. Thus by (3.1a), E_0^* already separates $-\infty$ from $+\infty$ over R. Thus we may replace E^* by E_0^*, and $\beta(k, l) \leq \alpha(k, l)$.

From now on we restrict ourselves to $\theta = \beta$. For the time being we restrict ourselves to $k = l$. Fix $0 < \varepsilon \leq \varepsilon_0/4$ and then $p = p(\varepsilon) \geq D_{12}(\varepsilon)$. Then for $m \geq k \geq D_4$ and $(p + 1)k \leq r < (p + 2)k$ (4.46) applies. If r also satisfies (4.52) then we obtain

$$f(k, k, m, \varepsilon) \leq \frac{9D_{11}m}{k}\left\{\frac{4D_{13}}{\log r}\right\}^{p-2} + 6mD_6\exp(-D_7k^2). \tag{4.53}$$

In view of (4.51), for all sufficiently large M we can find an $r \in [M^{1/2}, M]$ and a k in the interval

$$\left[(p + 2)^{-1}M^{1/2}, (p + 1)^{-1}M\right] \tag{4.54}$$

such that (4.53) holds for all $m \geq k$ (first choose r, then k). This is the "moderately good" estimate which we now improve by several iterations of (4.34). We start with any M and a k_0 in the interval (4.54) such that (4.53) holds for this k_0 and $m \geq k_0$. We set

$$k_1 = k_0(\log k_0)^{1/2p^2} \quad \text{and} \quad m = m_0 = D_3(4\varepsilon)k_1.$$

Then for some $M_1(\varepsilon)$, $M \geq M_1$ and $k_0 \geq (p+2)^{-1}M_1^{1/2}$ will imply $D_3(4\varepsilon)k_1 \geq k_0$, so that (4.53) gives

$$\begin{aligned} f(&k_0, k_0, D_3(4\varepsilon)k_1, \varepsilon) \\ &\leq 9D_{11}D_3(4\varepsilon)(\log M)^{1/2p^2}\left\{\frac{8D_{13}}{\log M}\right\}^{p^{-2}} \\ &\quad + 6D_3(4\varepsilon)D_6k_0(\log k_0)^{1/2p^2}\exp\left(-D_7k_0^2\right) \\ &\leq \frac{\varepsilon}{\nu}. \end{aligned}$$

(For the last inequality we may have to raise $M_1(\varepsilon)$.) We therefore have (4.35) for $s_1 = s_2 = k_0$, $k = k_1$. From (4.36) with $k = l = k_1$ we conclude that

$$\begin{aligned} f(k_1, k_1, m, 4\varepsilon) \leq \frac{9m}{k_1}&\exp\left(-C_8\varepsilon(\log k_0)^{p^{-2}}\right) \\ &+ 6mD_6(4\varepsilon)\exp\left(-D_7(4\varepsilon)k_1^2\right). \end{aligned}$$

In order to iterate (4.34) we want to use the last inequality to obtain (4.35) with $s_1 = s_2 = k_1$, and ε replaced by 4ε. The right hand side should therefore be no more than $4\varepsilon/\nu$. This allows us to take m of the order $k_1(\log k_0) \sim k_1(\log k_1)$. Indeed, substitution of $m_1 = D_3(16\varepsilon)k_1\log k_1$ for m gives

$$\begin{aligned} f(&k_1, k_1, D_3(16\varepsilon)k_1(\log k_1), 4\varepsilon) \\ &\leq 9D_3(16\varepsilon)\log k_1 \exp\left(-C_8\varepsilon(\log k_0)^{p^{-2}}\right) \\ &\quad + 6D_3(16\varepsilon)D_6(4\varepsilon)k_1\log k_1 \exp\left(-D_7(4\varepsilon)k_1^2\right) \leq \frac{4\varepsilon}{\nu}, \end{aligned}$$

provided $M \geq M_2$ for some $M_2 = M_2(\varepsilon) \geq M_1(\varepsilon)$. This gives (4.35) for $s_1 = s_2 = k_1$, k replaced by $k_2 := k_1(\log k_1)$ and ε replaced by 4ε. Next, in (4.36), we choose

$$m_2 = D_3(64\varepsilon)k_2^{\lambda+1}$$

for some $1 \le \lambda \le 4\delta^{-1}$. This yields

$$\begin{aligned} f\big(k_2, k_2, D_3(64\varepsilon)k_2^{\lambda+1}, 16\varepsilon\big) &\le 9D_3(64\varepsilon)k_2^{\lambda}\exp\big(-4C_8\varepsilon(\log k_1)^2\big) \\ &\quad + 6D_3(64\varepsilon)k_2^{\lambda+1}D_6(16\varepsilon)\exp\big(-D_7(16\varepsilon)k_2^{\frac{1}{2}}\big) \\ &\le \frac{16\varepsilon}{\nu}, \end{aligned}$$

provided $k_2 \ge \exp D_{14}(\varepsilon)\lambda$ and M is greater than or equal to some $M_3(\varepsilon, \delta) \ge M_2(\varepsilon)$. A final application of (4.36) with $k = n := k_2^{\lambda+1}$, $n^{2/(\lambda+1)} \le l \le n$ and $m \ge n$ gives

$$\begin{aligned} &f(n, l, m, 64\varepsilon) \\ &\quad \le \frac{9m}{n}\exp\big(-16C_8\varepsilon k_2^{\lambda-1}\big) + 6mD_6(64\varepsilon)\exp\big(-D_7(64\varepsilon)n^{l}\big). \end{aligned}$$

If we again replace 64ε by ε, we find that for $1/(\lambda+1) \le \delta$,

$$f(n, l, m, \varepsilon) \le \frac{9m}{n}\exp\big(-D_{15}(\varepsilon)n^{1-2\delta}l\big), n^{2\delta} \le l \le n, m \ge n. \tag{4.55}$$

Since we did not control k_0, k_1, k_2 or n carefully we must go back to check for which n's (4.55) holds. We merely know that there will be a k_0 in (4.54) to start the chain, provided $M \ge M_1(\varepsilon)$. Furthermore we need $k_2 = n^{1/(\lambda+1)} \ge \exp D_{14}(\varepsilon)\lambda$, but $1 \le \lambda \le 4\delta^{-1}$ arbitrary. If we use $k_1 = k_0(\log k_0)^{1/(2p^2)}$ and $k_2 = k_1 \log k_1$ we see that k_2 is some integer in

$$\left[\frac{M^{1/2}}{(p+2)}\left(\tfrac{1}{4}\log M\right), \frac{M}{p+1}(\log M)^2\right]. \tag{4.56}$$

The only restrictions on M, λ are

$$M \ge M_3(\varepsilon, \delta), \quad k_2 = n^{1/(\lambda+1)} \ge \exp D_{14}(\varepsilon)\lambda$$

and

$$4\delta^{-1} \ge \lambda \ge \max(1, \delta^{-1} - 1).$$

Without loss of generality take $\delta \le \frac{1}{2}$ so that we only need $4\delta^{-1} \ge \lambda \ge \delta^{-1} - 1$. Then for given n take $M = n^{\delta/2}$. For $n \ge n_1(\varepsilon, \delta)$ this M will be at least $M_3(\varepsilon, \delta)$. Next, find k_0, k_1, k_2 corresponding to this M. Since k_2 lies in (4.56) we find that

$$n^{\delta/4} \le k_2 \le n^{\delta}, \tag{4.57}$$

provided $n \ge n_2(\varepsilon, \delta)$ as well. Finally, choose $\lambda + 1 = (\log k_2)^{-1}\log n$ so that

$n = k_2^{\lambda+1}$. Then, by (4.57),

$$\delta^{-1} \le \lambda + 1 \le 4\delta^{-1},$$

and

$$k_2 \ge n^{\delta/4} \ge \exp\bigl(4D_{14}(\varepsilon)\delta^{-1} \ge \exp D_{14}(\varepsilon)\lambda,$$

provided $n \ge n_3(\varepsilon, \delta)$. Thus, all requirements can be satisfied if

$$n \ge n_0(\varepsilon, \delta) := \max\{n_i(\varepsilon, \delta) : i = 1, 2, 3\}.$$

In other words, (4.55) holds for *all* $n \ge n_0(\varepsilon, \delta)$.

(4.55) would be enough to estimate probabilities of cuts over squares. To deal with cuts over rectangles we must estimate $f(k, l, m, \varepsilon)$ for $k \ge l$. We cannot use (4.36) for this, as was done above, because each application of (4.36) raises ε to 4ε and, as we shall see, we need an unbounded number of applications of an analogue of (4.36). The required analogue is

$$(4.58) \quad f(k, l, m, \varepsilon) \le mC_1\exp(-C_2kl) + \left(\frac{m}{kl} + 1\right)f(k, l, C_{10}kl, \varepsilon), \quad \text{for } k \ge l \ge 1,\ m \ge 1.$$

(C_1, C_2 and C_{10} do *not* depend on ε.) (4.58) is trivial to prove, since $f(k, l, m, \varepsilon) - mC_1\exp(-C_2kl)$ is bounded in (4.13). Note that C_1 and C_2 in (4.13) are the same as in (2.1) and $C_5 = \Theta^{-1}\nu$; these constants depend on F only. In the left hand side of (4.13) we may restrict ourselves to connected cuts E^* (by (3.17)), and any such cut must contain a point in one of the at most $(kl)^{-1}m + 1$ segments

$$\{-\tfrac{1}{2}\} \times \{-\tfrac{1}{2}\} \times [jkl, (j+1)kl), 0 \le j \le (kl)^{-1}m.$$

But if E^* is connected and contains at most C_5kl plaquettes, one of which intersects $\{-\frac{1}{2}\} \times \{-\frac{1}{2}\} \times [jkl, (j+1)kl)$, then E^* is contained between the horizontal hyperplanes

$$\{x(3) = (j - C_5)kl\} \quad \text{and} \quad \{x(3) = (j + 1 + C_5)kl\}.$$

Thus, (4.13) and (4.3) yield

$$\begin{aligned} f(k, l, m, \varepsilon) &- mC_1\exp(-C_2kl) \\ &\le \sum_{0 \le j \le m/kl} P\{\text{there exists a cut } E^* \text{ which separates bottom from top} \\ &\qquad\qquad \text{in } [0, k] \times [0, l] \times [(j - C_5)kl, (j + 1 + C_5)kl] \text{ with} \\ &\qquad\qquad V(E^*) \le (\nu - 5\varepsilon)kl\} \\ &\le \left(\frac{m}{kl} + 1\right)f(k, l, (2C_5 + 1)kl, \varepsilon). \end{aligned}$$

(Note how much easier it is to prove (4.58) than (4.12), even though they seem to differ only in small details.)

Our starting point for the next step will be (4.55) with n equal to some large l. Fix $\varepsilon \le \varepsilon_0$, $\delta \le \frac{1}{2}$ and let $\eta_0 = \varepsilon$, $l_0 = l$,

$$\delta_i = i^{-2}\varepsilon,\ \eta_i = \varepsilon + \sum_{j=1}^{i} \delta_j,\ i \ge 1,\ l_1 = l^{1/\delta},\ l_2 = l^{2/\delta^2} \tag{4.59}$$

and[7]

$$\frac{l_i}{l_{i-1}} = \exp\left\lfloor \frac{C_7\varepsilon}{4i^2}\frac{l_{i-2}}{l_{i-3}} \right\rfloor \quad \text{for } i \ge 3.$$

It is not hard to see (by using induction on i) that—provided ε_0 is sufficiently small—there exists an

$$L = L(\varepsilon, \delta, F) \ge n_0\left(\varepsilon, \tfrac{1}{4}\right) + C_6\varepsilon^{-1} + C_{10},$$

such that for $l = l_0 \ge L$, (4.59) implies that for $i \ge 1$,

$$9C_{10}l_2\exp\left(-D_{15}(\varepsilon)l^{3/2}\right) \le \frac{\varepsilon}{\nu},$$

$$\frac{l_i}{l_{i-1}} \ge (i+2)^4\left(1 + C_6\varepsilon^{-1}\right), \quad \frac{l_{i+1}}{l_i} \ge \left(\frac{l_i}{l_{i-1}}\right)^{(i+1)/\delta}$$

(4.60) and

$$C_{10}C_1l_{i+2}l\exp(-C_2l_il) + \left(C_{10}\frac{l_{i+2}}{l_i} + 1\right)\exp\left(-\frac{C_7\varepsilon}{i^2}\frac{l_i}{l_{i-1}}\right) \le \frac{\varepsilon}{(i+1)^2\nu}.$$

We now show by induction that for $i \ge 1$, $m \ge 1$ one has

$$f(n, l, m, \eta_i) \le mC_1\exp(-C_2nl) + \left(\frac{m}{nl} + 1\right)\exp\left(-C_7\frac{n}{l_{i-1}}\delta_i\right), \quad l_i \le n \le l_{i+1}. \tag{4.61}$$

To prove (4.61) observe first that (by virtue of (4.58)) (4.61) will hold for a given i if

$$f(n, l, C_{10}nl, \eta_i) \le \exp\left(-C_7\frac{n}{l_{i-1}}\delta_i\right), \quad l_i \le n \le l_{i+1}.$$

[7] $\lfloor a \rfloor$ denotes the largest integer $\le a$.

In turn, by (4.33), this will hold if

$$f(l_{i-1}, l, C_{10}nl, \eta_{i-1}) \le \frac{\delta_i}{\nu}, \quad l_i \le n \le l_{i+1},$$

(4.62) and

$$l_{i-1} \ge C_6\delta_i^{-1}, \; l_{i-1}^{-1}l_i \ge C_6\delta_i^{-1}.$$

For $i = 1$, (4.62) and hence (4.61) holds by virtue of (4.55) and (4.60). Now assume that (4.61) holds for $i = I \ge 1$ and $l_I \le n \le l_{I+1}$. In particular we may then use (4.61) with $C_{10}nl$ and l_I substituted for m and n, respectively. For $n \le l_{I+2}$ this gives

$$\begin{aligned} f(l_I, l, C_{10}nl, \eta_i) &\le C_{10}C_1 nl \exp(-C_2 l_I l) \\ &\quad + \left(\frac{C_{10}n}{l_I} + 1\right)\exp\left(-C_7\frac{l_I}{l_{I-1}}\delta_I\right) \\ &\le \frac{\delta_{I+1}}{\nu}. \end{aligned}$$

(by (4.60)). Therefore (4.62) holds with $i = I + 1$ and so does (4.61). This completes the inductive proof of (4.61).

Theorem 2.10 is now immediate from (4.61) and the following observations:

$$\eta_i \le \varepsilon\left(1 + \sum_1^\infty i^{-2}\right) \le 3\varepsilon,$$

(4.63) $$P\{\beta(n,l) \le (\nu - 15\varepsilon)nl\} \le f(n, l, C_{10}nl, \eta_i) + C_1\exp(-C_2 nl).$$

(4.63) holds because a connected set E^* through $(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ with $V(E^*) \le \nu nl$ will contain at most $C_5 nl = \Theta^{-1}\nu nl$ plaquettes outside a set of probability $C_1\exp(-C_2 nl)$ (by (2.3)). If $|E^*| \le C_5 nl$ and E^* separates $-\infty$ from $+\infty$ over $[0, n] \times [0, l]$, then E^* contains a cut separating the bottom from the top of

$$[0, n] \times [0, l] \times [-C_5 nl, C_5 nl].$$

Thus, we can take $C_{10} = 2C_5 + 1$. Compare with the arguments for (4.58) and (4.13).

Finally, for fixed $0 < \delta < 1$ and $i \ge 1$,

$$l_i^\delta \ge \left(\frac{l_i}{l_{i-1}}\right)^\delta \ge \left(\frac{l_{i-1}}{l_{i-2}}\right)^i \ge \frac{l_{i-1}}{l_{i-2}}\frac{l_{i-2}}{l_{i-3}} \cdots \frac{l_1}{l_0}l_0 = l_{i-1}$$

(by (4.60)).

Thus, for $n \geq l_i$ one has $l_{i-1} \leq n^{\delta}$, and

$$(C_7\varepsilon)^{-1} i^2 l_{i-1} \leq 2(C_7\varepsilon)^{-1}\left(\frac{l_{i-1}}{l_{i-2}}\right)^{1/2} l_{i-1} \leq n^{2\delta}.$$

Consequently, by (4.61) and (4.63), for $i \geq 1$ and $l_i \leq n \leq l_{i+1}$,

$$\begin{aligned}(4.64)\quad P\{\theta((n,l)) \leq (\nu - 15\varepsilon)nl\} &\leq P\{\beta((n,l)) \leq (\nu - 15\varepsilon)nl\} \\ &\leq (C_{10}nl + 1)(C_1 + 1)\exp\left(-C_7\frac{n}{l_{i-1}}\delta_i\right) \\ &\leq \left(C_{10}n^2 + 1\right)(C_1 + 1)\exp - n^{1-2\delta}.\end{aligned}$$

Short of replacing 15ε by ε and 3δ by δ this proves (2.11) for $n \geq l_1$. For $l \leq n < l_1 = l^{1/\delta}$ we use (4.63) and (4.55), with δ replaced by $\delta/2$, directly. ∎

Proof of Theorem 2.12. This theorem also follows immediately from (4.61) and (4.55), by means of (1.6). Indeed (1.6) shows that

$$(4.65)\qquad \Phi(k,l,m) = \min\{V(E^*) : E^* \text{ is an } (F_0, F_m) \text{ cut of } [0,k]\times[0,l]\times[0,m]\}.$$

In particular, under the hypothesis (2.13), for $k \geq l$, $l_i \leq k \leq l_{i+1}$ and $i \geq 1$ (l_i and η_i as in the last proof) we have

$$\begin{aligned}P\{\Phi(k,l,m(k,l)) \leq (\nu - 15\varepsilon)kl\} &\leq f(k,l,m(k,l),\eta_i) \\ &\leq \exp(-k^{1-\delta} + o(k^{1-\delta}))\end{aligned}$$

(compare with (4.64) with δ replaced by $\delta/2$). Thus, under (2.13),

$$\liminf_{k,l\to\infty} \frac{1}{kl}\Phi(k,l,m(k,l)) \geq \nu \quad \text{w.p.1}$$

For the upper bound we can copy the two-dimensional proof of [8], Theorems 5.1 and 2.1b. Let

$$\begin{aligned}\tau^r(k,l) = \{\inf V(E^*) : &\ E^* \text{ is an } (F_0, F_{2r+1})\text{-cut of} \\ &B(k,l,2r+1) \text{ whose boundary } \partial E^* \text{ consists of} \\ &\text{the edges of } \mathscr{L}^* \text{ on the perimeter of} \\ &[-\tfrac12, k+\tfrac12]\times[-\tfrac12, l+\tfrac12]\times\{r+\tfrac12\}\end{aligned}$$

(compare with (1.11)). By (4.65), $\Phi(k,l,m) \leq \tau^r(k,l)$, whenever $m \geq 2r+1$.

Exactly as in (4.8) one shows that for each fixed r, $\tau^r(k, l)$ is subadditive in k, l, so that there exists a constant ν^r for which

$$\limsup_{k, l, m \to \infty} \frac{1}{kl} \Phi(k, l, m) \le \limsup_{k, l} \frac{1}{kl} \tau^r(k, l) = \nu^r \quad \text{w.p.1.}$$

Standard subadditivity arguments as in [15], p. 88 show that $\nu^r \downarrow \nu$ as $r \to \infty$, so that

$$\limsup_{k, l, m \to \infty} \frac{1}{kl} \Phi(k, l, m) \le \nu \quad \text{w.p.1.}$$

Proof of Theorem 2.18. We first prove (2.21) under the condition (2.22). Call an edge e of $\mathbf{Z}^3$ open (closed) if $t(e) = 0$ ($t(e) > 0$). Then

$$P\{e \text{ is closed}\} = 1 - F(0) < p_T(3). \tag{4.66}$$

Let $W(v)$ be the collection of all edges and vertices which can be connected to v by a closed path on $\mathbf{Z}^3$, i.e., a path all of whose edges are closed. Note that we define v to be a point of $W(v)$ always. Next we define

$$S(k, l) = \bigcup_{v \in [0, k] \times [0, l]} W(v)$$

and $F^*(k, l)$ as the collection of all plaquettes in

$$[-\tfrac{1}{2}, k + \tfrac{1}{2}] \times [-\tfrac{1}{2}, l + \tfrac{1}{2}] \times [\tfrac{1}{2}, \infty)$$

which are associated to some boundary edge e of $S(k, l)$, i.e., to some edge e with one endpoint in $S(k, l)$ and one endpoint outside $S(k, l)$. For later purposes note that such a boundary edge e is necessarily open (otherwise it should form part of $S(k, l)$). Thus, all plaquettes π^* in $F^*(k, l)$ have $t(\pi^*) = 0$ and

$$V(F^*(k, l)) = 0. \tag{4.67}$$

If

$$S(k, l) \subset \mathbf{R}^2 \times [-M + 1, M - 1], \tag{4.68}$$

then $F^*(k, l)$ separates the bottom from the top in $[0, k] \times [0, l] \times [0, M]$, for any path from $[0, k] \times [0, l] \times \{0\}$ to $[0, k] \times [0, l] \times \{M\}$ starts in $S(k, l)$ and ends outside $S(k, l)$, and hence contains a boundary edge of $S(k, l)$. In particular (4.68) implies that $F^*(k, l)$ contains an (F_0, F_m)-cut $E^*(k, l)$ of $B(k, l, M)$ with $V(E^*(k, l)) = 0$ (by (4.67)) and hence

$$\Phi(k, l, m) \le V(E^*(k, l)) = 0 \quad \text{for} \quad m \ge M. \tag{4.69}$$

Now, by [12], Theorem 5.1 or [9], Theorem 2, (4.66) implies that for suitable constants C_1, C_2,

$$P\{W(v) \text{ contains a point outside } \mathbf{R}^2 \times [-n, n]\} \leq C_1 e^{-C_2 n}, \quad n \geq 0, \tag{4.70}$$

whenever $v(3) = 0$. Consequently, for $C > 5C_2^{-1}$,

$$\begin{aligned} P\{S(k, l) &\subset \mathbf{R}^2 \times [-C\log(k + l), C\log(k + l)]\} \\ &\geq 1 - (k + 1)(l + 1)C_1 e^{-C_2 C \log(k+l)} \\ &\leq 1 - C_1 \frac{(k + 1)(l + 1)}{(k + l)^5}. \end{aligned}$$

By the Borel-Cantelli lemma this shows that w.p.1,

$$S(k, l) \subset \mathbf{R}^2 \times [-C\log(k + l), C\log(k + l)] \quad \text{for all large } k \text{ and } l.$$

Now, under the condition (2.22), (2.21) follows from (4.68), (4.69).

To prove (2.20) we take for $G^*(k, l)$ the collection of all plaquettes

$$[a - \tfrac{1}{2}, a + \tfrac{1}{2}] \times [b - \tfrac{1}{2}, b + \tfrac{1}{2}] \times \{\tfrac{1}{2}\}$$

with $a = -1$ or $k + 1$ and $-1 \leq b \leq l + 1$, or $b = -1$ or $l + 1$ and $-1 \leq a \leq k + 1$. Further we take for $H^*(k, l)$ all plaquettes π^* in one of the four vertical strips,

$$\begin{aligned} &\{a\} \times [-\tfrac{1}{2}, l + \tfrac{1}{2}] \times [\tfrac{1}{2}, \infty), \quad a = -\tfrac{1}{2} \quad \text{or} \quad a = k + \tfrac{1}{2}, \\ &[-\tfrac{1}{2}, k + \tfrac{1}{2}] \times \{b\} \times [\tfrac{1}{2}, \infty), \quad b = -\tfrac{1}{2} \quad \text{or} \quad b = l + \tfrac{1}{2}, \end{aligned}$$

with the property that π^* intersects an edge of $S(k, l)$. We claim that

$$F^*(k, l) \cup G^*(k, l) \cup H^*(k, l)$$

contains a cut over

$$[-1, k + 1] \times [-1, l + 1].$$

In fact it even separates

$$F_0 := [-1, k + 1] \times [-1, l + 1] \times \{0\}$$

from

$$F_M := [-1, k + 1] \times [-1, l + 1] \times \{M\}$$

in

$$B := [-1, k+1] \times [-1, l+1] \times [0, M]$$

whenever (4.68) holds. To see this consider any path ϕ on $\mathbf{Z}^3$ from F_0 to F_M in B, which lies in $[-1, k+1] \times [-1, l+1] \times (0, M)$ with the exception of its endpoints in F_0 and F_M, respectively. ϕ must start with a vertical edge from some $v = (v(1), v(2), 0)$. If $v(1) = -1$ or $k+1$ then the first edge in ϕ already intersects one of the plaquettes in $G^*(k, l)$. Similarly if $v(2) = -1$ or $l+1$. Thus, we may assume $0 \le v(1) \le k, 0 \le v(2) \le l$, so that ϕ starts in $S(k, l)$. If ϕ stays in $[0, k] \times [0, l] \times [0, M]$ and (4.68) holds, then ϕ must contain a boundary edge of $S(k, l)$ in $[0, k] \times [0, l] \times [0, M]$ and hence cross $F^*(k, l)$. Finally, if ϕ leaves $[0, k] \times [0, l] \times [0, M]$ before it has crossed $F^*(k, l)$, and e is the first edge of ϕ not contained in $[0, k] \times [0, l] \times [0, M]$, then either e is a boundary edge of $S(k, l)$ and hence crosses $F^*(k, l)$ or e is an edge of $S(k, l)$ and crosses $H^*(k, l)$. This proves our claim. As a corollary,

$$F^*(k, l) \cup G^*(k, l) \cup H^*(k, l)$$

contains a cut $\tilde{E}^*(k, l)$ over

$$[-1, k+1] \times [-1, l+1]$$

with

$$\begin{aligned} V(\tilde{E}^*(k, l)) &\le V(F^*(k, l)) + V(G^*(k, l)) + V(H^*(k, l)) \\ &= V(G^*(k, l)) + V(H^*(k, l)). \end{aligned}$$

Moreover, the only plaquettes in $F^*(k, l) \cup G^*(k, l) \cup H^*(k, l)$ which intersect the vertical boundary of $[-\frac{3}{2}, k+\frac{3}{2}] \times [-\frac{3}{2}, l+\frac{3}{2}] \times \mathbf{R}$ are those of $G^*(k, l)$. It follows from this and Lemma 3.6(a) (cf. the proof of Prop. 4.45) that $\partial\tilde{E}^*(k, l)$ consists exactly of the edges in the perimeter of $[-\frac{3}{2}, k+\frac{3}{2}] \times [-\frac{3}{2}, l+\frac{3}{2}] \times \{\frac{1}{2}\}$. Moreover $\tilde{E}^*$ is contained in $[-\frac{3}{2}, k+\frac{1}{2}] \times [-\frac{3}{2}, l+\frac{1}{2}] \times \mathbf{R}$. Thus, except for a translation by $(-1, 1)$, $\tilde{E}^*(k, l)$ is one of the E^*'s which figure in the definitions (1.11), (1.12), (2.5) and (2.6) of

$$\tau(k+2, l+2), \alpha(k+2, l+2), \sigma(k+2, l+2) \quad \text{and} \quad \beta(k+2, l+2),$$

respectively. Therefore, for all x and $\theta = \alpha, \beta, \sigma$ or τ,

$$\text{(4.71)} \quad P\{\theta(k+2, l+2) \ge x\} \le P\{V(G^*(k, l) + V(H^*(k, l)) \ge x\}.$$

Now, $G^*(k, l)$ contains $2(k+l+6)$ plaquettes, and we need an estimate for $|H^*(k, l)|$, the number of plaquettes in $H^*(k, l)$.

For any (i, j) let $N(i, j)$ be the number of $n \geq 0$ such that $(i, j, n) \in W(v)$ for some $v = (v(1), v(2), v(3))$ with $v(3) = 0$.

The definition of $H^*(k, l)$ shows that

$$(4.72) \qquad |H^*(k, l)| \leq 2 \sum_{k,l} N(i, j).$$

where $\Sigma_{k,l}$ is the sum over the pairs (a, j) with $a = 0$ or k and $0 \leq j \leq l$ and the pairs (i, b) with $b = 0$ or l and $0 \leq i \leq k$. To estimate (4.72) we need an approximation $\hat{N}(i, j)$ to $N(i, j)$. $\hat{N}$ is defined as follows.

Let $\hat{N}(i, j)$ be the number of $n \geq 0$ such that (i, j, n) is connected by a closed path ϕ to some $(v(1), v(2), 0)$ with ϕ contained in

$$\begin{aligned}[i - C\log(k + l), i + C\log(k + l)]& \\ \times [j - C\log(k + l), j + C\log(k + l)]& \times \mathbf{R}.\end{aligned}$$

Clearly $\hat{N}(i, j) \leq N(i, j)$ and

$$\begin{aligned}(4.73) \quad & P\{\hat{N}(i, j) \neq N(i, j)\} \\ & \leq \Sigma_{n \geq \kappa} P\{(i, j, n) \text{ is connected to some } (v(1), v(2), 0) \text{ by a closed path}\} \\ & \quad + \Sigma_{k=0}^{\kappa-1} P\{(i, j, n) \text{ is connected by a closed path to some point outside} \\ & \quad \times [i - C\log(k + l), i + C\log(k + l)] \\ & \quad \times [j - C\log(k + l), j + C\log(k + l)]\}.\end{aligned}$$

If we take $\kappa = 5C_2^{-1}\log(k + l)$, then we obtain from (4.70) that the right hand side of (4.73) is at most

$$\sum_{n \geq \kappa} C_1 \exp - C_2 n + \kappa C_1 \exp - C_2 C \log(k + l) \leq C_3 (k + l)^{-4}.$$

Consequently

$$P\left\{\sum_{k,l} \hat{N}(i, j) \neq \sum_{k,l} N(i, j)\right\} = 4C_3(k + l)^{-3}$$

and, for any α,

$$\begin{aligned}(4.74) \qquad & P\{|H^*(k, l)| \geq 2\alpha(k + l)\} \\ & \leq 4C_3(k + l)^{-3} + P\left\{\sum_{k,l} \hat{N}(i, j) \geq \alpha(k + l)\right\}.\end{aligned}$$

Now, the estimate (4.70) also shows that

$$\begin{aligned} &P\{\hat{N}(i,j) \ge x\} \\ &\quad \le P\{N(i,j) \ge x\} \\ &\quad \le \sum_{n \ge x} P\{(i,j,n) \text{ is connected to some } (v(1), v(2), 0) \text{ by a closed path}\} \\ &\quad \le C_1(1 - e^{-C_2})^{-1} e^{-C_2 x}, \end{aligned}$$

so that $\hat{N}$ has all moments. Moreover, by definition, any family $\{\hat{N}(i_r, j_r)\}$ which satisfies

$$|i_r - i_s| + |j_r - j_s| > 2C \log(k + l) \quad \text{for all } r \neq s$$

is independent. It is now easy to obtain

$$E\Big|\sum_{k,l} \{\hat{N}(i,j) - E\hat{N}(i,j)\}\Big|^6 \le C_4\{(k + l)\log(k + l)\}^3.$$

Together with (4.74) and the estimate on $|G^*(k,l)|$ this shows that

$$\sum_{k,l} P\{|G^*(k,l)| + |H^*(k,l)| \ge 3(EN(0,0) + 1)(k + l)\} < \infty. \tag{4.75}$$

Write C_5 for $3(EN(0,0) + 1)$. In view of (4.71) and (4.75) we will obtain (2.20) if we show that

$$\begin{aligned} &\sum_{k,l} P\{V(G^*(k,l)) + V(H^*(k,l)) \ge C_6(k + l) \\ &\quad \text{but} \\ &\quad |G^*(k,l)| + |H^*(k,l)| \le C_5(k + l)\} < \infty \end{aligned} \tag{4.76}$$

for suitable $C_6 < \infty$. Finally note that the sets $G^*(k,l)$ and $H^*(k,l)$ depend only on which edges are open or closed, but not on the actual values of the closed edges. Thus, conditional on $G^*(k,l)$ and $H^*(k,l)$, the values of the plaquettes in these sets are still independent. The distribution function of $v(\pi^*)$ for π^* in $H^*(k,l)$ is the conditional distribution function of $t(e)$, given $t(e) > 0$, i.e.,

$$(1 - F(0))^{-1}(F(x) - F(0)).$$

This distribution function of $v(\pi^*)$ for $\pi^* \in G^*(k,l)$ is simply $F(x)$. The summand in (4.76) is therefore bounded by

$$\begin{aligned} P\{&U(1) + \cdots + U(C_5(k + l)) + V(1) \\ &+ \cdots + V(C_5(k + l)) \ge C_6(k + l)\}, \end{aligned}$$

where all $U(i)$, $V(j)$ are independent and each $U(V)$ has distribution function $(1 - F(0))^{-1}(F - F(0))(F)$. We leave it to the reader to derive (4.76) by means of Chebyshev's inequality (with sixth moments). As mentioned above this completes the proof.

REFERENCES

1. D. ABRAHAM, J. CHAYES and L. CHAYES, *Random surface correlation functions*, to appear in Comm. Math. Phys. with follow up in Nuclear Phys. B.
2. M. AIZENMAN, J.T. CHAYES, L. CHAYES, J. FRÖHLICH and L. RUSSO, *On a sharp transition from area law to perimeter law in a system of random surfaces*, Comm. Math. Phys., vol. 92 (1983), pp. 19–69.
3. M.A. AKCOGLU and U. KRENGEL, *Ergodic theorems for superadditive processes*, J. Reine Angew. Math., vol. 323 (1981), pp. 53–67.
4. B. BOLLOBÁS, *Graph theory, An introductory course*, Springer-Verlag, 1979.
5. J.T. COX and R. DURRETT, *Some limit theorems for percolation processes with necessary and sufficient conditions*, Ann. Probab., vol. **9** (1981), pp. 583–603.
6. J. FRÖHLICH, *Quantum field theory in terms of random walks and random surfaces*, Lectures at Cargèse Summer School, 1983.
7. D.R. FULKERSON, "Flow networks and combinatorial operations research." in *Studies in Graph Theory*, D.R. Fulkerson, ed., Math Assoc. of America, 1975, pp. 139–171.
8. G. GRIMMETT and H. KESTEN, *First-passage percolation, network flows and electrical resistances*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, vol. 66 (1984), pp. 335–366.
9. J.M. HAMMERSLEY, *Percolation processes: Lower bounds for the critical probability*, Ann. Math. Statist., vol. 28 (1957), pp. 790–795.
10. J.M. HAMMERSLEY and D.J.A. WELSH, "First-passage percolation, subadditive processes, stochastic networks and generalized renewal theory" in *Bernoulli-Bayes-Laplace Anniversary Volume*, J. Neyman and L.M. LeCam eds., Springer-Verlag, New York, 1965, pp. 61–110.
11. H. KESTEN, *The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$*, Comm. Math. Phys., vol. 74 (1980), pp. 41–59.
12. ______, *Percolation theory for mathematicians*, Birkhäuser-Boston, 1982.
13. ______, *Aspects of first-passage percolation*, École d'été de calcul des probabilités, St. Flour 1984; in Springer Lecture Notes in Mathematics vol. 180, Springer-Verlag, New York.
14. R.T. SMYTHE, *Multiparameter subadditive processes*, Ann. Probability, vol. 4 (1976), pp. 772–782.
15. R.T. SMYTHE and J.C. WIERMAN, *First-passage percolation on the square lattice*, Lecture Notes in Mathematics, vol. 671, Springer-Verlag, New York, 1978.
16. H. WHITNEY, *Non-separable and planar graphs*, Trans. Amer. Math. Soc., vol. 34 (1932), pp. 339–362.
17. ______, *Planar graphs*, Fund. Math., vol. 21 (1933), pp. 73–84.

CORNELL UNIVERSITY
ITHACA, NEW YORK