ILLINOIS JOURNAL OF MATHEMATICS
Volume 31, Number 2, Summer 1987

ON THE HOMOTOPY TYPE OF DIFF(Σ^n)

BY

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Let M be a closed smooth n-manifold and let $\operatorname{Diff}_0(M)$ denote the identity component of the group $\operatorname{Diff}(M)$ of orientation preserving diffeomorphisms of M. It is shown in [5] that if M and N are homeomorphic, $n \neq 4$, then $\operatorname{Diff}_0(M)$ and $\operatorname{Diff}_0(N)$ have the same homotopy type away from a finite set of primes. More precisely, there is a subring Λ_n of the rationals Q obtained from the integers Z by inverting a finite set of primes so that $\operatorname{Diff}_0(M)$ and $\operatorname{Diff}_0(N)$ have the same Λ_n-homotopy type. If M and N are homotopy spheres, the result can be improved to state that $\operatorname{Diff}_0(M)$ and $\operatorname{Diff}_0(N)$ have the same $Z[1/m]$-homotopy type where m is the order of M-N in the group θ_n of homotopy n-spheres. (See Theorem 1.5 below.)

The object of this paper is to show that differences do indeed occur in the homotopy type of diffeomorphism groups of homeomorphic manifolds. In fact, one can take the manifolds to be homotopy spheres. Our principal results are the following.

THEOREM 1. *For each prime p, there is an integer n and an exotic sphere Σ^n such that*

$$\pi_i(\operatorname{Diff}_0(\Sigma^n)) \otimes Z\left[\frac{1}{p}\right] \simeq \pi_i(\operatorname{Diff}_0(S^n)) \otimes Z\left[\frac{1}{p}\right]$$

for all $i > 0$ but

$$\pi_j(\operatorname{Diff}_0(\Sigma^n)) \not\simeq \pi_j(\operatorname{Diff}_0(S^n))$$

for some $j > 0$.

THEOREM 2. *There are homotopy spheres Σ_1^n, Σ_2^n of dimension $n = 2202$ such that no two of the groups* $\operatorname{Diff}_0(S^n)$, $\operatorname{Diff}_0(\Sigma_1^n)$, $\operatorname{Diff}_0(\Sigma_2^n)$, *and* $\operatorname{Diff}_0(\Sigma_1^n \# \Sigma_2^n)$ *have the same homotopy type.*

It follows that the results of [5] are in general best possible.

Received July 29, 1985.
[1]Partially supported by a grant from the National Science Foundation.

Remark. In what follows, we will always take our base point to be the identity element for the homotopy groups of diffeomorphism groups. Thus

$$\pi_i(\mathrm{Diff}(M)) = \pi_i(\mathrm{Diff}_0(M)) \quad \text{for } i > 0.$$

The authors would like to thank Dan Burghelea, Don Davis, Bill Dwyer and Kyoshi Igusa for helpful conversations during the preparation of this paper.

1. Proof of the main theorems

Let Σ^n_β be the homotopy sphere defined by $\beta \in \pi_0(\mathrm{Diff}(S^{n-1}))$,

$$\Sigma^n_\beta = D^n_+ \cup_\beta D^n_- .$$

Denote by $\mathrm{Diff}_\partial(D^n)$ the group of diffeomorphisms of D^n which are the identity on ∂D^n and let

$$t^\beta\colon \mathrm{Diff}_\partial(D^n) \to \mathrm{Diff}(\Sigma^n_\beta)$$

be the mapping obtained by "extending with the identity on D^n". Let

$$t^\beta_k\colon \pi_k(\mathrm{Diff}_\partial(D^n)) \to \pi_k(\mathrm{Diff}(\Sigma^n_\beta))$$

be the induced homomorphism.

LEMMA 1.1. *If $i = 3 \bmod 4$ and $i < n$, then t^β_{i-1} is surjective for all $\beta \in \pi_0(\mathrm{Diff}(S^{n-1}))$. If $\beta = 0$ (so that $\Sigma^n_\beta = S^n$), then $t_{i-1} = t^0_{i-1}$ is bijective.*

Proof. Let $\mathrm{Emb}(D^n_+, \Sigma^n_\beta)$ be the space of embeddings of D^n_+ into Σ^n_β. Consider the fibration

$$\mathrm{Diff}_\partial(D^n) \xrightarrow{t^\beta} \mathrm{Diff}(\Sigma^n_\beta) \to \mathrm{Emb}(D^n_+, \Sigma^n_\beta) \tag{1.2}$$

and the fibration (up to homotopy)

$$O(n) \to \mathrm{Emb}(D^n_+, \Sigma^n_\beta) \to \Sigma^n.$$

The first part of Lemma 1.1 now follows from the fact that $\pi_{i-1}(O(n)) = \pi_{i-1}(O) = 0$ for $i < n$, $i \equiv 3 \bmod 4$. The fact that t_{i-1} is bijective follows from the existence of a section in the fibration (1.2) when $\Sigma^n_\beta = S^n$.

Lemma 1.1 implies that we have an epimorphism

$$t^\beta_{i-1} \circ t^{-1}_{i-1}\colon \pi_{i-1}(\mathrm{Diff}(S^n)) \to \pi_{i-1}(\mathrm{Diff}(\Sigma^n_\beta))$$

for $i < n$, $i \equiv 3 \bmod 4$. Our results will be proved by exhibiting elements in the kernel of this epimorphism for particular values of n and i for which the groups $\pi_{i-1}(\mathrm{Diff}(S^n))$ and $\pi_{i-1}(\mathrm{Diff}(\Sigma^n_\beta))$ are finitely generated.

Let ∂_i: $\pi_i(\mathrm{Diff}(S^{n-1})) \to \pi_{i-1}(\mathrm{Diff}_\partial(D^n))$ be the boundary homomorphism of the fibration

$$\mathrm{Diff}_\partial(D^n) \to \mathrm{Diff}(D^n) \xrightarrow{p} \mathrm{Diff}(S^{n-1})$$

and let $\tilde{\beta}$: $\mathrm{Diff}(S^{n-1}) \to \mathrm{Diff}(S^{n-1})$ be "conjugation with a representative of β". We write $\beta \cdot \gamma$ for $\tilde{\beta}_i(\gamma)$ where

$$\tilde{\beta}_i\colon \pi_i(\mathrm{Diff}(S^{n-1})) \to \pi_i(\mathrm{Diff}(S^{n-1}))$$

is induced by $\tilde{\beta}$ and denote by j: $SO(n) \to \mathrm{Diff}(S^{n-1})$ the inclusion mapping.

LEMMA 1.3. *For any $\alpha \in \pi_i(SO(n))$ and $\beta \in \pi_0(\mathrm{Diff}(S^{n-1}))$, we have*

$$t^\beta_{i-1}\partial_i(\beta \cdot j_*\alpha) = 0.$$

Proof. (See [10] and [11]). Let $\mathrm{Diff}(\Sigma^n_\beta, D^n_-)$ be the subgroup of $\mathrm{Diff}(\Sigma^n_\beta)$ consisting of all diffeomorphisms g with $gD^n_- = D^n_-$. Consider the diagram of fibrations

$$\begin{array}{ccccc}
\mathrm{Diff}_\partial(D^n_+) & \xleftarrow{\ \mathrm{id}\ } & \mathrm{Diff}_\partial(D^n_+) & \overset{t^\beta}{\searrow} & \\
\downarrow & & \downarrow{\scriptstyle l} & & \\
\mathrm{Diff}(D^n_+) & \longleftarrow & \mathrm{Diff}(\Sigma^n_\beta, D^n_-) & \xrightarrow{k} & \mathrm{Diff}(\Sigma^n_\beta) \\
\downarrow & & \downarrow & & \\
\mathrm{Diff}(S^{n-1}_+) & \xleftarrow{\ s\ } & \mathrm{Diff}(D^n_-) & &
\end{array}$$

where $s = \tilde{\beta} \circ p$ and k is the inclusion. Then, if

$$\partial_i'\colon \pi_i(\mathrm{Diff}(D^n_-)) \to \pi_{i-1}(\mathrm{Diff}_\partial(D^n_+))$$

is the boundary homomorphism of the right most fibration, we have

$$\begin{aligned} t^\beta_{i-1}(\partial_i(s_i(\alpha))) &= k_{i-1}l_{i-1}\partial_i(\beta \cdot p_*(\alpha)) \\ &= k_{i-1}l_{i-1}\partial_i'(\alpha) = 0. \end{aligned}$$

Lemma 1.3 now follows from the fact that $j = p \circ \tilde{j}$, $\tilde{j}$: $SO(n) \subset \mathrm{Diff}(D^n)$.

Let p be a prime and r, s integers with $0 \leq s < r < p$, $r - s \neq p - 1$. Define integers n and i depending on p by setting $n = 19$, $i = 3$ if $p = 2$ and

$$n = 2(rp^2 - (r - s)p - r), \qquad i = 2p - 3$$

if p is odd.

LEMMA 1.4. *For p a prime and n, i as above, there are p-torsion elements*

$$\beta \in \pi_0(\mathrm{Diff}(S^{n-1})) \quad \textit{and} \quad \alpha \in \pi_i(SO(n))$$

such that $\partial_i(\beta \cdot j_* \alpha)$ is a non-zero p-torsion element in $\pi_{i-1}(\mathrm{Diff}_\partial(D^n))$.

Proof. According to [1], pages 18 and 31 (see also [2] and [12]), we can find elements $\bar{\beta} \in \pi_n$, $\bar{\alpha} \in \pi_i$ with $\bar{\beta} \notin \mathrm{Im}\, J_n$, $\bar{\alpha} \in \mathrm{Im}\, J_i$, and $\bar{\beta} \circ \bar{\alpha} \notin \mathrm{Im}\, J_{n+i}$ where J_k: $\pi_k \to \pi_k(SO)$ is the stable J-homomorphism. It then follows that there are elements $\beta \in \pi_0(\mathrm{Diff}(S^{n-1}))$, $\alpha \in \pi_i(SO(n))$ such that $\rho(\beta) = \bar{\beta}$ and $J_i \alpha = \bar{\alpha}$ where ρ is the Kervaine-Milnor homomorphism (see [9]). Lemma 1.4 now follows as in [1] (see also [10] and [11]).

W.G. Dwyer [3] has shown that $\pi_i(\mathrm{Diff}_\partial(M^n))$ is finitely generated for i, n in the stability range for pseudo isotopy. (See also [6], page 16.) Igusa [8] has shown that for $n > 11$, $i \leq [(n - 4)/3]$ is within the stability range. (See also [7].) One checks easily that most n and i of Lemma 1.4 satisfy this condition. In particular, if we set $r = p - 1$, $s = p - 2$ for any odd prime p, then the corresponding n is greater than 11 and $i = 2p - 3$ satisfies $i \leq [(n - 4)/3]$. Thus $\pi_{i-1}(\mathrm{Diff}(\Sigma^n_\beta))$ is finitely generated for these values of i and n.

Remark. Let $C(D^n)$ be the pseudo isotopy space of D^n. One can prove that $\pi_i(\mathrm{Diff}_\partial(D^n))$ (and thus $\pi_i(\mathrm{Diff}(S^{n-1}))$) is finitely generated wherever $\pi_i(C(D^n))$ is by induction on i using the fibration

$$\mathrm{Diff}_\partial(D^{n+1}) \to C(D^n) \to \mathrm{Diff}_\partial(D^n).$$

To complete the proof of Theorem 1, we need the following result stated in [4]. It is proved using the same argument used to prove Theorem 1.1 of [5].

THEOREM 1.5 (Dwyer-Szczarba). *Let Σ^n_1 and Σ^n_2 be homotopy spheres and let m be the order of the element $\Sigma^n_1 - \Sigma^n_2$ in θ^n. Then $\mathrm{Diff}_0(\Sigma^n_1)$ and $\mathrm{Diff}_0(\Sigma^n_2)$ have the same $Z[1/m]$-homotopy type.*

Thus, since $\Sigma^n_\beta - S^n$ has order a power of p, we have

$$\pi_i(\mathrm{Diff}(\Sigma^n_\beta)) \otimes Z\left[\frac{1}{p}\right] \cong \pi_i(\mathrm{Diff}(S^n)) \otimes Z\left[\frac{1}{p}\right].$$

To prove Theorem 2, let $p_1 = 11$, $r_1 = 10$, $s_1 = 1$, and $p_2 = 17$, $r_2 = 4$, $s_2 = 1$. Then

$$n = 2202 = 2\left(r_1 p_1^2 - (r_1 - s_1)p_1 - r_1\right) = 2\left(r_2 p_2^2 - (r_2 - s_2)p_2 - r_2\right).$$

According to Lemma 1.4, there are elements

$$\beta_1, \beta_2 \in \pi_0\left(\mathrm{Diff}(S^{n-1})\right),\ \alpha_1 \in \pi_{19}\left(SO(n)\right),\ \alpha_2 \in \pi_{31}\left(SO(n)\right)$$

such that $\partial(\beta_1 \cdot i_*(\alpha_1))$ is a non-zero 11-torsion element in $\pi_{18}(\mathrm{Diff}_\partial(D^n))$ and $\partial(\beta_2 \cdot i_*(\alpha_2))$ is a non-zero 17-torsion element in $\pi_{30}(\mathrm{Diff}_\partial(D^n))$. If $\Sigma_1^n = \Sigma_{\beta_1}^n$ and $\Sigma_2^n = \Sigma_{\beta_2}^n$, we know from Lemma 1.3 that

$$t_*^{\beta_1} t_*^{-1}\colon \pi_{18}\left(\mathrm{Diff}(S^n)\right) \to \pi_{18}\left(\mathrm{Diff}(\Sigma_1^n)\right)$$

has non-trivial 11-torsion in its kernel and

$$t_*^{\beta_2} t_*^{-1}\colon \pi_{30}\left(\mathrm{Diff}(S^n)\right) \to \pi_{30}\left(\mathrm{Diff}(\Sigma_2^n)\right)$$

has nontrivial 17-torsion in its kernel. It now follows from Theorem 1.5 that $\pi_{18}(\mathrm{Diff}(\Sigma_1^n))$ and $\pi_{18}(\mathrm{Diff}(\Sigma_2^n))$ are not isomorphic so $\mathrm{Diff}(S^n)$, $\mathrm{Diff}(\Sigma_1^n)$, and $\mathrm{Diff}(\Sigma_2^n)$ have distinct homotopy types.

In the same way, applying Lemmas 1.3 and 1.4 to the elements

$$\partial\left((\beta_1 + \beta_2) \cdot i_*(\alpha_1)\right) \quad \text{and} \quad \partial\left((\beta_1 + \beta_2) \cdot i_*(\alpha_2)\right)$$

shows that

$$t_*^{(\beta_1+\beta_2)} t_*^{-1}\colon \pi_{18}\left(\mathrm{Diff}(S^n)\right) \to \pi_{18}\left(\mathrm{Diff}(\Sigma_1^n \# \Sigma_2^n)\right)$$

has nontrivial 11-torsion in its kernel and

$$t_*^{(\beta_1+\beta_2)} t_*^{-1}\colon \pi_{30}\left(\mathrm{Diff}(S^n)\right) \to \pi_{30}\left(\mathrm{Diff}(\Sigma_1^n \# \Sigma_2^n)\right)$$

has nontrivial 17-torsion in its kernel. Theorem 2 now follows easily from Theorem 1.5.

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